A product formula for homogeneous characteristic functions

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definitions

An operator T from a Hilbert space into itself is said to be *homogeneous*

if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb,

the group of bi-holomorphic automorphisms of the unit disc, which are analytic on the spectrum of T.

We say that a projective unitary representation σ of Möb is *associated* with an operator T if

$$\varphi(T) = \sigma(\varphi)^* T \sigma(\varphi)$$

for all φ in Möb. Clearly, if T has an associated representation then T is homogeneous.

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main results

We prove that if T is a cnu contraction with associated (projective unitary) representation σ, then there is a unique projective unitary representation $\widehat{\sigma}$, extending σ, associated with the minimal unitary dilation W of T.

Indeed, we have the formula for $\widehat{\sigma}$ in terms of σ, namely,

$$\widehat{\sigma} = (\pi \otimes D_1^+) \oplus \sigma \oplus (\pi_* \otimes D_1^-),$$

where $D_1^{\pm}$ are the two Discrete series representations (one holomorphic and the other anti-holomorphic) living on the Hardy space $H^2(\mathbb{T})$, and π, π_* are representations of the Mobius group living on the two defect spaces of T and explicitly defined in terms of σ.

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We find a *product formula* for the characteristic function θ of any cnu contractive homogeneous operator:

$$\theta(z) = \pi_*(\varphi_z)^* C\pi(\varphi_z), \quad z \in \mathbb{D}, \tag{1}$$

where φ_0 is the identity in Möb and, for $z \neq 0$ in $\mathbb{D}$, φ_z is the unique involution in Möb which interchanges 0 and z. Here, the two companion representations π, π_* are the ones that appear in the decomposition of $\hat{\sigma}$.

We make this formula explicit by describing the two representations π_* and π in the case of a large family of homogeneous contractions in the Cowen-Douglas class.

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the Möbius group

Möb $= \{\varphi_{\alpha,\beta} : \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where

$$\varphi_{\alpha,\beta}(z) = \alpha \frac{z - \beta}{1 - \bar{\beta} z}, \quad z \in \mathbb{D}. \tag{2}$$

Möb is the Möbius group of all biholomorphic automorphisms of $\mathbb{D}$. Recall that $\varphi_\beta := \varphi_{-1,\beta}$, $\beta \in \mathbb{D}$, is the unique involution in Möb which interchanges 0 and β.

Möb is topologised via the obvious identification with $\mathbb{T} \times \mathbb{D}$. With this topology, Möb becomes a topological group. Abstractly, it is isomorphic to $PSL(2, \mathbb{R})$ and to $PSU(1, 1)$.

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multipliers

Let G be a locally compact second countable topological group. Then a measurable function $\pi : G \to \mathcal{U}(\mathcal{H})$ is called a *projective representation* of G on the Hilbert space $\mathcal{H}$ if there is a function (necessarily Borel) $m : G \times G \to \mathbb{T}$ such that

$$\pi(1) = I,\ \pi(g_1 g_2) = m(g_1, g_2)\pi(g_1)\pi(g_2)$$

for all g_1, g_2 in G.

This requirement on a projective representation implies that its associated multiplier m satisfies

$$\begin{aligned} m(g,1) &= 1 = & m(1,g), \\ m(g_1,g_2)m(g_1g_2,g_3) &= & m(g_1,g_2g_3)m(g_2,g_3) \end{aligned}$$

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Sz.-Nagy – Foias model theory

Recall that an operator T is called a *contraction* if $\|T\| \leq 1$, and it is called *completely non-unitary* (cnu) if T has no non-trivial invariant subspace $\mathcal{M}$ such that the restriction of T to $\mathcal{M}$ is unitary. T is called a *pure contraction* if $\|Tx\| < \|x\|$ for all non-zero vectors x.

To any cnu contraction T on a Hilbert space, Sz.-Nagy and Foias associate a pure contraction valued analytic function

$$\theta_T : \mathbb{D} \to \mathcal{B}(\mathcal{D}, \mathcal{D}_*),$$

where $\mathcal{D} = \mathrm{clos}(\mathrm{ran}\sqrt{I - T^*T})$, and $\mathcal{D}_* = \mathrm{clos}(\mathrm{ran}\sqrt{I - TT^*})$, called the *characteristic function* of T.

