The Bergman kernel

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Let $\mathcal{D}$ be a bounded open connected subset of $\mathbb{C}^m$ and $\mathbb{A}^2(\mathcal{D})$ be the Hilbert space of square integrable (with respect to volume measure) holomorphic functions on $\mathcal{D}$. The Bergman kernel $B : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ is uniquely defined by the two requirements:

$$\begin{aligned} B_w &\in \mathbb{A}^2(\mathcal{D}) && \text{for all } w \in \mathcal{D} \\ \langle f, B_w \rangle &= f(w) && \text{for all } f \in A^2(\mathcal{D}). \end{aligned}$$

The existence of B_w is guaranteed as long as the evaluation functional $f \to f(w)$ is bounded.

We have $B_w(z) = \langle B_w, B_z \rangle$. Consequently, for any choice of $n \in \mathbb{N}$ and an arbitrary subset $\{w_1, \ldots, w_n\}$ of $\mathcal{D}$, the $n \times n$ matrix $(B_{w_i}(w_j))_{i,j=1}^n$ must be positive definite.

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Fourier series

Notice first that if $e_n(z), \quad n \geq 0$ is an orthonormal basis for the Bergman space $\mathbb{A}^2(\mathcal{D})$, then any $f \in \mathbb{A}^2(\mathcal{D})$ has the Fourier series expansion $f(z) = \sum_{n=0}^{\infty} a_n e_n(z)$. Assuming that the sum

$$B_w(z) := \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)},$$

is in $\mathbb{A}^2(\mathcal{D})$ for each $w \in \mathcal{D}$, we see that

$$\langle f(z), B_w(z) \rangle = f(w),\ w \in \mathcal{D}.$$

an example

For the Bergman space $\mathbb{A}^2(\mathbb{D}^m)$, of the polydisc $\mathbb{D}^m$, the orthonormal basis is $\{\sqrt{\prod_{i=1}^m (n_i+1)} z^I : I = (i_1, \ldots, i_m)\}$. Clearly, we have

$$B_{\mathbb{D}^m}(z,w) = \sum_{|I|=0}^{\infty} \Big(\prod_{i=1}^{m} (n_i+1)\Big) z^I \bar{w}^I = \prod_{i=1}^{m} (1 - z_i \bar{w}_i)^{-2}.$$

Similarly, for the Bergman space of the ball $\mathbb{A}^2(\mathbb{B}^m)$, the orthonormal basis is $\{\sqrt{\binom{-m-1}{|I|}\binom{|I|}{I}} z^I : I = (i_1, \ldots, i_m)\}$. Again, it follows that

$$B_{\mathbb{B}^m}(z,w) = \sum_{|I|=0}^{\infty} \binom{-m-1}{\ell} \Big(\sum_{|I|=\ell} \binom{|I|}{I} z^I \bar{w}^I\Big) = (1 - \langle z, w\rangle)^{-m-1}.$$

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Quasi-invariance of B

Any bi-holomorphic map $\varphi : \mathcal{D} \to \tilde{\mathcal{D}}$ induces a unitary operator $U_\varphi : \mathbb{A}^2(\tilde{\mathcal{D}} \to \mathbb{A}^2(\mathcal{D})$ defined by the formula

$$(U_\varphi f)(z) = (J(\varphi, z)\big(f \circ \varphi\big)(z),\ f \in \mathbb{A}^2(\tilde{\mathcal{D}}),\ z \in \mathcal{D}.$$

This is an immediate consequence of the change of variable formula for the volume measure on $\mathbb{C}^n$.

Consequently, if $\{\tilde{e}_n\}_{n\geq 0}$ is any orthonormal basis for $\mathbb{A}^2(\tilde{\mathcal{D}})$, then $\{e_n\}_{n\geq 0}$, where $\tilde{e}_n = J(\varphi, \cdot)(\tilde{e}_n \circ \varphi)$ is an orthonormal basis for the Bergman space $\mathbb{A}^2(\tilde{\mathcal{D}})$.

