DEHN SURGERY

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1. Introduction

Dehn surgery is a basic method for constructing 3-manifolds. It was introduced by Dehn to construct homology spheres. In the early 1960's, Lickorish and Wallace showed that all orientable 3-manifolds can be obtained using this construction.

Let $\mathcal{K}$ be a (tame) knot in S^3, i.e., an embedding of S^1 into S^3. We can remove a neighbourhood of this knot, which is a solid torus, and sew it back in a different way. This is the simplest case of a Dehn surgery.

More generally, given a link $\mathcal{L}$ in a 3-manifold M, we can remove neighbourhoods of each component of the link and sew back the solid tori in a different way. A theorem of Lickorish and Wallace asserts that we get every closed orientable 3-manifold by performing this operation on some link in S^3.

2. Co-ordinates for surgery

Let $\mathcal{K}$ be a knot in S^3 (or a 3-manifold M). Let T be the boundary of a regular neighbourhood of $\mathcal{K}$. There are two special homology classes of curves on T, called the *meridian* μ and the *longitude* λ.

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The meridian is represented by a simple curve in T that bounds a disc in $\mathcal{N}(K) = D^2 \times S^1$, for example $(\partial D^2) \times \{1\}\partial(D^2 \times S^1) \subset S^1 \times S^1$. The meridian is defined similarly for a knot $\mathcal{K}$ in

The longitude λ is a curve that intersects the meridian transversely in exactly one point, and bounds a surface in $S^3 \setminus \mathcal{N}(K)$. We shall see later that such a curve exists for a knot in S^3, or more generally in a homology sphere (all one needs is that $\mathcal{K}$ represents a trivial class in $H_1(M)$). In the general case, we simply take any curve that interects μ transversally in one point.

In terms of these co-ordinates, surgeries at K are parametrised by elements of $\mathbb{Z}^2$, or alternatively, by the *slope* in $\mathbb{Q}\cup\{\infty\}$. For, to sew in the solid torus $D^2 \times S^1$, we can first attach the disc $D = D^2 \times \{1\}$, and then its complement, which is a ball. The map is determined, up to isotopy, by the homology class of ∂D.

The disc D can be attached to any simple, closed, non-separating curve on T, to give an attaching map for the solid torus. Such curves are of the form $p\mu + q\lambda$, with $(p, q) = 1$. The homology class of ∂D is canonical up to sign.

As $(p, q) = 1$, and the surgery determined by (p, q) is the same as that determined by $(-p, -q)$, we can parametrise the Dehn surgeries by $p/q \in \mathbb{Q}\cup\{\infty\}$. This surgery is described as the p/q-surgery about the knot $\mathcal{K}$.

Remark 2.1. Some people call q/p the slope.

To specify a surgery about a link, one simply specifies the surgery at each component.

Example 2.1 (An unknot $\mathcal{K} \in S^3$). The results of surgery about an unknot are just manifolds with Heegard splittings of genus 1. For, the complement of an open neighbourhood of an unknot $\mathcal{K} \in S^3$ is a solid torus. Thus, the result of a surgery is just the result of gluing together two solid tori. More specifically, p/q-surgery gives $L(p, q)$

3. Some algebraic topology

Consider a knot $\mathcal{K}$ in a homology sphere M. We shall study in this section the homology of the knot complement and the surgered manifolds.

Proposition 3.1.
(1) $H_1(M \setminus int(\mathcal{N}(K))) = \mathbb{Z}$
(2) $H_2(M \setminus int(\mathcal{N}(K)), \partial\mathcal{N}(K)) = \mathbb{Z}$
(3) $ker(H_1(\partial\mathcal{N}(K)) \mapsto H_1(M \setminus int(\mathcal{N}(K)))) = \mathbb{Z}$

Thus the knot complement has the homology of a solid torus. Let λ be a curve representing a generator of $H_1(\partial\mathcal{N}(K))$. Let $T = \partial\mathcal{N}(K)$ and μ be as before.

Proposition 3.2. *The algebraic intersection number between μ and λ is ± 1.*

Proposition 3.3. *If N is the result of performing q/p surgery at K, then $H_1(N) = \mathbb{Z}/p\mathbb{Z}$*

Example 3.1 (Homology spheres)*.* We are now in a position to construct several homology sphere. Namely, we perform $1/n$-surgery about a knot $\mathcal{K}$ in S^3.

By performing 1/1-surgery on the trefoil knot, we get the Poincaré homology sphere. In fact performing $1/n$-surgery for various n at a fixed knot usually yields infinitely many homology spheres, but showing this requires deeper results in 3-manifold topology.

4. The theorem of Lickorish and Wallace

Any manifold can be obtained from S^3 by surgery about links, with all surgery slopes ± 1. We outline here Lickorish's proof of this fundamental fact.

The starting point is a beautiful theorem of Lickorish about surface diffeomorphisms. Given a simple closed curve γ on a surface S, we can perform a so called *Dehn twist*. This is a homeomorphism which is identity outside the neighbourhood of γ and is a full twist on an annular neighbourhood of γ (in other words, remove an annulus and glue it back with a full twist).

Theorem 4.1 (Lickorish)**.** *Any diffeomorphism of a surface is isotopic to a composition of Dehn twists.*

Lickorish's proof of this result is elementary and elegant. The reader isrefered to Lickorish's paper and to Roushan's lectures. This result is then used to prove the following theorem.

