Role of the curvature in Operator theory

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the Cowen-Douglas class

A class of operators which was introduced by Cowen and Douglas in the late seventies consists of those bounded commuting d-tuples of operators $\boldsymbol{T} = (T_1, \ldots, T_d)$ on a complex separable Hilbert space $\mathscr{H}$ which

- possess an open set $\Omega \subset \mathbb{C}^d$ of joint eigenvalues of constant multiplicity, say n, and
- admit a holomorphic choice of eigenvectors: $s_1(w), \ldots, s_n(w)$, $w \in \Omega$, that is,

$$T_i s_j(w) = w_i s_j(w),\ w \in \Omega,\ 1 \le i \le d,\ 1 \le j \le n.$$

Adjoint of the multiplication by the co-ordinate functions on a Hilbert space of holomorphic functions possessing a reproducing kernel are typical examples of operators in the Cowen-Douglas class.

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the Cowen-Douglas theorem

One of the striking results from the late seventies due to Cowen and Douglas says:

- There is a one to one correspondence between the unitary equivalence class of the operators T and the equivalence classes of the holomorphic Hermitian vector bundles E_T determined by them.
- Furthermore, they find a set of complete invariants, not very tractable unless $n = 1,$ for this equivalence. For $n = 1,$ as is well-known, the curvature

$$\mathsf{K}(w) = -\frac{\partial^2}{\partial w \bar{\partial} w} \log \|s(w)\|^2 dw \wedge d\bar{w}$$

of the line bundle L_T is a complete invariant of $L_T,$ or equivalently, that of the operator $T.$

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proof that the curvature is a complete invariant

Pick a holomorphic frame $s_i(w)$ for the line bundle E_i and let $\Gamma_i(w) = \langle s_i(w), s_i(w)\rangle$ be the Hermitian metric, $i = 1, 2.$ Suppose that the two curvatures K_E and K_F are equal on some open (simply connected) subset $\Omega_0 \subseteq \Omega.$ It then follows that $u = \log(\Gamma_1/\Gamma_2)$ is harmonic ensuring the existence of a harmonic conjugate v of u on $\Omega_0.$ Define $\tilde{s}_2(w) = e^{(u(w)+iv(w))/2} s_2(w).$ Then clearly, $\tilde{s}_2(w)$ is a new holomorphic frame for $F.$ Consequently, we have

$$\begin{aligned}
\tilde{\Gamma}_2(w) &= \langle \tilde{s}_2(w), \tilde{s}_2(w)\rangle \\
&= \langle e^{(u(w)+iv(w))/2} s_2(w), e^{(u(w)+iv(w))/2} s_2(w)\rangle \\
&= e^{u(w)} \langle s_2(w), s_2(w)\rangle \\
&= \Gamma_1(w).
\end{aligned}$$

kernel function

- The kernel function K is a complex valued function defined on $\Omega^* \times \Omega^*$ which is holomorphic in the first variable and anti - holomorphic in the second. Therefore, the map $w \to K(\cdot, w), w \in \Omega^*$, is holomorphic on $\Omega^* := \{\bar{w} : w \in \Omega\}$.
- It is Hermitian, $K(z,w) = \overline{K(w,z)}$, and positive definite, that is, $\big(\!\big(K(w^i, w^j)\big)\!\big)_{i,j=1}^n$ is positive definite for every subset $\{w^1, \ldots, w^n\}$ of Ω^*, $n \in \mathbb{N}$.
- The kernel K reproduces the value of functions in $\mathscr{H}$, that is, for any fixed $w \in \Omega^*$, the holomorphic function $K(\cdot, w)$ belongs to $\mathscr{H}$ and

$$f(w) = \langle f, K(\cdot, w)\rangle, f \in \mathscr{H}, w \in \Omega^*.$$

- The reproducing property of K ensures that $M_i^* K(\cdot, w) = \bar{w}_i K(\cdot, w)$. Therefore, we have a natural holomorphic frame $\gamma(w) := K(\cdot, w)$ on Ω^* for the commuting tuple $M_1^*, \ldots, M_d^*$.

