CHARACTERIZING QUOTIENT HILBERT MODULES

RONALD G. DOUGLAS AND GADADHAR MISRA

One approach to the study of multi-variate operator theory is through the study of Hilbert modules, that is, Hilbert spaces on which an algebra acts. Early work on this approach is presented in [8] and focuses on the commutative case in which the algebra consists of a natural space of holomorphic functions. Although there are some general results, most of the results to date concern concrete examples, in part because of our rather meager knowledge of what is possible and what might be true. Perhaps the most natural example of a Hilbert module over an algebra A of holomorphic functions on a domain Ω in $\mathbb{C}^n$ is a kernel Hilbert space $\mathcal{K}$ over Ω which is closed under multiplication by A. The critical property of a kernel Hilbert space of holomorphic functions is that evaluation at points in Ω is continuous in the Hilbert space norm.

The Hardy spaces and the Bergman spaces over any bounded domain Ω in $\mathbb{C}^n$ form natural families of examples over $A(\Omega)$, the supremum norm closure of the functions holomorphic on some neighborhood of the closure of Ω. Other natural examples are obtained by considering submodules of such kernel Hilbert modules. Specifically, we recall Beurling's characterisation of a submodule $\mathcal{M}$ of the Hardy module $H^2(\mathbb{D})$ which states that the submodule $\mathcal{M}$ must be of the form $\theta H^2(\mathbb{D})$, where θ is an inner function. It is then not hard to see, using the von Neumann - Wold decomposition for isometries, that all of these submodules of the Hardy module are isomorphic. If we enlarge the scope to include Hardy spaces of vector valued functions, there is a generalisation of Beurling's theorem due to Halmos and Lax which implies that a submodule is determined completely in this case by its multiplicity. Never the less, the quotient modules $H^2(\mathbb{D})/\mathcal{M}$ for $\mathcal{M} \subseteq H^2(\mathbb{D})$ are not isomorphic. Indeed, one way of describing the model theory of Sz.-Nagy and Foias is to say that a large class of contractive modules arise as such quotient modules. It is then natural to ask when two of these quotient modules are isomorphic and find complete invariants. This is done in the work of Sz-Nagy and Foias in the context of contractive modules over the disk algebra. In contrast to what happens in the case of the Hardy module for the disk, a great profusion of different examples is obtained in this manner in the several variables case as the rigidity phenomenon demonstrates [9]. If we attempt to obtain an appropriate generalisation of the Sz.-Nagy and Foias model to the multi-variate context then the following issues become aparent.

a) Describe all the submodules of a given module.
b) Decide if any of these are isomorphic.

    c) Describe the quotient modules.

    d) Find invariants for the quotient using the pair $\mathcal{M}_0 \subseteq \mathcal{M}$.

The first two issues have been addressed in the paper [9]. In this exposition, we assume that we are working with a fixed Hilbert module and a naturally occuring family of submodules. We describe some results concerning the quotient Hilbert modules in which the submodules are closely tied to the geometry of the domain Ω. In [5], some results in this direction were described using brute force calculations. Let us briefly elaborate.

Consider the bidisk Hardy module $H^2(\mathbb{D}^2)$ over the bidisk algebra $A(\mathbb{D}^2)$ and let w and z designate the two variables. Then $[w]$ and $[z]$ denote the closed submodules in $H^2(\mathbb{D}^2)$ generated by the functions w and z, respectively. They can also be viewed as the closures of the principal ideals in $\mathbb{C}[w,z]$ generated by w and z, respectively.

The quotients $H^2(\mathbb{D}^2)/[w]$ and $H^2(\mathbb{D}^2)/[z]$ can be readily described. The first module can be identified as the one-variable Hardy space of functions on the subvariety in $\mathbb{D}^2$ determined by $w = 0$ with the action of w being zero and the action of z being the usual one on the Hardy module. The quotient $H^2(\mathbb{D}^2)/[z]$ is the same with the roles of w and z interchanged. But what about the quotient module $H^2(\mathbb{D}^2)/[w-z]$ in which $[w-z]$ is the closure of the principal ideal generated by the function $w - z$? This requires a little more work and was the principal result in [5].

