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# HOMOGENEOUS TUPLES OF OPERATORS AND REPRESENTATIONS OF SOME CLASSICAL GROUPS

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ABSTRACT

Let $\mathbf{T} = (T_1 \ldots, T_n)$ be a $n$-tuple of bounded linear operators on a fixed Hilbert space $\mathscr{H}$ and let $\varphi$ be a biholomorphic automorphism of $\Omega$, the joint spectrum of $\mathbf{T}$. In this paper, we consider those $n$-tuples $\mathbf{T}$ for which the joint spectrum $\Omega$ is of the form $G/K$, a bounded symmetric domain. Let $\varphi$ be any biholomorphic automorphism of the domain $\Omega$. Define, $\varphi(\mathbf{T})$ via a suitable functional calculus and call a $n$-tuple of operators $\mathbf{T}$ homogeneous if $\varphi(\mathbf{T})$ is simultaneously unitarily equivalent to $\mathbf{T}$ for every automorphism $\varphi$ of $\Omega$. For each homogeneous operator $\mathbf{T}$, let $U_\varphi$ be a unitary operator implimenting this equivalence. We obtain a characterisation of all the homogeneous operators Cowen-Douglas class and show that it is possible to choose the unitary $U_\varphi$ in such a way that the map $\varphi \to U_{\varphi^{-1}}$ is a unitary representation of the group of biholomorpic automorphisms of $\Omega$.

## 1. INTRODUCTION

Let $\Omega$ be a domain in $\mathbf{C}^n$ of the form $G/K$ where $G$ is a semisimple complex Lie group and $K$ is a maximal compact subgroup of $G$ so that $G$ operates holomorphically on $\Omega$. These domains were classified by Cartan into four domains of classical type and two exceptional ones. In this paper by a bounded symmetric domain, we will always mean one of the first four domains of classical type. For details we refer the reader to [6].

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a pairwise commuting $n$-tuple of operators acting on a fixed Hilbert space $\mathscr{H}$. Assume that $\mathbf{T}$ admits the closure $\mathrm{cl}\,\Omega$ as a spectral set, that is, the map $\rho_{\mathbf{T}}\colon \mathscr{P}(\Omega) \to \mathscr{L}(\mathscr{H})$ defined by

$$\rho_{\mathbf{T}}(p) = p(T_1, \ldots, T_n)$$

is contractive, where $\mathscr{P}(\Omega)$ is the Banach algebra of all polynomials in $n$ variables with supremum norm over $\Omega$. Thus we can define $\varphi(\mathbf{T})$ for $\varphi$ in the closure $\mathscr{A}(\Omega)$ of $\mathscr{P}(\Omega)$ with respect to the supremum norm. For a biholomorphic automorphism in $G$ with coordinate functions $(\varphi^1, \ldots, \varphi^n)$, set $\varphi(\mathbf{T}) = (\varphi^1(\mathbf{T}), \ldots, \varphi^n(\mathbf{T}))$.

1.1. DEFINITION. Any $n$-tuple of pairwise commuting operators $\mathbf{T} = (T_1, \ldots, T_n)$ admitting $\mathrm{cl}\,\Omega$ as a spectral set will be called *homogeneous* if $\mathbf{T}$ is unitarily equivalent to $\varphi(\mathbf{T})$ for all $\varphi$ in $G$, that is there exists a unitary operator $U$ on $\mathscr{H}$ such that

$$U\varphi(\mathbf{T})U^* = (U\varphi^1(\mathbf{T})U^*, \ldots, U\varphi^n(\mathbf{T})U^*) = \mathbf{T}.$$

1.2. QUESTION. Given a domain $\Omega = G/K$ in $\mathbf{C}^n$, characterise the homogeneous $n$-tuples of operators.

This question is of course interesting on its own right. In addition, the following proposition which is modelled after Arveson et al. [1, Proposition 1.3] shows that each homogeneous $n$-tuple gives rise to a projective representation of the group $G$. The case of SU(1,1) was considered in [11]. We begin with some definitions.

1.3. DEFINITION. *A Polish space* is a topological space, which is homeomorphic to a separable complete metric space.

Let $G$ be a second countable locally compact group. Recall that a projective representation of $G$ is a mapping $\varphi \to U_\varphi$ of $G$ into the group $U(\mathscr{H})$ of unitary operators on a fixed Hilbert space $\mathscr{H}$ such that

(i) $U_e = I_{\mathscr{H}}$ where $e$ is the identity in $G$.

