A sheaf model for semi-Fredholm Hilbert modules

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Motivation

The Cowen - Douglas class

A `Hilbert module` over the polynomial ring $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \dots, z_m]$ is a Hilbert space $\mathcal{H}$ which is a $\mathbb{C}[\underline{z}]$ -module with the assumption

$$\|p \cdot f\| \leq C_p \|f\|, \quad f \in, \quad p \in \mathbb{C}[\underline{z}],$$

for some $C_p > 0$.

The multiplication M_j by the coordinate functions z_j, $M_j f := z_j \cdot f,\ 1 \leq j \leq m,$ then defines a commutative tuple $\mathbf{M} = (M_1, ..., M_m)$ of linear bounded operators acting on " and vice-versa.

A Hilbert module $\mathcal{H}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the `Cowen-Douglas class` $\mathrm{B}_n(\Omega)$, $n \in \mathbb{N}$, if

$\dim \mathcal{H}/\mathfrak{m}_w\mathcal{H} = n < \infty$ for all $w \in \Omega$

$\cap_{w \in \Omega} \mathfrak{m}_w \mathcal{H} = \{0\}$, where $\mathfrak{m}_w$ denotes the maximal ideal in $\mathbb{C}[\underline{z}]$ at w.

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Examples

A Hilbert module $\mathcal{M}$ in $\mathrm{B}_n(\Omega)$ determines a holomorphic Hermitian vector bundle on Ω.

Cowen and Douglas prove that isomorphic Hilbert modules correspond to equivalent vector bundles and vice-versa.

Also, they provide a model for the Hilbert modules in $\mathrm{B}_n(\Omega)$. Cowen and Douglas (Curto and Salinas, in general) show that these modules can be realized as a Hilbert space consisting of holomorphic functions on Ω possessing a reproducing kernel. The module action is then simply the pointwise multiplication.

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Not an example!

However, many natural examples of Hilbert modules fail to be in the class $\mathrm{B}_n(\Omega)$.

For instance, $H_0^2(\mathbb{D}^2) := \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$ is not in $\mathrm{B}_n(\mathbb{D}^2)$.

The problem is that the dimension of the joint kernel

$$\mathcal{H}/\mathfrak{m}_w\mathcal{H} \cong \cap_{j=0}^{m}\mathrm{Ker}(M_j - w_j)^*$$

is no longer a constant.

Indeed, we have (an easy calculation)

$$\dim\left(\mathcal{H}/\mathfrak{m}_w\mathcal{H}\right) = \begin{cases} 1 & \text{if } w \neq (0,0) \\ 2 & \text{if } w = (0,0). \end{cases}$$

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What about the kernel?

The computation of the dimension of the joint kernel for the module $H_0^2(\mathbb{D}^2)$ serves another purpose as well.

It shows that the module $H_0^2(\mathbb{D}^2)$ is not equivalent to the usual Hardy module. The dimension of the joint kernel for the Hardy module is 1 everywhere on the bi-disc.

This is a *gennuine* multi-variate phenomenon – for the unit disc, the Hardy module is equivalent to all its sub-modules.

Clearly, the dimension of the joint kernel is an important unitary invaraint for a module. However, in many instances, calculating this dimension, or other numerical invariants is possible only after determining the kernel itself.

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Definitions

A Hilbert module $\mathcal{M} \subset \mathcal{O}(\Omega)$ is said to be in `the class` $\mathfrak{B}_1(\Omega)$ if

it possesses a reproducing kernel K (we don't rule out the possibility: $K(w,w) = 0$ for w in some closed subset X of Ω) and

The dimension of $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ is finite for all $w \in \Omega$.

Most of the examples in $\mathfrak{B}_1(\Omega)$ are obtained by taking submodules of Hilbert modules $\mathcal{H}(\subseteq \mathcal{O}(\Omega))$ in the Cowen-Douglas class $\mathrm{B}_1(\Omega)$.

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A couple of questions

Let $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ be a Hilbert module and $\mathcal{I} \subseteq \mathcal{M}$ be a polynomial ideal. Assume without loss of generality that $0 \in V(\mathcal{I})$. Now, we ask

if there exists a set of polynomials $p_1, \ldots, p_t$ such that

$$p_i(\tfrac{\partial}{\partial \bar{w}_1}, \ldots, \tfrac{\partial}{\partial \bar{w}_m}) K_{[\mathcal{I}]}(z, w)|_{w=0},\ i = 1, \ldots, t,$$

spans the joint kernel of $[\mathcal{I}]$;

what conditions, if any, will ensure that the polynomials $p_1, \ldots, p_t$, as above, is a generating set for $\mathcal{I}$?

