DYNAMICS ON THE CIRCLE I

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Dynamics is the study of the motion of a body, or more generally evolution of a system with time, for instance, the motion of two revolving bodies attracted to each other by the force of gravity. To study such a system, we generally need to only consider a certain number of parameters, for instance, the positions and velocities of the bodies. These parameters taken together constitute the phase space X.

Further, the evolution in time is governed by some physical law. We call the phase space with the given law a *dynamical system*. These laws may be of different kinds, so we need to distinguish at this point between a variety of different dynamical systems.

The first distinction is whether we regard time as *discrete* or *continuous*. In this article, we shall take time to be discrete, i.e., we talk of the state of the system at times $t = 0$, $t = 1$, $t = 2$ etc. For systems with continuous time, we can consider the state at fixed intervals to obtain a discrete system. For instance, we may observe the position of a pendulum once every second. Hence most of the methods developed here work equally well for systems with continuous time. In fact often systems with continuous time are easier to understand than their discrete counterparts.

We thus have a dynamical system whose state at time $t+1$ is a point $x(t+1)$ in X that depends on its state $x(t)$ at time t. We make a further distinction between a *deterministic system* where $x(t+1)$ is determined by $x(t)$ and a *stochastic system* where $x(t+1)$ is a random variable whose probability distribution is determined by $x(t)$. We confine ourselves here to deterministic systems.

Assume moreover that absolute time does not matter. Then $x(t+1)$ is a function $f(x(t))$ of the state of the system at time t. Here f is a function from X to itself. Thus our setup is as follows.

Definition 0.1. A dynamical system is a phase space X together with a function $f : X \to X$.

Aside. Stochastic systems can also be treated in this framework by considering in place of the the phase space X the space $P(X)$ of probability distributions on X, since the probability distribution $P(t)$ of $x(t)$ determines the probability distribution $P(t+1)$ of $x(t+1)$.

1. Topological dynamics on the circle

So far we have said nothing about the nature of the function f or the space X. We shall take f to be a continuous function. This puts us in the realm of *topological dynamics*. We shall take our space X to be the circle S^1. Although as a space S^1 is simple its dynamics is very rich.

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In this article, we make the major simplifying assumption that our dynamical system is *reversible*, i.e., f has an inverse. More general dynamics of a circle is a complex and actively studied field.

Aside. There are other major branches of dynamics, for instance *smooth dynamics* where f is taken to be smooth, *conformal dynamics* where f is an analytical map on a complex domain and *measurable dynamics* where f is a measurable function on a probability space.

1.1. **Reparametrisation.** The first step in studying dynamics on the circle is to realise that we obtain the same dynamics if the circle is *reparametrised*, i.e., we make a change of co-ordinates. A reparametrisation is given by an invertible function $g : S^1 \to S^1$. On reparametrising, f is replaced by $g \circ f \circ g^{-1}$. Thus, we regard the dynamical systems given by f and $g \circ f \circ g^{-1}$ as equivalent. We formalise this.

Definition 1.1. Functions $f, h : X \to X$ are said to be *conjugate* if there exists an invertible continuous function $g : X \to X$ such that $h = g \circ f \circ g^{-1}$.

To understand dynamical systems on the circle thus amounts to classifying (invertible) continuous functions $f : S^1 \to S^1$ up to conjugacy. We first need to understand their structure a little better.

Aside. If we wished to study smooth dynamical systems (as most real life functions are smooth), we may assume f is smooth but we have to confine ourself to smooth re-parametrisations g to get a meaningful theory. This is actually *harder* than the topological case - what we gain here by allowing any continuous re-parametrisation exceeds the cost of not assuming smoothness for f.

1.2. **Fixed points.** A fixed point of f is a point $x \in S^1$ such that $f(x) = x$. We denote the set of fixed points as $Fix(f)$. This is a closed set of S^1.

We shall begin with the case when $Fix(f)$ is non-empty, i.e., there is at least one fixed point. This is the easier case.

Since the complement of the fixed point set is a non-empty open set of S^1, it is a (possibly infinite) collection of intervals $I_k = (a_k, b_k)$. The function f is determined by its restriction to each of these intervals, each of which we identify with the open interval $(0, 1)$. Further the restriction of f to I_k, with the above identification, extends to a map $f : [0, 1] \to [0, 1]$.

Thus, the dynamics in the case with fixed points reduces to dynamics on the interval $[0, 1]$.