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Quasi-invariance of B

Expressing the Bergman kernel $B_{\mathcal{D}}$ of the domains $\mathcal{D}$ as the infinite sum $\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$ using the orthonormal basis in $\mathbb{A}^2(\mathcal{D})$, we see that the Bergman Kernel B is *quasi-invariant*, that is, If $\varphi : \mathcal{D} \to \widetilde{\mathcal{D}}$ is holomorphic then we have the transformation rule

$$J(\varphi, z)B_{\tilde{\mathcal{D}}}(\varphi(z), \varphi(w))\overline{J(\varphi, w)} = B_{\mathcal{D}}(z, w),$$

where $J(\varphi, w)$ is the Jacobian determinant of the map φ at w .

If $\mathcal{D}$ admits a transitive group of bi-holomorphic automorphisms, then this transformation rule gives an effective way of computing the Bergman kernel. Thus

$$B_{\mathcal{D}}(z, z) = |J(\varphi_z, z)|^2 B_{\mathcal{D}}(0, 0), \; z \in \mathcal{D},$$

where φ_z is the automorphism of $\mathcal{D}$ with the property $\varphi_z(z) = 0$.

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Quasi-invariance of B

The quasi-invariance of B is equivalent to saying that the map $U_\varphi : \mathbb{A}^2(\widetilde{\mathcal{D}}) \to \mathbb{A}^2(\mathcal{D})$ defined by the formula:

$$(U_\varphi f)(z) = J_{\varphi^{-1}}(z)(f \circ \varphi^{-1})(z), f \in \mathbb{A}^2(\widetilde{\mathcal{D}}), z \in \mathcal{D}$$

is an isometry.

The quasi-invariance of the Bergman kernel $B_{\mathcal{D}}(z,w)$ also leads to a bi-holomorphic invariant. Let $\mathcal{K}_{B_{\mathcal{D}}}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} B_{\mathcal{D}}(z,z)$. Then

$$\frac{\det \mathcal{K}_{B_{\mathcal{D}}}(z)}{B_{\mathcal{D}}(z,z)}, \quad z \in \mathcal{D}$$

is a bi-holomorphic invariant for the domain $\mathcal{D}$.

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the multiplier

Consider the special case, where $\varphi : \mathcal{D} \to \mathcal{D}$ is an automorphism. Clearly, in this case, U_φ is unitary on $\mathbb{A}^2(\mathcal{D})$ for all $\varphi \in \mathrm{Aut}(\mathcal{D})$.

The map $J : \mathrm{Aut}(\mathcal{D}) \times \mathcal{D} \to \mathbb{C}$ satisfies the cocycle property, namely

$$J(\psi\varphi, z) = J(\varphi, \psi(z))J(\psi, z),\ \varphi, \psi \in \mathrm{Aut}(\mathcal{D}),\ z \in \mathcal{D}.$$

This makes the map $\varphi \to U_\varphi$ a homomorphism.

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examples of cocycles

The automorphism group Möb of the unit disc is the group

$$\{\varphi_{\theta,a} : 0 \le \theta < 2\pi, a \in \mathbb{D}\},$$

where $\varphi_{\theta,a}(z) = e^{i\theta}(z-a)(1-\bar{a}z)^{-1}$. As a topological group Möb is $\mathbb{T} \times \mathbb{D}$. More interesting is the two fold covering group $G = SU(1,1)$

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\},$$

which acts on the unit disc $\mathbb{D}$ according to the rule $g \cdot z = (az+b)(\bar{b}z+\bar{a})^{-1}$. For $\lambda > 0$, the map

$$j_g(z) = \left(\frac{\partial g}{\partial z}(z)\right)^{\lambda} = (\bar{b}z+a)^{-2\lambda}$$

defines a holomorphic multiplier on $SU(1,1) \times \mathbb{D}$.

more representations

Exploit the quasi-invariance of the Bergman kernel to construct unitary representations of the automorphism group $\mathrm{Aut}(\mathcal{D})$. Let $B^\lambda(z,w)$ be the polarization of the function $B(w,w)^\lambda$, $w \in \mathcal{D}$, $\lambda > 0$.