Let $\mathcal{P}_k$ for $k = 0, 1, 2 \ldots$ be the homogeneous polynomials in $\mathbb{C}[w,z]$ of degree k. A simple calculation shows that $\dim \mathcal{P}_k = k + 1$, $\dim \mathcal{P}_k \cap [w-k] = k$, and $\mathcal{P}_k/\mathcal{P}_k \cap [w-k] = \{e_k\} = \{w^k + w^{k-1}z + \ldots + z^k\}$. Moreover,

$$H^2(\mathbb{D}^2)/[w-z] = \mathcal{P}_0 \oplus \mathcal{P}_1/\mathcal{P}_1 \cap [w-z] \oplus \mathcal{P}_2/\mathcal{P}_2 \cap [w-z] \oplus \ldots .$$

Further, $\|e_k\| = \sqrt{k+1}$ and hence $\frac{\{e_k\}}{\sqrt{k+1}}$ is an orthonormal basis for $H^2(\mathbb{D}^2)/[w-z]$. These calculations enable us to determine the module action on the quotient. In particular, multiplication by w and z coincide and

$$w \cdot \frac{e_k}{\sqrt{k+1}} = z \cdot \frac{e_k}{\sqrt{k+1}} = \sqrt{\frac{k+1}{k+2}} \frac{e_{k+1}}{\sqrt{k+2}}$$

which is identical to that of the Bergman Hilbert module on the disk. In a different guise this result was first obtained by Rudin [11].

Another way to look at this result is by introducing new coordinates u and v, where $u = \frac{w+z}{2}$ and $v = \frac{w-z}{2}$. Now the quotient module $H^2(\mathbb{D}^2)/[w-z]$ can be identified with the one-variable Bergman module on the disk $v = 0$ with the usual action by the variable u and the zero action by v.

If one considers quotient modules defined by the closures of principal modules such as $(w-z)^n$, then more elaborate calculations enable one to describe the quotient modules and the actions but this approach is inadequate for the consideration of quotients by more general

principal ideals. Nonetheless, analogous results hold in great generality. To tackle the more general situation, we must give up brute force calculation and introduce other techniques.

Let Ω be a bounded domain in $\mathbb{C}^n$ and $A(\Omega)$ be the algebra obtained as the supremum norm closure of the functions holomorphic in some neighborhood of the closure of Ω. Now let $\mathcal{M}$ be a kernel Hilbert module over $A(\Omega)$. Thus

1) $\mathcal{M}$ consists of holomorphic functions on Ω;
2) there is a reproducing kernel function $K(\boldsymbol{w}, \boldsymbol{z})\colon \Omega \times \Omega \longrightarrow \mathbb{C}$ which is holomorphic in $\boldsymbol{w}$ and anti-holomorphic in $\boldsymbol{z}$;
3) $K(\cdot, \boldsymbol{z})$ is in $\mathcal{M}$ for $\boldsymbol{z}$ in Ω and $\langle f, K(\cdot, \boldsymbol{z})\rangle_{\mathcal{M}} = f(\boldsymbol{z})$ for f in $\mathcal{M}$ and $\boldsymbol{z}$ in Ω; and
4) $A(\Omega) \cdot \mathcal{M} \subset \mathcal{M}$.

A natural family of examples for the disk algebra are the modules $H^{(\lambda)}(\mathbb{D}), \lambda > 0$, which has the kernel function

$$K^{(\lambda)}(w, z) = \frac{1}{(1 - w\bar{z})^\lambda}, \ \lambda > 0.$$

The familiar Hardy and Bergman modules correspond to $\lambda = 1$ and $\lambda = 2$ respectively.

One may take tensor products of these modules to obtain $H^{(\lambda,\mu)}(\mathbb{D}^2), \lambda, \ \mu > 0$, which are examples of modules over the bidisk algebra. These possess the kernel function

$$K^{(\lambda,\mu)}(\boldsymbol{w}, \boldsymbol{z}) = \frac{1}{(1 - w_1\bar{z}_1)^\lambda} \frac{1}{(1 - w_2\bar{z}_2)^\mu}, \ \lambda, \ \mu > 0.$$

Notice that $\lambda = 1 = \mu$ corresponds to the usual Hardy module $H^2(\mathbb{D}^2)$ over the bi-disk algebra.

Now let $\mathcal{Z}$ be an analytic hyper-surface in Ω of the form $\mathcal{Z} = \{\boldsymbol{z} \in \Omega | \varphi(\boldsymbol{z}) = 0\}$, where φ is a non-zero holomorphic function on Ω. Let $\mathcal{M}_0$ be the submodule $\{f \in \mathcal{M} \mid f|_{\mathcal{Z}} \equiv 0\}$ of functions in $\mathcal{M}$ vanishing identically on $\mathcal{Z}$ and $\mathcal{M}_q$ be the quotient module $\mathcal{M}/\mathcal{M}_0$. For $\Omega = \mathbb{D}^2$, $\mathcal{M} = H^2(\mathbb{D}^2)$ and the functions $\varphi_1(w, z) = w$, $\varphi_2(w, z) = z$, and $\varphi_3(w, z) = w - z$, one obtains the quotient modules $H^2(\mathbb{D}^2)/[w]$, $H^2(\mathbb{D}^2)/[z]$, and $H^2(\mathbb{D}^2)/[w - z]$ studied earlier.