(ii) $U_\varphi U_\psi = c(\varphi, \psi)U_{\varphi\psi}$, where $|c(\varphi, \psi)| = 1$,

and

(iii) $\varphi \to \langle U_\varphi \zeta, \eta\rangle$ is a Borel function for each $\zeta$, $\eta$ in $\mathscr{H}$.

The function $c\colon G\times G \to \mathbf{T}$ is called a multiplier of $U$. It is uniquely determined by $U$ and is a Borel function on $G\times G$.

Also recall that a set $S$ of operators on a Hilbert space is irreducible if there is no common reducing subspace $\mathscr{M}$ for all of the $T_1, \ldots, T_n$.

For any two Polish spaces $X$ and $Y$, let

(i) $E$ be any subset of $X\times Y$,

(ii) $\pi\colon X\times Y \to X$, $\pi(x, y) = x$ be the projection onto the first coordinate,

(iii) $E_x = \{y \in Y : (x, y) \in E\}$ be the section at $x$,

and

(iv) $D_E = \{x \in X : E_x \neq 0\}$ be the domain of $E$.

1.4. DEFINITION. *A selection* for $E$ is a function $\varphi\colon D_E \to E$ which is contained in $E_x$, that is for all $x$ in $D$, $\varphi(x) \in E_x$.

The following is a powerful selection theorem due to Kenugi-Novikov [10, p. 471].

1.5. THEOREM. *If $X$ and $Y$ are Polish spaces and $E \subset X\times Y$ is a Borel set with $E_x$ compact for each $x$ in $X$ then $E$ admits a Borel selection.*

We now have all the tools to prove the following

1.6. THEOREM. *Any irreducible n-tuple of operators* **T** *admitting* $\mathrm{cl}\,\Omega$ *as a spectral set is homogeneous if and only if there is a projective representation* $\varphi \to U_\varphi$ *of G satisfying*

$$U_\varphi \mathbf{T} U_\varphi^* = \varphi(\mathbf{T}).$$

*Proof.* The if part is trivial. To prove the converse, note that the set

$$E = \{(\varphi,\ U) \in G \times U(\mathscr{H}) : U\mathbf{T}U^* = \varphi(\mathbf{T})\}$$

is a Borel subset of $G \times U(\mathscr{H})$. Each section $E_x$ is compact since **T** is irreducible. Thus the Kenugi-Novikov theorem guarantees the existence of a Borel map

$$\varphi \to U_\varphi.$$

Observe that

$$U_\varphi U_\psi \mathbf{T} U_\psi^* U_\varphi^* = U_\varphi(\mathbf{T}) U_\varphi^* =$$

$$= U_\varphi \lim_{n\to\infty} p_n(\mathbf{T}) U_\varphi^* = \psi(\varphi(\mathbf{T})) = U_{\psi\varphi} \mathbf{T} U_{\psi\varphi}^*$$

where we have chosen $p_n$ such that $p_n \to \psi$. Thus the unitary operator $U_{\psi\varphi}^* U_\varphi U_\psi$ commutes with the operator **T**, which is irreducible. Therefore

$$U_{\psi\varphi}^* U_\varphi U_\psi = c(\varphi,\ \psi) I, \quad |c(\varphi,\ \psi)| = 1$$

and it follows that $\varphi \to U_\varphi$ is a projective representation.

This proof of the theorem was suggested by E. Azoff to the first author.

We have not been able to obtain a complete characterization of homogeneous $n$-tuples of operators. However in this paper, we obtain a characterization of the homogeneous $n$-tuples **T** which are in the Cowen-Douglas class $\mathbf{P}_1(\Omega)$. This class of operators was introduced in [3, p. 334], see also [4].