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An alternative description of the Cowen-Douglas class

A `Hilbert module` over the polynomial ring $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \ldots, z_m]$ is a Hilbert space $\mathscr{H}$ which is a $\mathbb{C}[\underline{z}]$ -module if for some $C_p > 0$,

$$\|p \cdot f\| \leq C_p \|f\|, f \in \mathscr{H}, p \in \mathbb{C}[\underline{z}].$$

The multiplication M_j by the coordinate functions z_j, $M_j f := z_j \cdot f, 1 \leq j \leq m,$ then defines a commutative tuple $\mathbf{M} = (M_1, \ldots, M_m)$ of linear bounded operatorms acting on $\mathscr{H}$ and vice-versa.

A Hilbert module $\mathscr{H}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the `Cowen-Douglas class` $\mathrm{B}_n(\Omega)$, $n \in \mathbb{N}$, if

- $\dim \mathscr{H} / \mathfrak{m}_w \mathscr{H} = n < \infty$ for all $w \in \Omega$, where $\mathfrak{m}_w$ is the maximal ideal in $\mathbb{C}[\underline{z}]$ at w and
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tensor product

Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be Hilbert spaces of holomorphic functions on Ω so that they possess reproducing kernels K_1 and $K_2,$ respectively. Assume that the natural action of $\mathbb{C}[\underline{z}]$ on the Hilbert space $\mathscr{M}_1$ is continuous, that is, the map $(p,h) \to ph$ defines a bounded operator on M_p for $p \in \mathbb{C}[\underline{z}].$ (We make no such assumption about the Hilbert space $\mathscr{M}_2.$) Now, $\mathbb{C}[\underline{z}]$ acts naturally on the Hilbert space tensor product $\mathscr{M}_1 \otimes \mathscr{M}_2$ via the map

$$(p,(h \otimes k)) \to ph \otimes k, p \in \mathbb{C}[\underline{z}],\ h \in \mathscr{M}_1,\ k \in \mathscr{M}_2.$$

The map $h \otimes k \to hk$ identifies the Hilbert space $\mathscr{M}_1 \otimes \mathscr{M}_2$ as a reproducing kernel Hilbert space of holomorphic functions on $\Omega \times \Omega.$ The module action is then the point-wise multiplication $(p,hk) \to (ph)k,$ where $((ph)k)(z_1,z_2) = p(z_1)h(z_1)k(z_2),\ z_1,z_2 \in \Omega.$

a new kernel

Let $\mathscr{H}$ be the Hilbert module $\mathscr{M}_1 \otimes \mathscr{M}_2$ over $\mathbb{C}[\underline{z}]$. Let $\triangle \subseteq \Omega \times \Omega$ be the diagonal subset $\{(z,z) : z \in \Omega\}$ of $\Omega \times \Omega$. Let $\mathscr{S}$ be the maximal submodule of functions in $\mathscr{M}_1 \otimes \mathscr{M}_2$ which vanish on $\triangle$. Thus

$$0 \to \mathscr{S} \xrightarrow{X} \mathscr{M}_1 \otimes \mathscr{M}_2 \xrightarrow{Y} \mathscr{Q} \to 0$$

is a short exact sequence, where $\mathscr{Q} = (\mathscr{M}_1 \otimes \mathscr{M}_2)/\mathscr{S}$, X is the inclusion map and Y is the natural quotient map. One can appeal to an extension of an earlier result of Aronszajn to analyze the quotient module $\mathscr{Q}$ when the given modules are reproducing kernel Hilbert spaces. The reproducing kernel of $\mathscr{H}$ is then the pointwise product $K_1(z,w)K_2(u,v)$ for $z,w;u,v$ in Ω. Set $\mathscr{H}_{\rm res} = \{f_{|\triangle} : f \in \mathscr{H}\}$ and $\|f\|_{|\triangle} = \inf\{\|g\| : g \in \mathscr{H}, g_{|\triangle} \equiv f_{|\triangle}\}$.

- The quotient module is isomorphic to the module $\mathscr{H}_{\rm res}$ whose reproducing kernel is the pointwise product $K_1(z,w)K_2(z,w)$, $z,w \in \Omega$.

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another kernel!