One can appeal to an extension of an earlier result of Aronszajn [[1] to analyze the quotient module $\mathcal{M}_q$. Set $\mathcal{M}_{\mathrm{res}} = \{f|_{\mathcal{Z}} \mid f \in \mathcal{M}\}$ and $\|f|_{\mathcal{Z}}\|_{\mathrm{res}} = \inf\{\|g\|_{\mathcal{M}} \mid g \in \mathcal{M}, g|_{\mathcal{Z}} \equiv f|_{\mathcal{Z}}\}$.

Theorem (Aronszajn).   1) *$\mathcal{M}_{res}$ is a kernel Hilbert module over $\mathcal{Z}$ with $K_{res}(\cdot, \boldsymbol{z}) = K(\cdot, \boldsymbol{z})|_{\mathcal{Z}}$ for $\boldsymbol{z}$ in $\mathcal{Z}$.*
   2) *$\mathcal{M}_q$ is isometrically isomorphic to $\mathcal{M}_{res}$.*

In [6] this result was applied to the quotient $H^2(\mathbb{D}^2)/[w - z]$ to obtain the earlier result as follows. Since the kernel function for $H^2(\mathbb{D}^2)$ is $\frac{1}{(1-w_1\bar{z}_1)} \frac{1}{(1-w_2\bar{z}_2)}$, restricting the kernel function to $w - z = 0$ and using the (u, v) coordinates, we obtain that $K_q(u, u') = \frac{1}{(1-u\bar{u}')^2}$ for u, u' in $\{w - z = 0\}$. This is a multiple of the kernel function for the Bergman space $A(\Omega)$. Hence, the quotient module is isometrically isomorphism to the Bergman module

since multiplication by a constant doesn't change the isomorphism class of a kernel Hilbert module.

Thus the extension of Aronszajn's result enables one to obtain the kernel function for the quotient module and from it, one can construct the Hilbert space itself. However, different kernel functions can determine the same Hilbert module. To obtain invariants one approach is to appeal to an earlier theory obtained a couple of decades ago by the first author with M. Cowen [4] (cf. also [3]). A class $B_1(\Omega)$ of k-tuples of operators was introduced from which a Hermitian holomorphic vector bundle could be constructed. The coordinate functions for a kernel Hilbert module define a k-tuple which belongs to $B_1(\Omega)$. Thus the results from [4] apply to yield a complete set of unitary invariants which determine this k-tuple and hence the Hilbert module up to unitary equivalence. Without providing details, one can show for a kernel Hilbert space that $\boldsymbol{w} \to K(\cdot, \boldsymbol{w})$ determines an anti-holomorphic cross-section of the associated Hermitian holomorphic vector bundle. The invariant in this case is the curvature

$$\mathcal{K}(\boldsymbol{w}) = \sum_{i,j=1}^{k} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|K(\cdot, \boldsymbol{w})\| dw_i \wedge d\bar{w}_j$$

which is a complete unitary invariant.

To obtain the curvature for the quotient Hilbert module $\mathcal{M}_q$, there are two approaches. The first is to use the kernel function for $\mathcal{M}_q$ obtained from the Aronszajn result. The second approach uses the curvature for $\mathcal{M}$ directly and restricts it to obtain the curvature for $\mathcal{M}_q$. Restricting the curvature function for $\mathcal{M}$ to the subvariety $\mathcal{Z}$ involves two steps. First, one restricts the function to $\mathcal{Z}$. Second, one must also project the values of the curvature which lie in the two-forms defined by the tangent bundle to Ω to the corresponding two-forms for $\mathcal{Z}$. Combining the preceding results on restriction of kernel functions and the curvature, one obtains a complete and effective method of describing quotient Hilbert modules of the form $\mathcal{M}/\mathcal{M}_0$, where $\mathcal{M}$ is a kernel Hilbert module and $\mathcal{M}_0$ is the submodule of functions vanishing on a hyper-surface.