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Relation between $\mathfrak{B}_1(\Omega)$ and $\mathrm{B}_1(\Omega)$

The following Lemma isolates a very large class of elements from $\mathfrak{B}_1(\Omega)$ which belong to $\mathrm{B}_1(\Omega_0)$ for some open subset $\Omega_0 \subseteq \Omega$.

Lemma. *Suppose* $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ *is the closure of a polynomial ideal* $\mathcal{I}$. *Then* $\mathcal{M}$ *is in* $\mathrm{B}_1(\Omega)$ *if the ideal* $\mathcal{I}$ *is singly generated while if it is generated by the polynomials* $p_1, p_2, \ldots, p_t$ *, then* $\mathcal{M}$ *is in* $\mathrm{B}_1(\Omega \setminus X)$ *for* $X = \{z : p_1(z) = \ldots = p_t(z) = 0\}$.

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The sheaf model

Construction of the sheaf model

Following the correspondence of a vector bundle with a locally free sheaf, we construct a sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ for the Hilbert module $\mathcal{M}$.

The sheaf $\mathcal{S}^{\mathcal{M}}$ is the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ whose stalk $\mathcal{S}^{\mathcal{M}}_w$ at $w \in \Omega$ is

$$\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\}$$

For any Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, the sheaf $\mathcal{S}^{\mathcal{M}}$ is `coherent`.

This is essentially Noether's stationary lemma!

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The decomposition theorem

Theorem. *Suppose* $g_i^0, 1 \leq i \leq d,$ *be a minimal set of generators for the stalk* $\mathcal{S}^{\mathcal{M}}_{w_0}$. *Then there exists a open neighborhood* Ω_0 *of* w_0 *such that*

$$K(\cdot, w) := K_w = g_1^0(w) K_w^{(1)} + \cdots + g_n^0(w) K_w^{(d)},\, w \in \Omega_0$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathcal{M}$

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Outline of the proof of the Theorem

We point out that the linear span of the set of vectors $\{K_{w_0}^{(i)} \mid 1 \leq i \leq d\}$ in $\mathcal{M}$ is independent of the generators $g_1^0, \ldots, g_d^0$, and that the vectors $K_{w_0}^{(i)}, 1 \leq i \leq d$, are eigenvectors for the adjoint of the action of $\mathbb{C}[\underline{z}]$ on the Hilbert module $\mathcal{M}$ at w_0.

Key ingredients in the proof are the following observations.

There is a decomposition for a function in any submodule of $\mathcal{O}_{w_0}$ in terms of its generators valid over a small neighbourhood of w_0.

The coefficients in this decomposition satisfy uniform norm bounds in a even smaller compact neighbourhood of w_0.

$\mathcal{O}_{w_0}$ is a local ring to which Nakayama's lemma applies.

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There is a decomposition for a function in any submodule of $\mathcal{O}_{w_0}$ in terms of its generators valid over a small neighbourhood of w_0.

The coefficients in this decomposition satisfy uniform norm bounds in a even smaller compact neighbourhood of w_0.

$\mathcal{O}_{w_0}$ is a local ring to which Nakayama's lemma applies.

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We point out that the linear span of the set of vectors $\{K_{w_0}^{(i)} \mid 1 \leq i \leq d\}$ in $\mathcal{M}$ is independent of the generators $g_1^0, \ldots, g_d^0$, and that the vectors $K_{w_0}^{(i)}, 1 \leq i \leq d$, are eigenvectors for the adjoint of the action of $\mathbb{C}[\underline{z}]$ on the Hilbert module $\mathcal{M}$ at w_0.

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One easy consequence of the decomposition theorem is the inequality

$$\begin{aligned}\dim\ker D_{(\mathbf{M}-w_0)^*} &\geq \sharp\{minimal\ generators\ for\ S_{w_0}^{\mathcal{M}}\}\\ &\geq \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}_{w_0}^{\mathcal{M}}.\end{aligned}$$

One of the basic question is to ask if we have equality under additional hypothesis on the Hilbert module $\mathcal{M}$. Thus assuming $\mathcal{M}$ to be an analytic Hilbert module then Chen and Guo have shown that equality is forced. We show that this property continues to hold for submodules of analytic Hilbert modules.