2. Dynamics on the interval $[0, 1]$

We have reduced the study of the case with fixed points to considering functions $f : [0, 1] \to [0, 1]$ that are invertible with $f(0) = 0$ and $f(1) = 1$. Observe that if $[0, 1]$ is identified with a subinterval I of the circle, conjugation by an invertible function $g : [0, 1] \to [0, 1]$ that fixes 0 and 1 extends to conjugation by a function on S^1 by the function that is given by g on I and is identity elsewhere. More generally, if we break the circle into disjoint intervals I_j which are mapped to themselves by f and we have a function h which fixes I_j with f conjugate to h on each I_j, then f is conjugate to h on S^1.

Further, if f has fixed points in $(0, 1)$, the complement of the fixed point set is a collection of intervals. By studying f separately on each of these intervals, we can reduce to the case where f has no fixed points on $(0,1)$. We shall show that up to conjugation by g as above, there are *only two* classes of functions.

We begin with a simple observation.

Lemma 2.1. *If $f : [0,1] \to [0,1]$ has no fixed points on $(0,1)$, then f is either strictly increasing or strictly decreasing on $(0,1)$*

Proof. Consider the function $f(x)-x$ on $(0,1)$. This is continuous and, as f has no fixed points, it is never zero. Hence it must be everywhere positive or everywhere negative, i.e., $f(x) > x\ \forall x \in (0,1)$ or $f(x) < x\ \forall x \in (0,1)$. □

We shall show that all increasing functions are conjugate to each other. A typical example of such a function is $f(x) = \sqrt{x}$. A similar argument shows that all decreasing functions are also conjugate to each other.

Theorem 2.2. *If $f, h : [0,1] \to [0,1]$ are functions such that $f(0) = h(0) = 0$ and $f(1) = h(1) = 1$, $f(x) > x\ \forall x$ and $h(x) > x\ \forall x$ then f and h are conjugate.*

Proof. We construct a function g that gives the conjugacy. Pick points x and y in $(0,1)$. For $n \in \mathbb{Z}$, define $x_n = f^n(x)$ and $y_n = h^n(x)$. As f and h are increasing x_n and y_n are increasing sequences.

Lemma 2.3. $lim_{n\to\infty} x_n = lim_{n\to\infty} y_n = 1$ *and* $lim_{n\to-\infty} x_n = lim_{n\to-\infty} y_n = 0$

Proof. We prove $lim_{n\to\infty} x_n = 1$. First observe that as n goes to infinity, x_n is increasing and bounded above. Hence the limit $a = lim_{n\to\infty} x_n$ exists. Now

$$f(a) = f(lim_{n\to\infty} x_n) = lim_{n\to\infty} f(x_n) = lim_{n\to\infty} x_n = lim_{n\to\infty} x_{n+1} = a$$

and hence a is a fixed point of f. As the only fixed points of f are 0 and 1 and $a > x_0 \in (0,1)$, $a = lim_{n\to\infty} x_n = 1$. The other limits are deduced similarly. □

Since x_n and y_n are increasing sequences, we deduce the following.

Corollary 2.4.

$$\bigcup_{n=-\infty}^{\infty} [x_n, x_{n+1}] = \bigcup_{n=-\infty}^{\infty} [y_n, y_{n+1}] = (0,1)$$

We now define the map g. First define the restriction of g to $[x_0, x_1]$ to be any increasing one-to-one continuous function onto $[y_0, y_1]$ (for instance a linear map). We shall see that the condition $h = g \circ f \circ g^{-1}$, or equivalently,

$$h \circ g = g \circ f$$

now determines g.

Consider first a point $z_0 \in [x_0, x_1]$. Its image under f is a point $f(z_0) \in [x_1, x_2]$ and hence $g(f(z_0))$ has not been previously defined. As g satisfies $h \circ g = g \circ f$, we must have $g(f(z_0)) = h(g(z_0))$. We take this as the definition of g at $f(z_0)$. Since f maps $[x_0, x_1]$ to $[x_1, x_2]$, g has now been defined on $[x_1, x_2]$ and satisfies $h \circ g = g \circ f$ on $[x_0, x_2]$. Notice that the image under g of $[x_1, x_2]$ is $h([y_0, y_1]) = [y_1, y_2]$.