Now, as before,

$$J_\varphi(z)^\lambda B^\lambda(\varphi(z),\varphi(w))\overline{J_\varphi(w)}^\lambda = B^\lambda(z,w),\ \varphi \in \mathrm{Aut}(\mathcal{D}),\ z,w \in \mathcal{D}.$$

Let $\mathcal{O}(\mathcal{D})$ be the ring of holomorphic functions on $\mathcal{D}$. Define

$$U^{(\lambda)} : \mathrm{Aut}(\mathcal{D}) \to \mathrm{End}(\mathcal{O}(\mathcal{D}))$$

by the formula

$$(U_\varphi^{(\lambda)} f)(z) = \big(J_{\varphi^{-1}}(z)\big)^\lambda (f \circ \varphi^{-1})(z)$$

and note that $\varphi \mapsto U_\varphi$ is a homomorphism.

When is it unitarizable?

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quasi-invariant measures

An affirmative answer will ensure the existence of a unitary representation $U^{(\lambda)}$. Fortunately, there are two different ways in which we can obtain an answer to this question.

For the map $U_{\varphi}^{(\lambda)}$ to be isometric on a Hilbert space of the form $\mathbb{A}^2(\mathcal{D}, Q\,dV)$, we must have

$$\begin{aligned}
&\int_{\mathcal{D}} \overline{f(\varphi(z))}\,\overline{J_{\varphi}^{\lambda}(z)}\,Q(z) J_{\varphi}^{\lambda}(z) f(\varphi(z))\,dV(z) \\
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\end{aligned}$$

This amounts the transformation rule

$$Q(\varphi(z)) = \overline{J_\varphi^\lambda(z)} Q(z) (J_\varphi(z))^\lambda |J_\varphi(z)|^{-2}$$

for the weight function Q.

Example: In the case of the unit disc $\mathcal{D} = \mathbb{D}$, the automorphism group is transitive, picking a $\varphi := \varphi_z$ such that $\varphi_z(0) = z$, we see that $Q(z) = (1 - |z|^2)^{2\lambda - 2}$.

However, the Hilbert space

$$\mathbb{A}^2(\mathbb{D}, (1 - |z|^2)^{2\lambda - 2} dV(z))$$

is non-zero if and only if $2\lambda - 2 > -1$. Thus we must have $\lambda > \frac{1}{2}$. But if $\lambda = \frac{1}{2}$, the Hardy space appears!

No such luck if $\lambda < \frac{1}{2}$.

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Berizin-Wallach set

If for some, $\lambda > 0$, the Hilbert space $\mathbb{A}^2(\mathcal{D}, QdV) \neq \{0\}$, then the corresponding reproducing kernel must be B^λ. But what about λ for which this space is trivial? Even for such a λ, it is possible that B^λ is positive definite. In this case, there is a recipe to construct a Hilbert space $\mathcal{H}$ whose reproducing kernel is B^λ.

Define the Berizin-Wallach set

$$\mathcal{W}_{\mathcal{D}} := \{\lambda > 0 : B^\lambda \text{ is positive definite } \}.$$

In the case of the unit disc $\mathbb{D}$, the Wallach set $\mathcal{W}_{\mathcal{D}} = \mathbb{R}_+$. Thus there are representation spaces in this case for which the inner product is not given by an integral.

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If for some, $\lambda > 0,$ the Hilbert space $\mathbb{A}^2(\mathcal{D}, QdV) \neq \{0\},$ then the corresponding reproducing kernel must be $B^\lambda.$ But what about λ for which this space is trivial? Even for such a $\lambda,$ it is possible that B^λ is positive definite. In this case, there is a recipe to construct a Hilbert space $\mathcal{H}$ whose reproducing kernel is $B^\lambda.$

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What about $\mathcal{O}(\mathcal{D}, \mathbb{C}^n)$, the space of holomorphic functions on $\mathcal{D}$ taking values in $\mathbb{C}^n$?

