Classification of homogeneous operators in the Cowen-Douglas class

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systems of imprimitivity

- Let G be a topological group and $\Omega \subseteq \mathbb{C}^m$ be a G-space.
- Suppose that

  $U : G \to \mathscr{U}(\mathscr{H})$ is a unitary representation of the group G on the unitary operators acting on the Hilbert space $\mathscr{H}$ and that $\rho : C(\Omega) \to \mathscr{L}(\mathscr{H})$ is a $*$-homomorphism of the C^*-algebra of continuous functions $C(\Omega)$ on the algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators acting on the Hilbert space $\mathscr{H}$.
- The triple (G, U, ρ) is said to be a *system of imprimitivity* if

$$\rho(g \cdot f) = U(g)^* \rho(f) U(g), f \in C(\Omega), g \in G,$$

  where $(g \cdot f)(w) = f(g^{-1} \cdot w)$, $w \in \Omega$.
- The notion of imprimitivity was introduced by Mackey.

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The Möbius group $G = G_0/\{\pm I\}$, where

$$G_0 = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} ; |\, a\,|^2 + |\, b\,|^2 = 1 \right\},$$

acts naturally on the unit disc $\mathbb{D}$.

The map $U : G \to \mathscr{U}(L^2(\mathbb{D}))$ defined by $\left(U_{g^{-1}} f\right)(z) = g'(z) f(g(z))$ is a unitary representation of the Möbius group.

The map $\rho : C(\bar{\mathbb{D}}) \to \mathscr{L}(L^2(\mathbb{D}))$ defined by the formula $\rho(f) = M_f$ is a $*$ - homomorphism.

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a possible generalization

What if we start with a homomorphism of a function algebra rather than a $*$ - homomorphism of a C^* - algebra?

For instance, the multiplication operator on the Bergman space $\mathbb{A}^2(\mathbb{D})$ – the Hilbert space of square integrable holomorphic functions on the unit disc – defines a homomorphism of the disc algebra $\mathscr{A}(\mathbb{D})$ by the formula $\rho(f) = M_f, f$ in $\mathscr{A}(\mathbb{D})$.

Like before, $\big(U_{g^{-1}}f\big)(z) = g'(z)f(g(z))$ defines a unitary representation of the Möbius group on the Bergman space.

The algebra homomorphism ρ and the unitary representation U satisfy the imprimitivity relation, that is,

$$U_g^* M_{g^{-1}(z)} U_g = M_z.$$

We say that the operator of multiplication by z on the Bergman space $\mathbb{A}^2(\mathbb{D})$ is homogeneous.

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Construction of Homogeneous operators

Let $\mathbb{F}(J) := \{f_n : n \in J\}$, where$f_n : \mathbb{T} \to \mathbb{T}$ is the function $f_n(z) = z^n$ and J is some subset of $\mathbb{Z}$.

Declaring the vectors in the set $\mathbb{F}(J)$ to be orthogonal, as soon as the norms are prescribed, we can complete the linear span of the vectors in $\mathbb{F}(J)$ to form a Hilbert space, say $\mathscr{F}(J)$.

For $g \in G, f \in \mathscr{F}(J), z \in \mathbb{T}$ and complex parameters, $\lambda,\ \mu$, define

$$\big(\big(R_{\lambda,\mu}(g^{-1})\big)f\big)(z) = g'(z)^{\frac{\lambda}{2}} \mid g'(z) \mid^{\mu} \big(f \circ g\big)\big(z\big),$$

where $g'(z)^{\frac{\lambda}{2}} := \exp\left(\frac{\lambda}{2} \log g'(z)\right)$

There is no apriori guarantee that $R_{\lambda,\mu}$ is unitary (even bounded)! But if $R_{\lambda,\mu}$ unitary, then it defines a projective unitary representation on $\mathscr{F}(J)$.

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Multiplier representations

A representation of the form $R_{\lambda,\mu} : G \to \mathscr{F}(J)$,
(Or more generally, one of the form $M_{J_g} R_g$, where M_{J_g} is the multiplication by J_g and R_g is the composition by g with $J_{gh}(z) = J_g(h(z))J_h(z)$.)
is said to be a multiplier representation.
Let $\mathscr{H}$ be a space of functions, say, on the unit disc or the unit circle.
Suppose the operator T defined by the rule $(Tf)(x) = xf(x), f \in \mathscr{H}$ is bounded.
If there is a multiplier representation, say U, of the group G on the Hilbert space $\mathscr{H}$ then the operator T is homogeneous and U is the associated representation.