Corollary. *If $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathbb{C}[z]$, where $\mathcal{I}$ is an ideal in the polynomial ring $\mathbb{C}[z]$ and $w \in V(\mathcal{I})$ is a smooth point, then*

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Corollary. *If* $\mathcal{M} = [\mathcal{I}]$ *be a submodule of an analytic Hilbert module over* $\mathbb{C}[\underline{z}]$ *, where* $\mathcal{I}$ *is an ideal in the polynomial ring* $\mathbb{C}[\underline{z}]$ *and* $w \in V(\mathcal{I})$ *is a smooth point, then*

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The joint kernel of a Hilbert module

The characteristic space

Let $\mathcal{I}$ be an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$.

The characteristic space of an ideal $\mathcal{I}$ in $\mathbb{C}[\underline{z}]$ at the point w is the vector space

$$\mathbb{V}_w(\mathcal{I}) := \{q \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0,\, p \in \mathcal{I}\}.$$

The envolope $\mathcal{I}^e_w$ of the ideal $\mathcal{I}$ is

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If the zero set of the ideal $\mathcal{I}$ is $\{w\}$ then $\mathcal{I}^e_w = \mathbb{V}_w(\mathcal{I})$.

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An auxiliary space

Let $\tilde{\mathbb{V}}_w(\mathcal{I})$ be the `auxiliary space` $\mathbb{V}_w(\mathfrak{m}_w\mathcal{I})$. Then we have

$$\dim \cap \mathrm{Ker}(M_j - w_j)^* = \dim \tilde{\mathbb{V}}_w(\mathcal{I})/\mathbb{V}_w(\mathcal{I}).$$

Actually, we have something much more substantial.

Lemma. *Fix $w_0 \in \Omega$ and polynomials $q_1, \ldots, q_t$. Let $\mathcal{I}$ be a polynomial ideal and K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then the vectors*

$$q_1(\bar{D})K(\cdot, w)|_{w=w_0}, \ldots, q_t(\bar{D})K(\cdot, w)|_{w=w_0}$$

form a basis of the joint kernel $\cap_{j=1}^m \ker(M_j - w_{0j})^*$ if and only if the classes $[q_1^*], \ldots, [q_t^*]$ form a basis of $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$.

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A canonical set of generators

Theorem. *Let* $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ *be a homogeneous ideal and* $\{p_1, \ldots, p_v\}$ *be a minimal set of generators for* $\mathcal{I}$ *consisting of homogeneous polynomials. Let* K *be the reproducing kernel corresponding to the Hilbert module* $[\mathcal{I}]$ *, which is assumed to be in* $\mathfrak{B}_1(\Omega)$. *Then there exists a set of generators* $q_1, \ldots, q_v$ *for the ideal* $\mathcal{I}$ *such that the set*

$$\{q_i(\bar{D})K(\cdot, w)|_{w=0} : 1 \leq i \leq v\}$$

is a basis for $\cap_{j=1}^{m} \ker M_j^*$.

We note that the new set $\{q_1, \ldots, q_v\}$ of generators for $\mathcal{I}$ is more or less ``canonical''. It is uniquely determined modulo a linear transformation as shown below.

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An Example

Let $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ be the ideal generated by $z_1 + z_2$ and z_2^2. We have $V(\mathcal{I}) = \{0\}$. The reproducing kernel K for $[\mathcal{I}] \subseteq H^2(\mathbb{D}^2)$ is

$$\begin{aligned} K_{[\mathcal{I}]}(z, w) &= \frac{1}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} - \frac{(z_1 - z_2)(\bar{w}_1 - \bar{w}_2)}{2} - 1 \\ &= \frac{(z_1 + z_2)(\bar{w}_1 + \bar{w}_2)}{2} + i + j \geq 2^{\infty} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j. \end{aligned}$$

The vector $\bar{\partial}_2^2 K_{[\mathcal{I}]}(z, w)|_0 = 2z_2^2$ is not in the joint kernel of $P_{[\mathcal{I}]}(M_1^*, M_2^*)|_{[\mathcal{I}]}$ since $M_2^*(z_2^2) = z_2$ and $P_{[\mathcal{I}]} z_2 = (z_1 + z_2)/2 \neq 0$.