Does there exist a positive definite kernel

$$\mathbf{B} : \mathcal{D} \times \mathcal{D} \to \mathbb{C}^{n \times n}$$

satisfying the quasi-inavariance:

$$\mathbf{B}(\varphi(z), \varphi(w)) = J_\varphi(z)^{-1} \mathbf{B}(z, w) (J_\varphi(w)^*)^{-1},$$

for some cocycle $J : \mathrm{Aut}(\mathcal{D}) \times \mathcal{D} \to \mathbb{C}^{n \times n}$?

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the unit disc, again!

A co-cycle $J : Möb \times \mathbb{D} \to \mathbb{C}^{(m+1)\times(m+1)}$ is given by the formula:

$$J_{\mathbf{m}}(\varphi, z) = (\varphi')^{2\lambda - \frac{m}{2}}(z) D(\varphi)^{\frac{1}{2}} \exp(c_\varphi S_m) D(\varphi)^{\frac{1}{2}},$$

where S_m is the forward shift with weights $\{1, 2, \ldots, m\}$ and $D(\varphi)$ is a diagonal matrix whose diagonal sequence is $\{(\varphi')^m(z), (\varphi')^{m-1}(z), \ldots, 1\}$.

We now have the Hilbert space $\mathcal{H}^{(\lambda,\mathbf{m})}$ of square integrable holomorphic functions on the unit disc with respect to the measure $Q(z)dV(z)$, where

$$Q(z) := J_{\varphi_z}(0)^* Q(0) J_{\varphi_z}(0) |\varphi_z'(z)|^{-2},$$

where $\varphi_z(0) = z$. For this Hilbert space $\mathcal{H}^{(\lambda,\varrho)}$ to be non-zero, it is necessary and sufficient that $\lambda > \frac{m+1}{2}$.

the unit disc, again!

A co-cycle $J : Möb \times \mathbb{D} \to \mathbb{C}^{(m+1)\times(m+1)}$ is given by the formula:

$$J_{\mathbf{m}}(\varphi, z) = (\varphi')^{2\lambda - \frac{m}{2}}(z) D(\varphi)^{\frac{1}{2}} \exp(c_\varphi S_m) D(\varphi)^{\frac{1}{2}},$$

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the reproducing kernel

The reproducing kernel $\mathbf{B}^{(\lambda,\mathbf{m})}$ for the Hilbert space $\mathcal{H}^{(\lambda,\mathbf{m})}$ is obtained by polarizing the identity

$$\mathbf{B}^{(\lambda,\mathbf{m})}(z,z) = J_{\varphi_z}(0)^{-1}\mathbf{B}^{(\lambda,\mathbf{m})}(0,0)(J_{\varphi_z}(0)^*)^{-1}.$$

The possible values for the positive diagonal matrix $\mathbf{B}^{(\lambda,\varrho)}(0,0)$ are completely determined by $Q(0)$. Also, $\mathbf{B}^{(\lambda,\varrho)}$ is a positive definite kernel for each choice of $Q(0)$.

Are there any other quasi-invariant kernels??
One sees that

$$\mathbf{B}^{(\lambda,\mathbf{m})} = B^{(2\lambda-m)} + \mu_1(Q(0))B^{(2\lambda-m+2)} + \cdots, \mu_m(Q(0))B^{(2\lambda+m)},$$

where $\mu_1(Q(0)),\ldots,\mu_m(Q(0))$ are some positive real numbers and $B^{(2\lambda-m+2j)}$ is a positive definite matrix which can be computed explicitly:

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left(\!\left(\partial^\ell\bar\partial^k B^{2\lambda-m+2j}(z,w)\right)\!\right)_{\ell,k=0}^{m-j} \end{pmatrix}.$$

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Now, for any $\lambda : \lambda > \frac{m}{2}$ and an arbitrary m-tuple $\mu_1, \ldots, \mu_m$ of positive numbers, the Hilbert space $\mathbf{B}^{(\lambda,\varrho)}$ is non-trivial.

Thus we have unitarized a much larger class of representations than what would be possible if we insist on integral inner products. But this is an interesting issue on its own right.

Are the multiplication operators on these spaces bounded?
Answer: yes!
What is more, they are *homogeneous*.

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Thank you!