## 2. HOMOGENEOUS $n$-TUPLES IN COWEN-DOUGLAS CLASS $\mathbf{P}_1(\Omega)$

Following Cowen-Douglas [3], we define $\mathbf{P}_1(\Omega)$ to be the class of those pairwise commuting operators **T** acting on $\mathscr{H}$ such that

(i) $\dim \bigcap_{j=1}^{n} \ker(T_j - \omega_j) = 1$ for all $(\omega_1, \ldots, \omega_n)$ in $\Omega$;

(ii) The operator $T_\omega : \mathscr{H} \to \mathscr{H} \oplus, \ldots, \oplus \mathscr{H}$ defined by

$$T_\omega x = \bigoplus_{j=1}^{n} (T_j - \omega_j x)$$

has closed range; and

(iii) $\bigvee_{\omega\in\Omega}\left\{\bigcap_{j=1}^{n}\ker(T_j-\omega_j)\right\}=\mathscr{H}$.

For $\mathbf{T}$ in $\mathbf{P}_1(\Omega)$, let $H(T_1,\ldots,T_n)$ denote $\bigcap_{j,k=1}^{n}\ker(T_jT_k)$ and define

$$N_j(\omega)=(T_j-\omega_j)|H(T_1-\omega_1,\ldots,T_n-\omega_n).$$

2.1. THEOREM (Cowen and Douglas). *The n-tuples* $(T_1,\ldots,T_n)$ *and* $(\tilde{T}_1,\ldots,\tilde{T}_n)$ *in* $\mathbf{P}_1(\Omega)$ *are unitarily equivalent if and only if* $\operatorname{tr}(\tilde{N}_j(\omega)\tilde{N}_k(\omega)^*)$ *is identically equal to* $\operatorname{tr}(N_j(\omega)N_k(\omega)^*)$.

It was shown in [3] that each $n$-tuple in $\mathbf{P}_1(\Omega)$ determines a nonzero holomorphic map $\gamma:\Omega\to\mathscr{H}$ such that $\gamma(\omega)\in\bigcap_{j=1}^{n}\ker(T_j-\omega_j)$ for all $\omega$ in $\Omega$ and the curvature of $\mathbf{T}$ is

$$\mathscr{K}_{\mathbf{T}}(\omega)=\left[\frac{\partial^2}{\partial\omega_i\partial\bar{\omega}_j}\log\|\gamma(\omega)\|^2\right].$$

As in [3, p. 336–337] it can be verified that

$$\mathscr{K}_{\mathbf{T}}=(N_j(\omega)N_k(\omega)^*)^{-1}.$$

Thus the curvature is a complete unitary invariant of $\mathbf{T}$.

2.2. We now recall some well known results about the Bergman kernel on $\Omega$. Most of what follows can be found in Helgason [5]. However, the following is from Inoue [7].

Since $G$ is simply connected we can uniquely define, for each $t\in\mathbf{R}$, the power $j(\varphi,z)^t$ with $j(e,z)^t=1$ ($e$ is the identity element in $G$) for all $z$ in $\Omega$. As usual $j(\varphi,z)$, denotes the Jacobian of $\varphi$ at $z$. For $z,\omega$ in $G$, let $K(z,\omega)$ be the Bergman kernel for $\Omega$. We can define $K(z,\omega)^t$, so that $K(z,z)^t>0$ for all $z$ in $\Omega$.

Note that

$$j(\varphi\psi,z)^t=j(\varphi,\psi z)^t j(\psi,z)^t \quad \text{for } \varphi,\psi \text{ in } G,\ z \text{ in } \Omega;$$

$$K(\varphi z,\psi\omega)^t=j(\varphi,z)^{-t}K(z,\omega)^t\overline{j(\psi,\omega)}^{-t}$$

for $\varphi$ in $G$ and $z,\omega$ in $\Omega$; and for $\varphi$ with $\varphi(0)=z$,

$$\left[\frac{\partial^2}{\partial\omega_i\partial\bar{\omega}_j}\log K(z,z)^2\right]=\mathrm{D}\varphi(0)\mathrm{D}\varphi(0)^*.$$

Let $\mu$ be the Lebesgue measure on $\Omega$. Then we have

$$\int f(\varphi z)\mathrm{d}\mu(z) = \int f(z)|j(\varphi^{-1}, z)|^2 \mathrm{d}\mu(z)$$

for all integrable $f$ on $\Omega$ and $\varphi$ in $G$. For $t$ in $\mathbf{R}$ define a measure $\mu_t$ on $\Omega$ by

$$\mathrm{d}\mu_t(z) = K(z, z)^{-t+1}.$$

It follows from the above that $\mu_t$ is invariant under the action of $G$.