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However, we have $q_1 = z_1 + z_2$ and $q_2 = (z_1 - z_2)^2$ and they generate the ideal $\mathcal{I}$ as well. Moreover, $\{(\bar{\partial}_1 + \bar{\partial}_2)K(\cdot, w)|_0, (\bar{\partial}_1 - \bar{\partial}_2)^2 K(\cdot, w)|_0\}$ forms a basis of the joint kernel.

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Indeed, our Theorem provides an effective tool for deciding when an ideal is a monomial ideal.

Let $\{q_1, \ldots, q_v\}$ be a canonical set of generators for $\mathcal{I}$. Let Λ be the collection of monomials in the expressions of $\{q_1, \ldots, q_v\}$ that are in $\mathcal{I}$. If the number of algebraically independent monomials in Λ is v , then $\mathcal{I}$ is a monomial ideal.

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Let $\{q_1, \ldots, q_v\}$ be a canonical set of generators for $\mathcal{I}$. Let Λ be the collection of monomials in the expressions of $\{q_1, \ldots, q_v\}$ that are in $\mathcal{I}$. If the number of algebraically independent monomials in Λ is v, then $\mathcal{I}$ is a monomial ideal.

Example contd.

However, we have $q_1 = z_1 + z_2$ and $q_2 = (z_1 - z_2)^2$ and they generate the ideal $\mathcal{I}$ as well. Moreover, $\{(\bar{\partial}_1 + \bar{\partial}_2)K(\cdot, w)|_0, (\bar{\partial}_1 - \bar{\partial}_2)^2 K(\cdot, w)|_0\}$ forms a basis of the joint kernel.

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Back to $H^2_0(\mathbb{D}^2)$

In the example of the module $H^2_0(\mathbb{D}^2)$, we have

$$\mathcal{S}_w^{H^2_0(\mathbb{D}^2)} = \begin{cases} \mathcal{O}_w & \text{if } w \neq (0,0) \\ \mathfrak{m}_{(0,0)}\mathcal{O}_{(0,0)} & \text{if } w = (0,0). \end{cases}$$

While the germs of holomorphic function $\mathcal{O}_w$ at $w \in \mathbb{D}^2$ is singly genarated (even if $w = (0,0)$), the ideal $\mathfrak{m}_{(0,0)}\mathcal{O}_{(0,0)} \subseteq \mathcal{O}_{(0,0)}$ is 2-generated.

Thus the number of generators match the dimension of the joint eigenspace, in this case.

The reproducing kernel $K_{H^2_0(\mathbb{D}^2)}(z,w)$ is easy to compute:

$$K_{H^2(\mathbb{D}^2)}(z,w) - 1 \quad = \quad \frac{z_1\bar{w}_1 + z_2\bar{w}_2 - z_1z_2\bar{w}_1\bar{w}_2}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)}.$$

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Back to our favorite example

How do we find the unique pair of vectors $K_0^{(1)}$ and $K_0^{(2)}$?

set $\bar{w}_1\theta_1 = \bar{w}_2$ for $w_1 \neq 0$, and take the limit:

$$\lim_{(w_1,w_2)\to 0} \frac{K_{H_0^2(\mathbb{D}^2)}(z,w)}{\bar{w}_1} = K_0^{(1)}(z) + \theta_1 K_0^{(2)}(z) = z_1 + \theta_1 z_2$$

to obtain $K_0^{(1)}$ and $K_0^{(2)}$ by the uniqueness in the Decomposition Theorem. Similarly, for $\bar{w}_2\theta_2 = \bar{w}_1$ with $w_2 \neq 0,$ we have

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A resolution of the singularity at 0

Thus we have a Hermitian line bundle on the complex projective space $\mathbb{P}^1$ given by the frame $\theta_1 \mapsto z_1 + \theta_1 z_2$ and $\theta_2 \mapsto z_2 + \theta_2 z_1$.

The curvature of this line bundle is then an invariant for the Hilbert module $H_0^2(\mathbb{D}^2)$. This curvature is easily calculated and is given by the formula $\mathcal{K}(\theta) = (1 + |\theta|^2)^{-2}$.

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The decomposition theorem yields similar results in many other examples.