Theorem 4.2 (Lickorish)**.** *Any closed, orientable 3-manifold M can be obtained by performing surgeries along a link in S^3 with slopes ± 1.*

Proof. The manifold M has a Heegard splitting $M = H_1 \cup H_2$, where the H_i are handlebodies. Identify these with the handlebodies in a Heegard splitting of S^3 of the same genus, so that we have an identification $\partial H_1 = \partial H_2 = S$. Then the Heegard splitting is determined by a gluing map $f : S = \partial H_2 \to del H_1 = S$, which is the identity map from S to itself in the case of S^3.

Observe that if $f : S \to S$ extends to a homeomorphism $F : H_1 \to H_1$, then we $M = S^3$. For, we define the homeomorphism between M and S^3, regarded as $S^3 = H_1 \cup H_2$ with the gluing map being the identity, to be F on H_1 and the identity on H_2.

More generally, if H_1' is obtained from H_1 by a sequence of surgeries, and f extends to H_1', then M is homeomorphic to the manifold obtained by performing the corresponding surgeries on S^3. The following lemma now suffices.

Lemma 4.3. *Let $(N, \partial N)$ be a manifold with boundary $S = \partial N$ and let $f : S \to S$ be an orientation preserving homeomorphism. Then there is a manifold $(N', \partial N'), \partial N' = S$, obtained by performing integral surgeries about a link in N, such that f extends to a homeomorphism $F : N' \to N'$.*

Proof. Express f as a sequence of $f = C_1 \circ C_2 \cdots \circ C_n$ of Dehn twists. The proof is by induction on the number of Dehn twists.

Suppose the first Dehn twist is about a curve γ, and let K be obtained by pushing γ sligtly into M. Then we can easily extend the Dehn twist to $N \setminus \mathcal{N}(K)$. The extension restricts to a Dehn twist on $\partial\mathcal{N}(K)$. Thus if a solid torus is glued in to $N \setminus \mathcal{N}(K)$ so that the curve about which the twisting takes place is a meridian, the map still extends. The resulting manifold, N'', is obtained from N by an integral Dehn surgery, and C_1 extends to N''. By applying the induction hypothesis to $C_2 \circ \cdots \circ C_n$, the result follows. □

□

5. Surgeries and Cobordisms

We have seen that all oriented 3-manifolds can be obtained from S^3 by integral surgeries. However the surgery is far from unique. As we shall see, comparing different surgeries that give the same manifold is a question in 4-manifold topology.

The boundaries of 4-manifolds are 3-manifolds. We see below that any oriented 3-manifold is the boundary of an oriented 4-manifold. First, we need to understand the relation, which holds in all dimensions, between (integral) surgeries and cobordisms.

Surgery is the replacement of the neighbourhood of a sphere with trivial normal bundle by a disc bundle over another sphere, usually of different dimension. In the case of Dehn surgery, we replace the neighbourhood of S^1 with a disc bundle over S^1.

This operation is based on the fact that

$$\partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1})$$

Consider a sphere $S^k \subset M$ embedded in a manifold with trivial normal bundle. Its neighbourhood is diffeomorphic to $S^k \times D^{n-k}$. We can delete the interior of this neighbourhood and sew in $D^{k+1} \times S^{n-k-1}$, since this has the same boundary.

The sewing is not canonical, but depends on the identification $S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1})$. One can change the identifications corresponding to elements of $\pi_{n-k-1}(SO(k+1))$.

This operation corresponds to handle addition in the sense that if a k-handle is added to a $n+1$-dimensional manifold, the effect on the boundary is a surgery.

Dehn surgery is a generalisation of 1-surgery on a 3-manifold. The 1-surgeries are parametrised by $\pi_1(SO(2)) = \mathbb{Z}$. It is easy to see that these are precisely integral surgeries.

6. The Kirby Calculus

We have seen that any 3-manifold can be obtained by integral surgery about a link. However, many surgeries give the same manifold. For instance, ± 1 surgeries about unknots have no effect on the manifold.

Regarding surgeries as handle-additions makes clear another move on surgeries that does not change the manifold. Given a handlebody decomposition of a manifold, one can get another decomposition using *handle-slides.* Purely in terms of surgery, this corresponds to forming *band connected sum.* Namely, push a copy of a component of the link off itself with linking equal to the slope of the surgery to be performed. Take an arc joining another link component to this copy and connect the two components along this arc.

A fundamental theorem of Kirby says that these are the only moves required to pass between two surgery descriptions.

Theorem 6.1 (Kirby)**.** *Two integral surgery descriptions of a* 3*-manifold are equivalent under adding or deleting a* ± 1 *framed unknot unlinked with other components and handle slides.*

7. Constructing knots using surgery

An important application of Dehn surgery is to construct knots with desired propertes. The construction is based on the fact that $1/n$-surgeries about an unknot do not change the ambient manifold.

Suppose K is a knot, and γ is an unknot which is linked with K. Then $1/n$-surgery about γ does not change the ambient manifold, but in general changes the embedding of the knot K. Indeed, such surgeries can be used to change crossings, and thus to obtain any knot.

This is a very useful construction. See, for instance, Rolfsen, for several examples.

8. Surgeries about knots

We have seen that many surgeries about links give a given manifold. However, there are some deep results asserting that in the case of S^3 and $S^2 \times S^1$, only the obvious surgeries about knots give these manifolds.

Theorem 8.1 (Gabai)**.** *No surgery about a non-trivial knot in S^3 gives $S^2 \times S^1$.*

An equivalent formulation of this result is that no surgery about a non-trivial knot in $S^2 \times S^1$ gives S^3. The proof involves the theory of taut-foliations, which has come to play a major role in 3-manifold toplogy.

Theorem 8.2 (Gordon-Leucke (see [1]))**.** *No non-trivial surgery about a non-trivial knot in S^3 gives S^3.*

Equivalently, *a knot is determined by its complement.*

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