We extend this process to the whole interval. Observe that by the above corollary, any point in $z \in (0,1)$ can be expressed as $f^n(z_0)$ for some $n \in \mathbb{Z}$ and z_0 in $[x_0, x_1]$ and this is unique except that $f^n(x_0) = f^{n-1}[x_1]$. The equation $h = g \circ f \circ g^{-1}$ implies $h^n = g \circ f^n \circ g^{-1}$ or equivalently $g \circ f^n = h^n \circ g$. This can once more be taken as the defining equation, i.e., we define $g(z) = h^n(f(z_0))$. One can readily verify that though $f^n(x_0) = f^{n-1}[x_1]$, we get a well defined function.

By the above corollary, this determines g on $(0,1)$ and, as $h^n(f(z_0)) \in [y_n, y_{n+1}]$ for $z_0 \in [x_0, x_1]$. Finally, we define $g(0) = 0$ and $g(1) = 1$ to obtain the required conjugation.

It is now straightforward to verify that $g \circ f = h \circ g$, i.e. $g(f(z)) = h(g(z))\ \forall z \in [0,1]$. For, any $z \in (0,1)$ can be expressed as $f^n(z_0)$ for some $n \in \mathbb{Z}$ and z_0 in $[x_0, x_1]$. Hence $g(f(z)) = g(f^{n+1}(z_0)) = h^{n+1}(g(z_0))$ by the definition of g. Further, by the definition of g, $h^{n+1}(g(z_0)) = h(h^n(g(z_0))) = h(g(f^n(z_0))) = h(g(z))$. Thus, $g(f(z)) = h(g(z))$ □

Thus, up to conjugation, there are two classes of fixed point free invertible maps of $[0,1]$, namely the class of increasing functions and the class of decreasing functions. Returning to the case of the circle, we see that if $Fix(f)$ is non-empty, then f is either increasing or decreasing in each interval of $S^1 \setminus Fix(f)$. This completely determines the dynamics of f.

Here is an application of the above.

Exercise. *Show that any invertible continuous function $f : [0,1] \to [0,1]$ has a square root, i.e. a function g such that $f = g \circ g$*

3. Rotation numbers

We now turn to the case without fixed points.

The simplest examples of maps from the circle to itself are *rotations* and these have no fixed points. Thus it is natural to try to understand when an invertible map $f : S^1 \to S^1$ is conjugate to *some* rotation. As is often the case in mathematics, half the battle is won if one can recognise to which rotation one expects f to be conjugate.

In this vein, Poincaré introduced the notion of the *rotation number $rot(f)$* of an invertible map $f : S^1 \to S^1$. This is well defined modulo 2π (as with angles) and for a rotation by θ the rotation number is θ. Moreover, if f is conjugate to a map g, then $rot(f) = rot(g)$. It immediately follows that rotations by different angles are not conjugate (which can be shown by more elementary means).

Furthermore, it follows that the only rotation to which f can be conjugate is the one by the angle $rot(f)$, and we have to decide whether f is conjugate to this rotation, or more generally what are all the maps (up to conjugacy) with a given rotation number. This turns out to be subtle - for instance it turns out to depend crucially on whether $rot(f)$ is rational.

Before getting to the definition of the rotation number, we need some preliminaries.

3.1. Orientations.

Invertible maps of the circle are of two kinds - those that are *orientation preserving* and those that are *orientation reversing*. We formalise this notion (in one of the many possible ways).

Given 3 points a, b and c on S^1, we shall say $c \in [a,b]$ if we pass c while moving from a to b in the counter-clockwise direction. We shall define f to be *orientation-preserving* if for points a, b and c in S^1, $c \in [a,b] \implies f(c) \in [f(a), f(b)]$.

3.2. Lifting maps.

We can view points on the circle as representing angles. Any angle $\theta \in \mathbb{R}$ has a unique point associated to it, but to each point on the circle we have associated infinitely many angles differing by 2π. We formalise this by considering the map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi t}$. This is just the 'angle map' rescaled to have periodicity 1 rather than 2π.

Now suppose we have a function $F : \mathbb{R} \to \mathbb{R}$ with the periodicity property $F(t+1) = F(t) + 1\ \forall t$. Then we get a corresponding map $f : S^1 \to S^1$ of the circle which takes the point $p(t)$ (the point with angle t) to the point $p(F(t))$ (the point with angle $F(t)$). The periodicity property of F means that f is well defined.