The isomorphism of Modules

For any two Hilbert module $\mathcal{M}_1$ and $\mathcal{M}_2$ in the class $\mathcal{B}_1(\Omega)$ and $L : \mathcal{M}_1 \to \mathcal{M}_2$ a module map between them, let $\mathcal{S}^L : \mathcal{S}^{\mathcal{M}_1}(V) \to \mathcal{S}^{\mathcal{M}_2}(V)$ be the map defined by

$$\mathcal{S}^L \sum_{i=1}^{n} f_i|_V g_i := \sum_{i=1}^{n} L f_i|_V g_i, \text{ for } f_i \in \mathcal{M}_1, g_i \in \mathcal{O}(V), n \in \mathbb{N}.$$

The map $\mathcal{S}^L$ is well defined: if $\sum_{i=1}^n f_i|_V g_i = \sum_{i=1}^n \hat{f}_i|_V \hat{g}_i,$ then $\sum_{i=1}^n L f_i|_V g_i = \sum_{i=1}^n L\hat{f}_i|_V \hat{g}_i.$

Suppose $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$ via a unitary module map $L.$ Now, it is easy to verify that $(\mathcal{S}^L)^{-1} = \mathcal{S}^{L^*}.$ It then follows that $\mathcal{S}^{\mathcal{M}_1}$ is isomorphic, as a sheaf of modules over $\mathcal{O}_\Omega,$ to $\mathcal{S}^{\mathcal{M}_2}$ via the map $\mathcal{S}^L.$

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The Rigidity Theorem

For $w_0 \in X,$ the common zero set of the two modules $\mathcal{M}_1$ and $\mathcal{M}_2,$ the stalks are not just isomorphic but equal:

$$\begin{aligned}\mathcal{S}_{w_0}^{\mathcal{M}_1} &= \{i = 1^n h_i g_i : g_i \in \mathcal{M}_1, h_i \in {}_m\mathcal{O}_{w_0}, 1 \le i \le n, n \in \mathbb{N}\} \\ &= \{i = 1^n h_i \phi f_i : f_i \in \mathcal{M}_2, h_i \in {}_m\mathcal{O}_{w_0}, 1 \le i \le n, n \in \mathbb{N}\} \\ &= \{i = 1^n \hat{h}_i f_i : f_i \in \mathcal{M}_2, \hat{h}_i \in {}_m\mathcal{O}_{w_0}, 1 \le i \le n, n \in \mathbb{N}\} = \mathcal{S}_{w_0}^{\mathcal{M}_2}.\end{aligned}$$

Theorem. *Let* $\mathcal{M} \subseteq \mathcal{O}(\Omega)$ *and* $\hat{\mathcal{M}} \subseteq \mathcal{O}(\Omega)$ *be two Hilbert modules of the form* $[\mathcal{I}]$ *and* $[\hat{\mathcal{I}}],$ *respectively, where* $\mathcal{I}, \hat{\mathcal{I}}$ *are polynomial ideals. Assume that* $\mathcal{M}, \hat{\mathcal{M}}$ *are in* $\mathfrak{B}_1(\Omega)$ *and that the dimension of the zero set of these modules is at most* $m-2.$ *Furthermore, also assume that every algebraic component of* $V(\mathcal{I})$ *and* $V(\hat{\mathcal{I}})$ *intersects* $\Omega.$ *If* $\mathcal{M}$ *and* $\hat{\mathcal{M}}$ *are equivalent, then* $\mathcal{I} = \hat{\mathcal{I}}.$

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Applications

Example. For $j = 1, 2$, let $\mathcal{I}_j \subset \mathbb{C}[z_1, \ldots, z_m]$, $m > 2,$ be the ideals generated by z_1^n and $z_1^{k_j} z_2^{n-k_j}$. Let $[\mathcal{I}_j]$ be the submodule in the Hardy module $H^2(\mathbb{D}^m)$. Now, from the Theorem proved above, it follows that $[\mathcal{I}_1]$ is equivalent to $[\mathcal{I}_2]$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$. We conclude that these two ideals are same only if $k_1 = k_2$.