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Representations of Möb

Principal Series:

$P_{\lambda,s}$, $-1 < \lambda \le 1$, $s = iy$, $y \in \mathbb{R}$, $\mu = \frac{1-\lambda}{2} + s$, $J = \mathbb{Z}$, $\|f_n\| = 1, n \in \mathbb{Z}$.

Holomorphic Discrete Series:

D^+_λ, $\lambda \ge 0$, $\mu = 0 J = \mathbb{N} \cup \{0\}$, $\|f_n\|^2 = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}, n \in \mathbb{N}$.

The Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D})$, usually called the weighted Bergman space, is a space of holomorphic functions on $\mathbb{D}$ with reproducing kernel $(1 - z\bar{w})^{-\lambda}$.

Anti-holomorphic Discrete Series: D^-_λ. This is easily obtained from the holomorphic Discrete series. We have $D^-_\lambda(g) := D^+_\lambda(g^*)$, where $g^*(z) = \overline{g(\bar{z})}$.

Complimentary Series: $C_{\lambda,\sigma}$, $-1 < \lambda < 1, 0 < \sigma < \frac{1-|\lambda|}{2}$, $\mu = \frac{1}{2}(1-\lambda) + \sigma, J = \mathbb{Z}, \|f_n\|^2 = \prod_{k=0}^{|n|-1} \frac{k \pm \lambda/2 + 1/2 - \sigma}{k \pm \lambda/2 + 1/2 + \sigma}, n \in \mathbb{Z}$.

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Examples

The Principal Seies Examples: These give the unweighted bi-lateral shift!

The Discrete Series Examples: For any real $\lambda > 0$, the unilateral shift with weight sequence $\sqrt{\frac{n+1}{n+\lambda}}$

(up to unitary equivalence, this is the operator $M^{(\lambda)}$ of multiplication by the coordinate function z on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D})$)

is homogeneous.

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One other Example: The bi-lateral shift with 1 except in any one slot, where it is allowed to be any complex number.

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Block shifts

We can choose U_g such that $\mathbb{K} \ni k \mapsto U_k$ is a representation of $\mathbb{K} \subseteq G$, the rotation group. Write K_θ for $z \mapsto e^{i\theta} z$. If

$$\mathscr{H}(n) = \{x \in \mathscr{H} : U_k x = e^{i(n-1)\theta} x\},$$

then $T : \mathscr{H}(n) \to \mathscr{H}(n+1)$ is a block shift.

A complete classification of these for $\dim \mathscr{H}(n) \leq 1$ was obtained in [1] using the representation theory of $\tilde{G}$. First examples for $\dim \mathscr{H}(n) = 2$ appeared in [3]. Recently [2], a m - parameter family of examples with $\dim \mathscr{H}(n) = m$ was constructed. This leads to a compleet classification of the homogeneous operators in the Cowen - Douglas class.

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Holomorphic Hermitian vector bundles

Definition

A bounded operator T on a Hilbert space $\mathscr{H}$ is said to be in the Cowen - Douglas class of the domain $\Omega \subseteq \mathbb{C}$ if its eigenspaces $E_w,\, w \in \Omega$ are of constant finite dimension.

- Cowen and Douglas show that $E \subseteq \Omega \times \mathscr{H}$ with fiber E_w is a holomorphic Hermitian vector bundle,
- isomorphism classes of E correspond to unitary equivalence classes of T,
- E is irreducible as a holomorphic Hermitian vector bundle if and only if T is irreducible.

Important to note here is that E has a reproducing kernel. Indeed, $\mathrm{ev}_w : \mathscr{H} \to E_w^*$ induced by the map $f \mapsto \langle f, \cdot \rangle$ is continuous and hence $K(z,w) = \mathrm{ev}_w^* \circ \mathrm{ev}_z$ is a reproducing kernel for E^*.

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Reproducing Kernel

Here we will always use trivialization of the bundles with standard Euclidean inner product. The Hilbert space $\mathscr{H} \subseteq \mathrm{Hol}(\Omega, \mathbb{C}^n)$ has a reproducing kernel $K_w(z) : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\langle f, K_w \xi \rangle = \langle f(w), \xi \rangle, f \in \mathscr{H}, \xi \in \mathbb{C}^n.$$

The operators in the Cowen-Douglas class can be realized as the adjoint of the multiplication operator M defined by $(Mf)(z) = zf(z)$ on a Hilbert space with holomorphic functions possessing a reproducing kernel.