In our situation we have an orientation preserving invertible map $f : S^1 \to S^1$. We shall see that this comes from a periodic map $F : \mathbb{R} \to \mathbb{R}$ such that $F(t+1) = F(t) + 1\ \forall t$ as above.

Definition 3.1. A *lift* of a map $f : S^1 \to S^1$ is a map $F : \mathbb{R} \to \mathbb{R}$ such that $p(F(t)) = f(p(t))\ \forall t \in \mathbb{R}$.

Proposition 3.1. *Every orientation preserving invertible map $f : S^1 \to S^1$ has a lift $F : \mathbb{R} \to \mathbb{R}$. Furthermore F is a monotonically increasing function and $F(t+1) = F(t) + 1\ \forall t$.*

Proof. We will first construct F on $[0, 1]$. Let $a \in \mathbb{R}$ be such that $p(a) = f(p(0))$ and set $F(0) = a$ and $F(1) = a + 1$. As x increases from 0 to 1, $p(x)$ moves counterclockwise once around the circle. As f is orientation preserving, $f(p(x))$ makes a single counterclockwise rotation beginning at $f(p(0))$. For each $x \in (0, 1)$ define $F(x)$ to be the unique point y in $[a, a+1]$ such that $p(y) = f(p(x))$.

We now extend F to $\mathbb{R}$. Any point $z \in \mathbb{R}$ is of the form $x + k$ with $x \in [0, 1]$, $k \in \mathbb{Z}$. We define $F(z) = F(x) + k$. We leave the verification of the properties of F as an exercise to the reader. □

The lift is essentially unique.

Proposition 3.2. *Suppose F and G are lifts of $f : S^1 \to S^1$. Then there is an integer $n \in \mathbb{Z}$ such that $\forall t \in \mathbb{R}$, $G(t) = F(t) + n$*

Proof. For each $x \in \mathbb{R}$, $p(F(x)) = f(p(x)) = p(G(x))$, hence $F(x) - G(x) \in \mathbb{Z}$. As $F - G$ is continuous and takes only integer values, it must be a constant $n \in \mathbb{Z}$. □

3.3. **Rotation number.** Suppose f is the rotation by an angle $2\pi\alpha$, then a lift of f is given by $F(t) = t + \alpha$. Thus, each point is translated by α. In general the lift will not be of such a nice form and each point may be translated by different amounts. But if we apply iterations F^n of the lift F, we see that the amount of translations (which in the case of the rotation by α is $n\alpha$) averages out.

Theorem 3.3. *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function such that $F(t+1) = F(t) + 1$. Then $lim_{t\to\infty} \frac{F^n(x)}{n}$ exists and is independent of x.*

Proof. We shall first consider the case when $x = 0$ and then show that we obtain the same limit for arbitrary x. We need the following lemma.

Lemma 3.4. *Suppose $k \le F^m(0) < k+1$. Then for any $n \in \mathbb{N}$, $nk \le F^{mn}(k) \le n(k+1)$*

Proof. Note that from the hypothesis we get $F^n(x+m) = F^n(x)+m\ \forall x \in \mathbb{R}, m \in \mathbb{Z}$. We prove the lemma in the case when $n = 2$. The general case follows similarly by induction.

As $k \le F^m(0) < k+1$ and F is increasing, by periodicity $F^m(k) \le F^{2m}(0) < F^m(k+1)$, i.e. $F^m(0) + k \le F^{2m}(0) < F^m(0) + k + 1$. Using $k \le F^m(0) < k+1$ once more, we obtain $2k \le F^{2m}(0) < 2(k+1)$ □

We next show that the sequence $F^n(0)/n$ is Cauchy. This follows from the following lemma.

Lemma 3.5. *Suppose $m, n > N$ for some $N \in \mathbb{N}$. Then* $|\frac{F^m(0)}{m} - \frac{F^n(0)}{n}| < \frac{2}{N}$

Proof. We show that $|\frac{F^m(0)}{m} - \frac{F^{mn}(0)}{mn}| < \frac{1}{m}$. Note that for some integer k, we must have $k \leq F^m(0) < k+1$. By the previous lemma, it follows that $nk \leq F^{mn}(0) < n(k+1)$. Hence $\frac{k}{m} < \frac{F^m(0)}{m} < \frac{k+1}{m}$ and $\frac{k}{m} < \frac{F^{mn}(0)}{mn} < \frac{k+1}{m}$, hence $|F^m(0)m - \frac{F^{mn}(0)}{mn}| < \frac{1}{m} < \frac{1}{N}$. Interchanging the role of m and n we get a similar inequality. Using the triangle inequality the lemma follows. □