Let $L^2(\Omega, \mu_t)$ be the $L^2$ space of square integrable functions on $\Omega$ with respect to the measure $\mu_t$. Denote the space of holomorphic functions on $\Omega$ by $\mathbf{H}(\Omega)$ and the space $L^2(\Omega, \mu_t) \cap \mathbf{H}(\Omega)$ by $H^2(\Omega, \mu_t)$. The following proposition was proved in [7, Lemma 2.13].

2.3. PROPOSITION. *For any $t \geqslant 1$, $H^2(\Omega, \mu_t)$ is nonzero and is a closed subspace of $L^2(\Omega, \mu_t)$. Furthermore, it possesses a kernel function, which is a constant multiple of $K(z, w)^t$.*

The following theorem shows that for a homogeneous $n$-tuple the curvature function is determined once its value at zero is known. The proof however is elementary and can be viewed as a change of variable formula for the curvature.

2.4. THEOREM. *If $(T_1, \dots, T_n)$ is a homogeneous n-tuple of operators in $\mathbf{P}_1(\Omega)$ admitting $\mathrm{cl}\,\Omega$ as a spectral set, then*

$$\mathscr{K}_{\mathbf{T}}(\omega) = \mathrm{D}\varphi(\omega)\mathscr{K}_{\mathbf{T}}(0)\mathrm{D}\varphi(\omega)^*,$$

*where $\varphi$ is an automorphism of $\Omega$ which carries $\omega$ to zero and $\mathscr{K}_{\mathbf{T}}(0)$ must be of the form $cI$.*

*Proof.* Let $\omega \to \gamma_\omega$ be a holomorphic map from $\Omega$ to $\mathscr{H}$ such that $\gamma_\omega$ is in $\bigcap_{j=1}^{n} \ker(T_j - \omega_j)$ for each $\omega = (\omega_1, \dots, \omega_n)$ in $\Omega$. It is easy to verify that $\omega \to \gamma_{\varphi(\omega)}$ is a holomorphic map such that $\gamma_{\varphi(\omega)}$ is in $\bigcap_{j=1}^{n} \ker(\varphi^j(\mathbf{T}) - \varphi^j(\omega))$ for each $\varphi$ in $G$. Thus $\omega \to \gamma_{\varphi^{-1}(\omega)}$ is holomorphic and $\gamma_{\varphi^{-1}(\omega)}$ is in $\bigcap_{j=1}^{n} \ker(\varphi^j(\mathbf{T}) - \omega_j)$. Applying the chain rule we obtain

$$\mathscr{K}_{\varphi(\mathbf{T})}(\omega) = \mathrm{D}_j \mathrm{D}_k \log\|\gamma^{-1}(\omega)\|^2 =$$

$$= ((\mathrm{D}\varphi^{-1}))(\omega)\mathscr{K}_{\mathbf{T}}(\varphi^{-1}(\omega))((\mathrm{D}\varphi^{-1})(\omega))^*.$$

Evaluate both sides at zero and observe that the Cowen-Douglas theorem implies the equality of $\mathscr{K}_{\varphi(\mathbf{T})}(0)$ and $\mathscr{K}_{\mathbf{T}}(0)$ for each $\varphi$ in $G$, whenever $\mathbf{T}$ is homogeneous.

Thus,

$$\mathscr{K}_{\mathbf{T}}(\varphi^{-1}(0)) = ((\mathrm{D}\varphi)(\varphi^{-1}(0)))\mathscr{K}_{\mathbf{T}}(0)((\mathrm{D}\varphi)(\varphi^{-1}(0)))^*$$

and so $\mathscr{K}_{\mathbf{T}}(0)$ commutes with $\mathrm{D}\psi(0)$ for each $\psi$ in $G$ such that $\psi(0)=0$. In each of the four classical domains of interest here straightforward calculations imply that $\mathscr{K}_{\mathbf{T}}(0)$ must be a constant multiple of the identity. Since $G$ acts transitively on $\Omega$ the proof is complete.