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Goal:

Describe all homogeneous holomorphic Hermitian vector bundles! Determine which ones of these correspond to operators in the Cowen-Douglas class.

What we know: The homogeneous CD operators are (upto isomorphism) all of the form M^*, with M the multiplication by z.

What we have to do: Successively find:

1. All holomorphic homogeneous vector bundles
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The description of homogeneous vector bundles via holomorphic induction is well-known. Here we write down explicit trivialization of the vector bundle.

Let $\mathfrak{t} \subseteq \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ be the algebra $\mathbb{C}h + \mathbb{C}y$, where

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Linear representations (ρ, V) of the algebra $\mathfrak{t} \subseteq \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ satisfying $[\rho(h), \rho(y)] = -\rho(y)$ give all the homogeneous holomorphic vector bundles.

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The $\tilde{G}$ - invariant Hermitian structures on the homogeneous holomorphic vector bundle (making it into a homogeneous holomorphic Hermitian vector bundle), if they exist, are given by $\rho(\tilde{\mathbb{K}})$ - invariant inner products on the representation space.

An inner product is $\rho(\tilde{\mathbb{K}})$ - invariant if and only if $\rho(h)$ is diagonal with real diagonal elements in an appropriate basis.

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$$\rho(h) = \begin{pmatrix} -\eta I_0 & & \\ & \ddots & \\ & & -(\eta+m)I_m \end{pmatrix}, \text{with } I_j = I \text{ on } V_{-(\eta+j)} = \mathbb{C}^{n_j}$$

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the cocycle

The sections of homogeneous holomorphic vector bundle $E^{(Y,\eta)}$ are holomorphic functions $\mathbb{D} \to \mathbb{C}^n$. The $\tilde{G}$ action is given by $f \mapsto J_{g^{-1}}^{(Y,\eta)}\left(f \circ g^{-1}\right)$ with multiplier

$$\left(\!\left(J_g^{(Y,\eta)}(z)\right)\!\right)_{p,\ell} = \begin{cases} \binom{p}{\ell}(-c_g)^{p-\ell}(g')(z)^{\eta+\frac{p+\ell}{2}} Y_p \cdots Y_{\ell+1} & \text{if } p \geq \ell \\ 0 & \text{if } p < \ell \end{cases},$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c_g = c$.

Hermitian structure

A Hermitian structure appears as an inner product $\langle\cdot,\cdot\rangle_z$ on $\mathbb{C}^n$ for each $z\in\mathbb{D}$. We can write

$$\langle\zeta,\xi\rangle_z=\langle H(z)\zeta,\xi\rangle,\ \text{with}\ H(z)\succ 0.$$

We have homogeneity of the holomorphic Hermitian vector bundle if and only if

$$J_g(z)H(g\cdot z)^{-1}J_g(z)^*=H(z)^{-1},\ \text{for all}\ z\in\mathbb{D},\ g\in G.$$

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Hermitian structure

There are equivalences: $(E^{(\eta,Y)}, H) \approx (E^{(\eta,Y')}, H')$ if $Y' = AYA^{-1}$, $H' = A^{*-1}HA$ with block-diagonal A.

Hence each class has a representative $(E^{(\eta,Y)}, I)$. The classes can be parametrized as $E^{\eta,[Y]}$, with $[Y]$ the equivalence class under block-diagonal unitary conjugation.

Theorem

We have $E^{(Y,\eta)} \equiv E^{(Y',\eta')}$ if and only if $\eta = \eta'$ and $Y' = AYA^{-1}$ with a block diagonal matrix A.

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basic question

When there is a reproducing kernel?

If there is one, it gives a canonical Hermitian structure by setting $H(z) = K(z,z)^{-1}$.

Let $p_z = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \in G$, so $p_z \cdot 0 = z$. Writing $J_z^{(Y,\eta)}$ for $J_{p_z}^{(Y,\eta)}(z)$, we have

$$K(z,z) = J_z^{(Y,\eta)} K(0,0) J_z^{(Y,\eta)^*}.$$

So, the question is only: what can $K(0,0)$ be?

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construction of the reproducing kernel

Let $\mathbb{A}^{(\lambda)}$ be the Hilbert space of holomorphic functions on the unit disc with reproducing kernel $(1-z\bar{w}))^{-2\lambda}$. This is homogeneous under the multiplier ${g'}^{\lambda}$ for the $\tilde{G}$ action.