Suppose we consider quotients by submodules of higher multiplicity. Simple examples of this phenomenon are $H^2(\mathbb{D}^2)/[w^2]$, $H^2(\mathbb{D}^2)/[z^2]$, and $H^2(\mathbb{D}^2)/[(w-z)^2]$. The first two examples present little difficulty. For the first one, we obtain the orthogonal direct sum of two copies of the one-variable Hardy space on the disk with the z-action being the usual one while the w-action is a nilpotent shift of order two taking the first space to the second and the second space to zero. The second example is the same except the roles of w and z are interchanged. But what about the quotient by $[(w-z)^2]$?

Proceeding as before using the spaces $\mathcal{P}_k$ of homogeneous polynomials, we again have

$$H^2(\mathbb{D}^2)/[(w-z)^2] = \Sigma \oplus \mathcal{P}_k/\mathcal{P}_k \cap [(w-z)^2] \text{ and } \mathcal{P}_k/\mathcal{P}_k \cap [w-z] \subseteq \mathcal{P}_k/\mathcal{P}_k \cap ([w-z]^2).$$

In this case

$$\dim \mathcal{P}_k / \mathcal{P}_k \cap [(w - z)^2] = 2 \quad \text{for} \quad k > 0$$

and one can identify $H^2(\mathbb{D}^2)/[(w - z)^2]$ as the non-orthogonal direct sum of two copies of the Bergman module. Again using the new coordinates one sees that the u-action is given by the matrix $\begin{pmatrix} \sqrt{\frac{k+1}{k+2}} & 0 \\ \sqrt{\frac{3}{(k+2)(k+3)}} & \sqrt{\frac{k}{k+3}} \end{pmatrix}$ and the v-action is nilpotent of order two.

This calculation also appears in [5]. One can guess what happens for $H^2(\mathbb{D}^2)/[w^k]$, $H^2(\mathbb{D}^2)/[z^k]$ and $H^2(\mathbb{D}^2)/[(w - z)^k]$ but the calculation in the last case, or for that matter, in the case of $H^{(\lambda,\mu)}(\mathbb{D}^2)/[(w - z)^k]$ would be formidable. Again this kind of result is true in great generality but one must extend and develop further the earlier ideas, including some new ones, to show that. We will attempt to describe these results.

Again, let $\mathcal{M}$ be a kernel Hilbert module over the algebra $A(\Omega)$ for Ω a bounded domain in $\mathbb{C}^N$ but this time let $\mathcal{M}_k$ be the submodule of functions in $\mathcal{M}$ vanishing to order $k > 0$ on the analytic hyper-surface $\mathcal{Z}$ in Ω which is the zero set of a holomorphic function φ in $A(\Omega)$. A function f on Ω is said to vanish to order k on $\mathcal{Z}$ if it can be written $f = \varphi^k g$ for some holomorphic function g. We seek to characterize the quotient module $\mathcal{M}_q = \mathcal{M}/\mathcal{M}_k$. Now $\mathcal{M}_q$ no longer satisfies the hypotheses of the Aronszajn result and hence it doesn't apply. To proceed we must generalize this approach to allow vector-valued kernel Hilbert modules. The basic result in [7] is that $\mathcal{M}_q$ can be characterized as such a vector-valued kernel Hilbert space over the algebra $A(\Omega)|_{\mathcal{Z}}$ of the restriction of functions in $A(\Omega)$ to $\mathcal{Z}$ and multiplication by φ acts as a nilpotent operator of order k.

The definition of a vector-valued kernel Hilbert space over $A(\Omega)$ is straightforward. For a fixed integer $n > 0$, $\mathcal{M}$ consists of $\mathbb{C}^n$-valued holomorphic functions, and there is an $M_n(\mathbb{C})$-valued function $K(\boldsymbol{w}, \boldsymbol{z})$ on $\Omega \times \Omega$ which is holomorphic in $\boldsymbol{w}$ and anti-holomorphic in $\boldsymbol{z}$ such that

(1) $K(\cdot, \boldsymbol{z})v$ is in $\mathcal{M}$ for $\boldsymbol{z}$ in Ω and v in $\mathbb{C}^n$;
(2) $\langle f, K(\cdot, \boldsymbol{z})v \rangle_{\mathcal{M}} = \langle f(\boldsymbol{z}), v \rangle_{\mathbb{C}^n}$ for f in $\mathcal{M}$, $\boldsymbol{z}$ in Ω and v in $\mathbb{C}^n$; and
(3) $A(\Omega)\mathcal{M} \subset \mathcal{M}$.