Let $\mathbf{M}_z = (M_{z_1}, \ldots, M_{z_n})$ denote the multipication operators given by $(M_{z_j}f)(z) = z_j f(z)$. It was pointed out in [4] that $\mathbf{T}$ in $\mathbf{P}_1(\Omega)$ is unitarily equivalent to $\mathbf{M}_z = (M_{z_1}, \ldots, M_{z_n})$ on a Hilbert space with a kernel function $K_\gamma$. We recall from [4] that, if $\mathbf{T}$ is in $\mathbf{P}_1(\Omega)$ then there exists a holomnrphic map $\gamma: \Omega_0 \subset \Omega \to \mathscr{H}$ such that $\gamma(\omega)$ is in $\bigcap_{j=1}^{n} \ker(T_j - \omega_j)$. Define $U: \mathscr{H} \to \mathrm{Hol}(\Omega)$ by

$$(Ux)(\omega) = \langle x, \gamma(\omega)\rangle, \quad x \in \mathscr{H}, \ \omega \in \Omega.$$

Let $\mathscr{H}_\gamma = \mathrm{range}\, U$ and define the bilinear form $\langle\ ,\ \rangle_\gamma$ on $\mathscr{H}_\gamma$ by

$$\langle Ux, Uy\rangle_\gamma = \langle x, y\rangle; \quad x, y \text{ in } \mathscr{H}.$$

The map $U$ is linear and injective, $\mathscr{H}_\gamma$ is a Hilbert space with inner product $\langle\ ,\ \rangle_\gamma$ and $U$ is a Hilbert space isomorphism. Furthermore, the space $\mathscr{H}_\gamma$ is invariant under multiplication by the coordinate functions $z_j$ and the $n$-tuple $\mathbf{M}_z = (M_{z_1}, \ldots, M_{z_n})$ of these multiplication operators belongs to $\mathscr{L}(\mathscr{H}_\gamma)$. Indeed $U$ intertwines $\mathbf{T}$ and $\mathbf{M}_z^*$. Evaluation at each point is a bounded linear functional from $\mathscr{H}_\gamma$ to $\mathbf{C}$. Moreover, there exists a reproducing kernel for the space $\mathscr{H}_\gamma$ given by $K_\gamma(\lambda, \mu) = \langle\gamma(\lambda), \gamma(\mu)\rangle$ for $\lambda$, $\mu$ in $\Omega_0$.

2.4. THEOREM. *For $t \geqslant 1$, let $\mathbf{T}_t$ denote the $n$-tuple $\mathbf{M}_z^* = (M_{z_1}^*, \ldots, M_{z_n}^*)$ on $H^2(\Omega, \mu_t)$. Then $\mathbf{T}_t$ is in $\mathbf{P}_1(\Omega)$ and $\mathbf{T}_t$ is homogeneous. Moreover, there exists $U$ satisfying $U_\varphi^*(\mathbf{T})U_\varphi = \varphi(\mathbf{T})$ which is of the form*

$$(U_\varphi f)(z) = j(\varphi^{-1}, \varphi^{-1}(z))^{-t} f(\varphi^{-1}(z)).$$

*Proof.* The fact that $\mathbf{T}_t$ is in $\mathbf{P}_1(\Omega)$ follows from [12, Proposition 4.1]. There it is shown that if $\Omega$ is a pseudoconvex domain and $H^2(\Omega, \nu)$ is the closed subspace of $L^2(\Omega, \mathrm{e}^{-\varphi}\mathrm{d}\nu)$ consisting of analytic functions on $\Omega$ then the Koszul complex determined by $\mathbf{T}_t$ is exact except at the end point, where

$$\dim H_n(\Omega, \nu) = 1.$$

Since the domain $\Omega$ we are looking at is a bounded symmetric domain, it is convex. In particular it is pseudoconvex. Also the measure $\mu_t$ can be written in the form $\exp(-\log K(z, z)^{t-1})$. Thus, $\varphi(z) = (t - 1)\log K(z, z)$ is continuous and plurisubharmonic since

$$[\partial_{z_i}\partial_{\bar{z}_j}] = \log K(z, z) \geqslant 0,$$

cf. [5, p. 368]. This shows that $\mathbf{T}_t$ is in $\mathbf{P}_1(\Omega)$, cf. [4, Remark 2.4C].