Let $\mathbf{A}^{(\eta)} = \oplus_{j=0}^{m} n_j \mathbb{A}^{(\eta+j)}$. For Y as before and $\eta > 0$, define

$$\left(\Gamma^{(Y,\eta)} f_j\right)_\ell = \begin{cases} \frac{1}{(\ell-j)!}\frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} D^{\ell-j} f_j & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j \end{cases}.$$

Let $\mathscr{H}^{(Y,\eta)}$ denote the image of $\Gamma^{(Y,\eta)}$ in the space of holomorphic functions $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^n)$.

Define a Hilbert space structure on $\mathscr{H}^{(Y,\eta)}$ by stipulating $\Gamma^{(Y,\eta)}$ to be unitary. We thus have a reproducing kernel Hilbert space.

construction of the reproducing kernel

Let $\mathbb{A}^{(\lambda)}$ be the Hilbert space of holomorphic functions on the unit disc with reproducing kernel $(1-z\bar{w}))^{-2\lambda}$. This is homogeneous under the multiplier g'^{λ} for the $\tilde{G}$ action.
Let $\mathbf{A}^{(\eta)} = \oplus_{j=0}^{m} n_j \mathbb{A}^{(\eta+j)}$. For Y as before and $\eta > 0$, define

$$\left(\Gamma^{(Y,\eta)} f_j\right)_\ell = \begin{cases} \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} D^{\ell-j} f_j & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j \end{cases}.$$

Let $\mathscr{H}^{(Y,\eta)}$ denote the image of $\Gamma^{(Y,\eta)}$ in the space of holomorphic functions $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^n)$.

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This actions lifts to a multiplier representation on $\mathscr{H}^{(Y,\eta)}$ with multiplier $J_g^{(Y,\eta)}(z)$.

The reproducing kernel for $K^{(Y,\eta)}(z,w)$ for the Hilbert space $\mathscr{H}^{(Y,\eta)}$ is of the form $J_g^{(Y,\eta)}(z)K(0,0)J_z^{(Y,\eta)^*}$ with

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Theorem

These are all the homogeneous holomorphic vector bundles with a reproducing kernel.

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These are all the homogeneous holomorphic vector bundles with a reproducing kernel.

Proof

Given any holomorphic homogeneous Hermitian vector bundle (assume irreducible), as a holomorphic homogeneous vector bundle it is of the form $E^{(\eta,Y)}$. Since it has a reproducing kernel, the $\widetilde{G}-$ action on it is unitary.

One can see that this unitary representation is a sum $\oplus U_j^{(\eta+j)} \otimes I_{n_j}$ of irreducible discrete series representations. So there exists some intertwining operator from (some) $\mathbb{A}^{(\eta)}$ to our Hilbert space.

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For holomorphic homogeneous Hermitian vector bundles, we had the convenient parametrization $E^{(\eta,[Y])}$, $\eta \in \mathbb{R}$, Y as before. Now we can also tell which ones among these have their Hermitian structure come from a reproducing kernel.

Theorem

$E^{(\eta,[Y])}$ *is a holomorphic homogeneous Hermitian vector bundle if and only if* $\eta > 0$ *and*

$$I - Y_j\left(\sum_{k=0}^{j-1}\frac{(-1)^{j+k-1}}{(j-k)!(2\eta+j+k-1)_{j-k}}Y_{j-1}\dots Y_{k+1}Y_{k+1}^*\dots Y_{j-1}^*\right)Y_j^*$$

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All the homogeneous holomorphic Hermitian vector bundles with a reproducing kernel correspond to homogeneous operators in the Cowen – Douglas class. The irreducible ones are the adjoint of the multiplication operator M *on the space* $\mathscr{H}^{(Y,\eta)}$ *for some* $\eta > 0$ *and irreducible* Y. *The block matrix* Y *is determined up to conjugacy by block diagonal unitaries.*

Proof.

There is a simple orthonormal system for the Hilbert space $\mathbb{A}^{(\lambda)}$. Hence we can find such a system for $\mathbf{A}^{(\eta)}$ as well. Transplant it using $\Gamma^{(Y,\eta)}$ to the Hilbert space $\mathscr{H}^{(Y,\eta)}$.

The multiplication operator in this basis has a block diagonal form with $M_n := M : \mathscr{H}(n) \to \mathscr{H}(n+1)$. This description is sufficiently explicit to see:

$M_n \sim I + 0(\frac{1}{n})$. Hence M is the sum of an ordinary block shift operator and a Hilbert Schmidt operator. This completes the proof. □

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Thank you!

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Thank you!