Let $\mathcal{M}$ be a Hilbert module in $\mathfrak{B}_1(\Omega),$ which is the closure, in $\mathcal{M},$ of some polynomial ideal $\mathcal{I}$. Let K denote the corresponding reproducing kernel. Let $w_0 \in V(\mathcal{M})$. Set

$$t = \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0} = \dim \cap_{j=1}^m \ker(M_j - w_{0j})^* = \dim \hat{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I}).$$

By the Decomposition Theorem, there exists a minimal set of generators $g_1, \cdots, g_t$ of $\mathcal{S}_0^{\mathcal{M}_1}$ and a $r > 0$ such that

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New vector bundles

Consider the open set $U_1 = (\Delta(w_0; r) \times \{u_1 \neq 0\}) \cap \widehat{\Delta}(w_0; r)$. Let $\frac{u_j}{u_1} = \theta_j^1$, $2 \leq j \leq t$. On this chart $g_j(w) = \theta_j^1 g_j(w)$. From the decomposition for the $K(\cdot, w)$, we have

$$K(\cdot, w) = \overline{g_1(w)}\{K^{(1)}(\cdot, w) + \sum_{j=2}^{t} \bar{\theta}_j^1 K^{(j)}(\cdot, w)\}.$$

This decomposition then yields a section on the chart U_1, of the line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$:

$$s_1(w, \theta) = K^{(1)}(\cdot, w) + \sum_{j=2}^{t} \bar{\theta}_j^1 K^{(j)}(\cdot, w).$$

The vectors $K^{(j)}(\cdot, w)$ are not uniquely determined. However, there exists a canonical choice of these vectors starting from a basis, $\{v_1, \ldots, v_t\}$, of the joint kernel $\cap_{i=1}^{n} \ker(M_j - w_j)^*$:

$$K(\cdot, w) = \sum_{j=1}^{t} \overline{g_j(w)} P(\bar{w}, \bar{w}_0) v_j, \; w \in \Delta(w_0; r)$$

for some $r > 0$ and generators [illegible] of the stalk $\mathcal{S}^{\mathcal{M}}$

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Restriction to the exceptional set

Let $\mathcal{L}(\mathcal{M})$ be the line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$ determined by the section $(w, \theta) \mapsto s_1(w, \theta),$ where

$$s_1(w, \theta) = P(\bar{w}, \bar{w}_0)v_1 + \sum_{j=2}^{t} \bar{\theta}_j^1 P(\bar{w}, \bar{w}_0)v_j, \; (w, \theta) \in U_1.$$

In general, Z need not be a complex manifold. However, the restriction of s_1 to $p^{-1}(w_0)$ for $w_0 \in Z$ determines a holomorphic line bundle $\mathcal{L}_0(\mathcal{M})$ on $p^{-1}(w_0)^*$ which is the set $\{(w_0, \pi(\bar{u})) : (\bar{w}_0, \pi(u)) \in p^{-1}(w_0)\},$. Thus $s_1 = s_1(w, \theta)|_{\{w_0\} \times \{u_i \neq 0\}}$ is given by the formula

$$s_1(\theta) = K^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \bar{\theta}_j^1 K^{(j)}(\cdot, w_0).$$

Since the vectors $K^{(j)}(\cdot, w_0), 1 \leq j \leq t$ are uniquely determined by the generators $g_1, \ldots, g_t,$ s_1 is well defined.

many more examples

Theorem. *Let* $\mathcal{M} \subseteq \mathcal{O}(\Omega)$ *and* $\hat{\mathcal{M}} \subseteq \mathcal{O}(\Omega)$ *be two Hilbert modules of the form* $[\mathcal{I}]$ *and* $[\hat{\mathcal{I}}]$*, respectively and* $\mathcal{I}, \hat{\mathcal{I}} \subseteq \mathbb{C}[\underline{z}]$*. Assume that* $\mathcal{M}, \hat{\mathcal{M}}$ *are in* $\mathfrak{B}_1(\Omega)$ *and that the dimension of the zero set of these modules is at most* $m-2$*. If the modules* $\mathcal{M}$ *and* $\hat{\mathcal{M}}$ *are equivalent, then the corresponding bundles* $\mathcal{L}_0(\mathcal{M})$ *and* $\mathcal{L}_0(\hat{\mathcal{M}})$ *they determine on the projective space* $p^{-1}(w_0)^*$ *for* $w_0 \in Z$*, are equivalent as Hermitian holomorphic line bundle.*

Example. Let $\mathbb{B}^2$ be the unit ball in $\mathbb{C}^2$. For $-1 < \alpha, \beta, \theta < +\infty$, let $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ be the Hilbert space of functions on $\mathbb{B}^2$ satisfying

$$\| f \|^2_{\alpha,\beta,\theta} = \int_{\mathbb{B}^2} |f(z)|^2 d\mu(z_1, z_2) < +\infty,$$

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Many more examples

The weighted Bergman space $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is the subspace of $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ consisting of the holomorphic functions on $\mathbb{B}^2$. The Hilbert space $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is non-trivial if we assume that the parameters α, β, θ satisfy the additional condition: $\alpha + \beta + \theta + 2 > 0$.