Considering $\varphi(\mathbf{T}_t)$, we find that

$$\gamma_{\varphi(\mathbf{T}_t)}(\omega) - \gamma_{\mathbf{T}_t}(\varphi^{-1}(\omega)).$$

Thus, the map $U: H^2(\Omega, \mu_t) \to \mathrm{Hol}(\Omega)$ defined by

$$Ux(\omega) = \langle x, \gamma_{\varphi(\mathbf{T}_t)}\rangle$$

intertwines $\varphi(\mathbf{T}_t)$ and $\mathbf{M}_z^*$ on $\mathscr{H}_\gamma$. The kernel for $\mathscr{H}_\gamma$ is

$$K_\gamma(\lambda, \mu) = \langle\gamma(\lambda), \gamma(\mu)\rangle = (\varphi^{-1}(\lambda), \varphi^{-1}(\mu)) = K(\varphi^{-1}(\lambda), \varphi^{-1}(\mu)).$$

However,

$$K(\varphi^{-1}(\mu), \varphi^{-1}(\lambda)) = j(\varphi^{-1}, \mu)K(\lambda, \mu)f(\varphi^{-1}, \lambda).$$

Lemma 4.8 of [4] implies that $\mathbf{M}_z^*$ on $\mathscr{H}_\gamma$ is unitarily equivalent to $\mathbf{T}_t$. Furthermore by Lemma 3.9 of the same article [4] the map $U_\gamma : \mathscr{H} \to \mathscr{H}_\gamma$ defined by

$$U_\gamma f(\omega) = j(\varphi^{-1}, \omega)f(\omega)$$

intertwines $\mathbf{M}_z^*$ on $H^2(\mu_t)$ and $\mathbf{M}_z^*$ on $\mathscr{H}_\gamma$. Thus $U_\varphi = U_\gamma U$ is an unitary map intertwining $\mathbf{M}_z^*$ on $H^2(\mu_t)$ and $\varphi(\mathbf{M}_z^*)$. Observe that

$$U_\varphi f(\omega) = U_\gamma Uf(\omega) = \langle Uf, \gamma(\omega)\rangle = Uf(\gamma(\omega)) = j(\varphi^{-1}, \varphi^{-1}(\omega))^{-t}f(\varphi^{-1}(\omega)).$$

2.5. REMARK. When $t$ is an integer greater than $n$, the map $\varphi \to U_{\varphi^{-1}}$ is an irreducible representation of $G$ which is in the discrete series.

## 3. THE CASE OF THE UNIT BALL

In the following $I = (i_1, \ldots, i_n)$ will always denote a multi-index of positive integers. Let $\varepsilon_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the multi-index having $i_j = 1$ or $0$ according as $j = k$ or otherwise. The multi-index $I + k$ denotes $\langle i_1, \ldots i_k + k, \ldots, i_n\rangle$. Let $(e_I)$ be an orthogonal basis for a complex Hilbert space $\mathscr{H}$ and let $\omega_{I,j}$, $j = 1 \ldots, n$, be a bounded sequence of complex numbers such that

$$\omega_{I,k}\omega_{I+\varepsilon_k,l} = \omega_{I,l}\omega_{I+\varepsilon_l,k}.$$

3.1. DEFINITION. A system of $n$-variable weighted shifts is a family of $n$ operators $(T_1, \ldots, T_n)$ on $\mathscr{H}$ such that

$$Te_I = \omega_{I,j} e_{I+\varepsilon_j}.$$

As in the single operator case, a commuting system of $n$-variable weighted shifts is an $n$-tuple of multiplication operators on a suitable Hilbert space consisting of formal power series in $n$ variables defined as follows.

3.2. DEFINITION. Let $\{\beta_I : I \geqslant 0\}$ be a set of strictly positve numbers with

$$H^2(\beta) = \{f(z) = \sum f_I z^I : \|f\|^2 = \sum |f_I|^2 \beta_I^2 < \infty\}.$$

Clearly, $H^2(\beta)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum f_I \bar{g}_I \beta_I^2.$$

The set $\mathbf{M}_z^*$ is a commuting system of $n$-variable weighted shifts with $\omega_{I,j} = \beta_{I+\varepsilon_j}(\beta_I)^{-1}$ and it is possible to go the other way round cf. [8].