Thus, the limit $lim_{t\to\infty}\frac{F^n(0)}{n}$ exits. Now for any $x \in \mathbb{R}$, there is an integer k such that $-k < x < k$. By monotonicity of F^n, $F^n(-k) < F^n(x) < F^n(k)$, i.e. $F^n(0) - k < F^n(x) < F^n(0) + k$. It follows that $lim_{t\to\infty}\frac{F^n(x)}{n}$ exits and equals $lim_{t\to\infty}\frac{F^n(0)}{n}$

□

Any other lift of f is of the form $G(t) = F(t) + k$ for a fixed integer k as we have seen, and hence $G^n(t) = F^n(t) + k$. It follows that $lim_{t\to\infty}\frac{G^n(x)}{n} = lim_{t\to\infty}\frac{F^n(x)}{n}$. Hence we can make the following definition.

Definition 3.2. The rotation number $rot(f)$ of f is defined by $lim_{t\to\infty}\frac{F^n(x)}{n}$ where F is a lift of f.

The importance of the rotation number lies in its being a *dynamical invariant*, that is, the rotation number of two conjugate maps are equal.

Theorem 3.6. *If f and g are conjugate then $rot(f) = rot(g)$*

Proof. Let $g = hfh^{-1}$ and let F and G be lifts of F and G. Then $H = GFG^{-1}$ is a lift of h and hence we can use H to compute the rotation number of h.

Observe that $H^n = GF^nG^{-1}$. We shall show that there is a fixed M, independent of n, such that $|F^n(x) - H^n(x)| < M\ \forall x \in \mathbb{R}$. The theorem follows immediately from this.

Firstly, by the periodicity of G, the function $P(x) = G(x) - x$ is bounded. This follows as for all integers k, $P(x+k) = P(x)$. As P is continuous it is bounded on $[0, 1]$. The periodicity thus implies that P is bounded on $\mathbb{R}$. It follows similarly that $G^{-1}(x) - x$ is bounded. Thus, we can find k such that $|G(x) - x| < k\ \forall x \in \mathbb{R}$ and $|G^{-1}(x) - x| < k\ \forall x \in \mathbb{R}$.

Next, we claim that if $|y - z| < k$ then $|F^n(y) - F^n(z)| < k$. For, we may assume without loss of generality that $y \leq z < y + k$. As F^n is an increasing function it follows that $F^n(y) \leq F^n(z) < F^n(y + k) = F^n(y) + k$.

Thus, by what we have seen above, for any real number x, $|G^{-1}(x) - x| < k$ and hence $|F^n(G^{-1}(x)) - F^n(x)| < k$. Further, by the above results applied to $F^n(G^{-1}(x))$, $|F^n(G^{-1}(x)) - G(F^n(G^{-1}(x)))| < k$. By the triangle inequality, $|G(F^n(G^{-1}(x))) - x| < 2k$. Taking $M = 2k$ we obtain $|F^n(x) - H^n(x)| < M\ \forall x \in \mathbb{R}$. This completes the proof of the theorem.

□

Corollary 3.7. *Rotations by α and β are not conjugate unless $\alpha - \beta$ is divisible by 2π (in which case the rotations are equal).*

Thus, we see that (as one would expect) rotations by different angles are not conjugate. Furthermore, each orientation preserving invertible map $f : S^1 \to S^1$ has a rotation number $rot(f)$, and if f is conjugate to a rotation, it must be the rotation by $rot(f)$.

This still leaves us with the problem of classifying orientation preserving invertible maps $f : S^1 \to S^1$ with a fixed rotation number.

To get an idea of the solution, we consider the case when the rotation number is zero.

Exercise. *Show that orientation preserving invertible map $f : S^1 \to S^1$ has rotation number zero if and only if f has a fixed point.*

The rotation by angle zero is the identity, and we have seen that there are plenty of maps besides these with fixed points. The case when the rotation number is rational is similar to this case. The situation for irrational angles is more subtle in some ways but nicer in other ways. We consider the problem of classifying orientation preserving invertible maps $f : S^1 \to S^1$ with a fixed rotation number in more detail in the sequel.

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