Proposition. *Suppose $\mathcal{I}$ is an ideal in $\mathbb{C}[z_1, z_2]$ with $V(\mathcal{I}) = \{0\}$. Then the Hilbert modules $[\mathcal{I}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ and $[\mathcal{I}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ are unitarily equivalent if and only if $\alpha = \alpha', \beta = \beta'$ and $\theta = \theta'$.*

Another set of Invariants

Local construction of vector bundles

Let $\mathbb{P}_0$ be the orthogonal projection onto the joint kernel $\mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M}$

Lemma. *The dimension of* $\ker \mathbb{P}_0\big(\mathcal{M}/\mathfrak{m}_w\mathcal{M}\big)$ *is constant in a suitably small neighbourhood* Ω_0 *of* $w_0 \in \Omega$.

Thus

$$\mathcal{P}^{\mathcal{M}}_{w_0} := \{(w,f) \in \Omega \times \mathcal{M} : f \in \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}\} \text{ and } \pi(w,f) = w$$

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Existence of holomorphic structure

Existence of the operator $R_{\mathbf{M}}(w)$ satisfying

$$\begin{aligned} R_{\mathbf{M}}(w) D_{(\mathbf{M}-w)^*} &= I - P_{\ker D_{(\mathbf{M}-w)^*}} \\ D_{(\mathbf{M}-w)^*} R_{\mathbf{M}}(w) &= P_{\operatorname{ran} D_{(\mathbf{M}-w)^*}} \end{aligned}$$

on Ω_0 is established.

(Here, $D_{(\mathbf{M}-w)^*} : \mathcal{M} \to \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ is the operator $f \mapsto \big((M_1 - w_1)^* f, \ldots, (M_m - w_m)^* f\big)$)

Then the operator

$$P(\bar{w}, \bar{w}_0) = I - \{I - R_{\mathbf{M}}(w_0) D_{\bar{w}-\bar{w}_0}\}^{-1} R_{\mathbf{M}}(w_0) D_{(\mathbf{M}-w)^*},$$

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Main Theorem

Theorem. *If any two Hilbert modules* $\mathcal{M}$ *and* $\tilde{\mathcal{M}}$ *from* $\mathfrak{B}_1(\Omega)$ *are equivalent, then the corresponding holomorphic Hermitian vector bundles* $\mathcal{P}_{w_0}^{\mathcal{M}}$ *and* $\mathcal{P}_{w_0}^{\tilde{\mathcal{M}}}$*, they determine on* Ω_0 *are equivalent.*

Examples, calculation of the invariant

For $\lambda, \mu > 0$, let $K^{(\lambda,\mu)}$ denote the positive definite kernel $\frac{1}{(1-z_1\bar{w}_1)^\lambda(1-z_2\bar{w}_2)^\mu}$, $z, w \in \mathbb{D}^2$ on the bi-disc. Let $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := \{f \in H^{(\lambda,\mu)}(\mathbb{D}^2) : f(0,0) = 0\}$ be the corresponding Hilbert module in $\mathfrak{B}_1(\mathbb{D}^2)$. The normalized metric $h_0(w,w)$, which is real analytic, is of the form

$$h_0(w,w) = I + \begin{pmatrix} \frac{\lambda+1}{2}|w_1|^2 + \frac{\lambda^2\mu}{(\lambda+\mu)^2}|w_2|^2 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_1\bar{w}_2 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_2\bar{w}_1 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}|w_1|^2 + \frac{\mu+1}{2}|w_2|^2 \end{pmatrix}$$

$$+O(|w|^3),$$

where $O(|w|^3)_{i,j}$ is of degree $\geq 3.$

The final outcome of these calculations

The curvature for $\mathcal{P}$ at $(0,0)$ is given by the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0 \end{pmatrix},$$

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$H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent if and only if $\lambda = \lambda'$ and $\mu = \mu'$.

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Thank you!