Let $\beta_t(I)^{-2}$ denote the coefficient of $\omega^I \bar{\omega}^I$ in the multivariable binomial expansion of

$$(1 - |\omega|^2)^{-r}, \quad r \text{ real and } \omega \text{ in } \mathbf{B}^n \subset \mathbf{C}^n$$

It is then evident that the kernel function for $H^2(\mu_t)$ is $(1 - \langle z, \omega \rangle)^{-r}$. Following general considerations of Jewell and Lubin [8], we see that if $\omega = (\omega_1, \ldots, \omega_n)$ is in the ball, then $\omega_j$ is an eigenvalue for $\mathbf{M}_z^*$ with joint eigenvector

$$K(z, \omega) = (1 - \langle z, \omega \rangle)^{-r} = (1 - \langle z, \omega \rangle)^{-(n+1)t}, \quad t = r/(n+1).$$

Of course $(1 - \langle z, \omega \rangle)^{-(n+1)}$ is the Bergman kernel for the ball in $\mathbf{C}^n$.

3.3. THEOREM. $\mathbf{T}_t$ *is in* $\mathbf{P}_1(\mathbf{B}^n)$ *for* $t \geqslant 1/(n+1)$.

*Proof.* In view of [4], we have to only verify that for the weighted shift $\varphi(\mathbf{T}_t) - \omega I$ satisfies

$$D \leqslant \sum_{j=1}^{n} |\omega_{I-\varepsilon_j, j}|^2 \leqslant C$$

for all $\omega$ in the unit ball $\mathbf{C}^n$. We consider the case of $\omega = 0$ and immediately see that

$$\beta(I) = \frac{|I|!\, i_1! \ldots i_n!}{t(t-1) \ldots (t+|I|-1)(i_1 + \ldots + i_n)} =$$

$$= \frac{i_1! \ldots i_n!}{t(t-1) \ldots (t+|I|-1)} =$$

$$\omega_{J,j} = \left(\frac{i_1! \dots (i_j + 1)! \dots i_n!}{t(t-1)\dots(t+|I|)}\right) \Big/ \left(\frac{i_1! \dots i_n!}{t \dots (t+|I|-1)}\right) = \frac{i_j + 1}{t + |I|},$$

and

$$\sum_{j=1}^{n} |\omega_{I,j}|^2 = \sum_{j=1}^{n} \frac{i_j}{t + |I| - 1} = \frac{|I|}{t + |I| - 1},$$

which is both bounded below and above. Writing down the homogeneous expansion for $K(z, \omega)$ around the point $(z_0, \omega_0)$, we can verify that $(\mathbf{M}_z^* - \omega_0 I)$ also satisfies similar inequalities. Thus $\mathbf{M}_z^*$ is in $\mathbf{P}_1(\mathbf{B}^n)$.

3.4. REMARKS. a) In the case of the ball in $\mathbf{C}^n$ for $t$ in the set $\{1/(n+1), \dots$ $\dots, n/(n+1)\}$ we do get irreducible unitary representations of $\mathrm{SU}(n, 1)$ in a very simple form. The fact that these representations are irreducible follows from a rather general result of Kunze [9]. However, these representations are no longer in the discrete series [13]. Of course the case of $t = n/(n+1)$ corresponds to familiar Hardy space on the ball. Note that if $t = k/(n+1)$, $1 \leqslant k \leqslant n-1$, it is not clear that $\mathbf{M}_z^*$ admits $\mathbf{B}^n$ as a spectral set, however $\varphi(\mathbf{M}_z^*)$ can still be defined to be $(M^*_{\varphi^1(z)}, \dots, M^*_{\varphi^n(z)})$. To see that $\varphi(\mathbf{M}_z^*)$ defines an $n$-tuple of bounded linear operators, we merely note that $\varphi(\mathbf{M}_z^*)$ is unitarily equivalent to $\mathbf{M}_z^*$ on the Hilbert space $\mathscr{H}$ with kernel function $K(\varphi^{-1}(\lambda), \varphi^{-1}(\mu))$, where the kernel function is some power of the Bergman kernel function; transformation properties of the kernel function (see, Section 2.2) imply that $\varphi(\mathbf{M}_z^*)$ is a bounded $n$-tuple of operators.

b) To treat the case of an arbitrary real $t$, we have to use the notion of a Wallach set, which will be taken up in a subsequent paper.

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