Two pure contraction valued analytic functions $\theta_i : \mathbb{D} \to \mathcal{B}(\mathcal{D}_i, \mathcal{D}_{*i})$, $i = 1, 2$, are said to *coincide* if there exist two unitary operators $\tau_1 : \mathcal{D}_1 \to \mathcal{D}_2$, $\tau_2 : \mathcal{D}_{*1} \to \mathcal{D}_{*2}$ such that

$$\theta_2(z)\tau_1 = \tau_2\theta_1(z), \; z \in \mathbb{D}.$$

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unitary equivalence of contractions

The theory of Sz.-Nagy and Foias shows that

- two cnu contractions are unitarily equivalent if and only if their characteristic functions coincide,
- any pure contraction valued analytic function is the characteristic function of some cnu contraction.

In general, the model for the operator associated with a given function θ is difficult to describe.

transformation rule

Theorem

A pure contraction valued analytic function θ on $\mathbb{D}$ is the characteristic function of a homogeneous cnu contraction if and only if $\theta \circ \varphi$ coincides with θ for every φ in Möb.

As an interesting particular case of this theorem, one finds that any cnu contraction with a constant characteristic function is necessarily homogeneous.

Question

Are there cnu contractions with nonconstant characteristic function?

The holomorphic discrete series examples $(\lambda \geq 1)$ provide many examples of cnu contractions with non-constant characteristic function.

One may ask if it is possible to obtain a characterization of homogeneous cnu contractions using the characteristic functions?

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the product formula

Theorem

If T is an irreducible homogeneous contraction then its characteristic function $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{D}, \mathcal{D}_*)$ is given by

$$\theta(z) = \pi_*(\varphi_z)^* C\pi(\varphi_z), \quad z \in \mathbb{D}$$

where π and π_* are two projective representations of Möb (on the Hilbert spaces $\mathcal{D}$ and $\mathcal{D}_*$ respectively) with a common multiplier. Further, $C : \mathcal{D} \to \mathcal{D}_*$ is a pure contraction which intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$.

the converse

Theorem

Conversely, whenever π, π_* are projective representations of Möb with a common multiplier and C is a purely contractive intertwiner between $\pi_*|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$ such that the function θ defined by $\theta(z) = \pi_*(\varphi_z)^* C \pi(\varphi_z)$ is analytic on $\mathbb{D}$, then θ is the characteristic function of a homogeneous cnu contraction (not necessarily irreducible).

(Here φ_z is the involution in Möb which interchanges 0 and z. Also, $\mathbb{K} = \{\varphi \in \text{Möb} : \varphi(0) = 0\}$ is the standard maximal compact subgroup of Möb.)

This theorem provides a recipe for classifying homogeneous contractions.

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isometric dilation

Let T be a contraction on$\mathcal{H}$ and $V : \mathcal{D} \otimes H^2 \oplus \mathcal{H} \to \mathcal{D} \otimes H^2 \oplus \mathcal{H}$ be the operator

$$V = \begin{pmatrix} I \otimes S & iD_T \\ 0 & T \end{pmatrix}, \; D_T : \mathcal{H} \to \mathcal{D}, \; D_T = \sqrt{I - T^* T},$$

and $i : \mathcal{D} \to \mathcal{D} \otimes H^2$, $x \mapsto x \otimes 1$, $S : H^2 \to H^2$, $(Sf)(z) = zf(z)$, $f \in H^2$.

One easily verifies that V is an isometric dilation of T, that is, $P_{\mathcal{H}} V^n_{|\mathcal{H}} = T^n$.

unitary dilation

Let $\mathcal{D}_* := \mathrm{clos}(\mathrm{ran}\sqrt{I - TT^*})$. It is possible to construct a unitary dilation $U : \mathcal{D} \otimes H^2 \oplus \mathcal{H} \oplus \mathcal{D}_* \otimes H^2 \to \mathcal{D} \otimes H^2 \oplus \mathcal{H} \oplus \mathcal{D}_* \otimes H^2$ in a similar manner:

$$U = \begin{pmatrix} I \otimes S & iD_T & * \\ 0 & T & D_{T^*}^* i_*^* \\ 0 & 0 & I \otimes S^* \end{pmatrix},$$

where $D_{T^*} : \mathcal{H} \to \mathcal{D}_*$, $D_{T^*} = \sqrt{I - TT^*}$ and $i_* : \mathcal{D}_* \to \mathcal{D}_* \otimes H^2$, $x \mapsto x \otimes 1$.

new representations from old ones

Set c : Möb $\times \mathbb{D} \to \mathbb{C}$ will denote the function $c(\varphi, z) = (\varphi')^{1/2}(z)$.