Suppose $\Omega \subseteq \mathbb{C}^d$ is open connected and bounded. Let $K : \Omega \times \Omega$ be a non-negative definite kernel. Then $\widetilde{K}$ defined by

$$\widetilde{K}(z,w) = \big(\big(K^2 \partial_i \bar{\partial}_j \log K(z,w)\big)\big)_{1 \le i,j \le d}$$

is a $\mathbb{C}^{d \times d}$ valued non-negative definite kernel.

- We point out that $\sum_{i,j} \partial_i \bar{\partial}_j \log K(w,w) dw_i \wedge d\bar{w}_j$ is the curvature of the metric $K(w,w)$.

To see that $\widetilde{K}$ defines a positive definite kernel on Ω, set

$$\phi_i(w) := K_w \otimes \bar{\partial}_i K_w - \bar{\partial}_i K_w \otimes K_w, 1 \le i \le m$$

and note that each $\phi : \Omega \to \mathscr{H}$ is holomorphic. A simple calculation then shows that

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what is the Hilbert module?

How to describe the Hilbert space, or more importantly, the Hilbert module $\mathscr{H}(\widetilde{K})$? May be, it is a quotient of the Hilbert module $\mathscr{H} \otimes \mathscr{H}$? If so, How do we identify the corresponding submodule?

Let $\mathscr{H}_0$ be the subspace of $\mathscr{H}(K) \otimes \mathscr{H}(K)$ given by $\overline{\bigvee}\{\phi_i(w) : w \in \Omega, 1 \leq i \leq m\}$.

From this definition, it is not clear which functions belong to the subspace. We give an explicit description.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the submodules defined by

$$\mathscr{H}_1 = \{f \in \mathscr{H}(K) \otimes \mathscr{H}(K) : f|_\Delta = 0\}$$

and

$$\mathscr{H}_2 = \{f \in \mathscr{H}(K) \otimes \mathscr{H}(K) : f|_\Delta = \partial_1 f|_\Delta = \partial_2 f|_\Delta = \ldots = \partial_m f|_\Delta = 0\}.$$

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$$\mathscr{H}_1 = \{f \in \mathscr{H}(K) \otimes \mathscr{H}(K) : f|_\Delta = 0\}$$

and

$$\mathscr{H}_2 = \{f \in \mathscr{H}(K) \otimes \mathscr{H}(K) : f|_\Delta = \partial_1 f|_\Delta = \partial_2 f|_\Delta = \ldots = \partial_m f|_\Delta = 0\}.$$

We have

- $\mathscr{H}_{11} = \mathscr{H}_2^\perp \ominus \mathscr{H}_1^\perp$

what is the Hilbert module?

How to describe the Hilbert space, or more importantly, the Hilbert module $\mathscr{H}(\widetilde{K})$? May be, it is a quotient of the Hilbert module $\mathscr{H} \otimes \mathscr{H}$? If so, How do we identify the corresponding submodule?

Let $\mathscr{H}_0$ be the subspace of $\mathscr{H}(K) \otimes \mathscr{H}(K)$ given by $\overline{\bigvee}\{\phi_i(w) : w \in \Omega, 1 \le i \le m\}$.

From this definition, it is not clear which functions belong to the subspace. We give an explicit description.

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a limit computation

The point of what we have said so far is that we can explicitly describe the Hilbert modules $\mathscr{H}_2^{\perp}$ and $\mathscr{H}_1^{\perp}$, upto an isomorphism of modules. Using the jet construction followed by the restriction map, one may also describe the direct sum $\mathscr{H}_2^{\perp} \oplus \mathscr{H}_1^{\perp}$, again upto an isomorphism.

But what is the module $\mathscr{H}_{11}$ as a submodule of $\mathscr{H}$? To answer this question, one must find the kernel function for $\mathscr{H}_{11}$. Set K_1 to be the kernel function of the module $\mathscr{H}_1$. Assuming $d = 1$, we have

$$\left(\frac{K_1(z,u,v,w)}{(z-u)(\bar{w}-\bar{v})}\right)\Bigg|_{\substack{z=u,z\neq u\\ w=v,w\neq v}} = \tfrac{1}{2}K(z,w)^2\partial\bar{\partial}\log K(z,w).$$

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Thank You!