Theorem 1

Let T be a homogeneous contraction with associated representation σ and W be the minimal unitary dilation. Then the unique representaion $\hat{\sigma}$ extending σ and associated with W is of the form

$$(\pi \otimes D_1^+) \oplus \sigma \oplus (\pi_* \otimes D_1^-),$$

where π : Möb $\to \mathcal{U}(\mathcal{D})$ and π_* : Möb $\to \mathcal{U}(\mathcal{D}_*)$ are given by the formula

$$\pi(\varphi)D = D\sigma(\varphi)c(\varphi, T)^{-1},\ \pi_*(\varphi)D_* = D_*\sigma(\varphi)c(\varphi, T)^{-1*}.$$

surprise

The representations π, π_* appear both in the product formula for the characteristic function of the operator T and the representation $\hat{\sigma}$ of its minimal unitary dilation.

new representations from old ones

Set c : Möb $\times \mathbb{D} \to \mathbb{C}$ will denote the function $c(\varphi, z) = (\varphi')^{1/2}(z)$.

Theorem 2

Let T be a homogeneous contraction with associated representation σ and W be the minimal unitary dilation. Then the unique representaion $\hat{\sigma}$ extending σ and associated with W is of the form

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the discrete series examples

For positive real numbers λ, let $\mathcal{H}^{(\lambda)}$ denote the Hilbert space of holomorphic functions on $\mathbb{D}$ with reproducing kernel $B^{(\lambda)}$, $B^{(\lambda)}(z,w) = (1-\overline{w}z)^{-\lambda}$. Let $D_\lambda^+(\varphi) : \text{Möb} \to \mathcal{U}(\mathcal{H}^{(\lambda)})$ be the operator

$$D_\lambda^+(\varphi) = (\varphi')^{\frac{\lambda}{2}} f \circ \varphi,\; f \in \mathcal{H}^{(\lambda)},\; \varphi \in \text{Möb}.$$

D_λ^+ is the holomorphic Discrete series representation of Möb living on the Hilbert space $\mathcal{H}^{(\lambda)}$.

the Gamma map

Let $\mathcal{H}_n^{(\lambda)}$ denote the Hilbert space $\oplus_{i=0}^{n-1}\mathcal{H}^{(\lambda_i)}$, where $\lambda_i = \lambda + 2i$ and $n \in \mathbb{N}$.
Given an n - tuple of strictly positive numbers $\underline{\mu} := (\mu_0, \ldots, \mu_{n-1})$, let

$$\Gamma^{(\lambda,\underline{\mu})} : \mathcal{H}_n^{(\lambda)} \to \mathrm{Hol}(\mathbb{D}, \mathbb{C}^n)$$

be the map defined by

$$\left(\Gamma^{(\lambda,\underline{\mu})}(\underline{f})\right)_\ell = \sum_{0\leq j\leq \ell} \frac{\sqrt{\mu_j}\binom{\ell}{j}}{(\lambda+2j)_{\ell-j}} f_j^{(\ell-j)},\ 0 \leq \ell < n,\ f = \oplus_{0\leq j<n} f_j.$$

Let $\mathcal{H}^{(\lambda,\underline{\mu})}$ be the image of $\Gamma^{(\lambda,\underline{\mu})}$. The operator $M^{(\lambda,\underline{\mu})}$ of multiplication by the coordinate function on $\mathcal{H}^{(\lambda,\underline{\mu})}$ is said to be a *generic contraction* if $\lambda > 1$ and $\frac{\mu_{k+1}}{\mu_k} > \frac{(k+1)^2}{(\lambda+2k-1)(\lambda+2k)}$ for $0 \leq k \leq n-2$.

explicit formula

Theorem

Let $M^{(\lambda,\underline{\mu})}$ be a generic contraction. Then the characteristic function of $M^{(\lambda,\underline{\mu})}$ coincides with the function

$$\theta^{(\lambda,\underline{\mu})} : \mathbb{D} \to \mathcal{B}\big(\oplus_{0\le k<n} \mathcal{H}^{(\lambda+2k+1)}, \oplus_{0\le j<n} \mathcal{H}^{(\lambda+2j-1)} \big)$$

given by the formulae

$$\begin{aligned} \theta^{(\lambda,\underline{\mu})}(z) &= \big(\oplus_{0\le j<n} D^{+}_{\lambda+2j-1}(\varphi_z)^{*} \big) C \big(\oplus_{0\le k<n} D^{+}_{\lambda+2k+1} \big) \\ &= \big(\!\big(\theta_{jk}(z)\big)\!\big)_{0\le j,k<n}, \ z \in \mathbb{D}, \end{aligned}$$

where C is an explicitly determined block operator.

Here's wishing Bhaskar Bagchi all the best in the coming years!