Natural examples are the Hardy and Bergman spaces of $\mathbb{C}^n$-valued holomorphic functions or, equivalently, the orthogonal direct sum of k-copies of $H^2(\mathbb{D}^n)$ or $B^2(\mathbb{D}^n)$.

Let $\mathcal{M}$ be a vector-valued kernel Hilbert module over $A(\Omega)$ for Ω a bounded domain in $\mathbb{C}^n$, $\mathcal{Z}$ an analytic hyper-surface in Ω, and $\mathcal{M}_0$ be the submodule of functions in $\mathcal{M}$ that vanish identically on $\mathcal{Z}$. Let $\mathcal{M}_q$ be the quotient module $\mathcal{M}/\mathcal{M}_0$ and $\mathcal{M}_{res}$ the restriction module defined as before. One can extend the Aronszajn theorem to this case as follows:

Theorem ([7]).   1) $\mathcal{M}_{res}$ *is a vector-valued kernel Hilbert module over* $\mathcal{Z}$ *with kernel function* $K_{res}(\cdot, \boldsymbol{z}) = K(\cdot, \boldsymbol{z})|_{\mathcal{Z}}$ *for z in $\mathcal{Z}$.*
    2) $\mathcal{M}_q$ *is isometrically isomorphic to* $\mathcal{M}_{res}$.

Although this extension is rather routine, applying it to the higher-multiplicity case we have been considering is not so obvious. In particular, the kernel Hilbert module $H^2(\mathbb{D}^2)$ is not presented as a vector-valued kernel Hilbert module but the submodule $[(w-z)^2]$ does not consist of all functions vanishing on $\{(w,z) \mid w-z = 0\}$ either. We solve both problems by showing that $H^2(\mathbb{D}^2)$ can also be realized as a vector-valued kernel Hilbert space. In this representation, the submodule will also have the desired form. To accomplish that we define the J operator

$$J\colon\ H^{(\lambda,\mu)}(\mathbb{D}^2) \longrightarrow H^{(\lambda,\mu)}(\mathbb{D}^2)\dot{+}H^{(\lambda,\mu)}(\mathbb{D}^2),$$

where the symbol $\dot{+}$ denotes a non-orthogonal direct sum, such that

$$Jf = f\dot{+}\frac{\partial f}{\partial v}$$

and $\frac{\partial}{\partial v}$ is the normal derivative in the direction orthogonal to the zero set $\{(z,w)|\varphi(z,w) = z - w = 0\}$. The inner product on the space $JH^{(\lambda,\mu)}(\mathbb{D}^2)$ of vector-valued holomorphic functions is obtained by requiring that the map J be isometric. Now, the kernel function, denoted $JK^{(\lambda,\mu)}$, for this space is obtained as follows

$$JK^{(\lambda,\mu)}\colon\ \mathbb{D}^2 \times \mathbb{D}^2 \longrightarrow M_2(\mathbb{C})$$

$$(JK^{(\lambda,\mu)})_{i+1,j+1}(\boldsymbol{w},\boldsymbol{z}) = \left(\frac{\partial^i}{\partial v^i}\frac{\overline{\partial^j}}{\partial v^j}K^{(\lambda,\mu)}\right)(\boldsymbol{w},\boldsymbol{z}), \quad 0 \le i,j \le 1.$$

Finally, the module action for $JH^{(\lambda,\mu)}(\mathbb{D}^2)$ is defined via the map $(f, Jh) \rightarrow \begin{pmatrix} f & 0 \\ \frac{\partial f}{\partial v} & f \end{pmatrix}\begin{pmatrix} h \\ \frac{\partial h}{\partial v} \end{pmatrix}$, for f in the bi-disk algebra. Clearly, the module $H^{(\lambda,\mu)}(\mathbb{D}^2)$ and the range $JH^{(\lambda,\mu)}(\mathbb{D}^2)$ of J in $H^{(\lambda,\mu)}(\mathbb{D}^2)\dot{+}H^{(\lambda,\mu)}(\mathbb{D}^2)$ are isomorphic via the module map J.

We now set

$$(JH^{(\lambda,\mu)}(\mathbb{D}^2))_0 = \left\{f\dot{+}\frac{\partial f}{\partial v} : \left(f\dot{+}\frac{\partial f}{\partial v}\right)\Big|_{u=0} \equiv 0\right\}$$

and observe that the space equals the image of $J(H^{(\lambda,\mu)}(\mathbb{D}^2)_0)$ under J of the space of functions in $H^{(\lambda,\mu)}(\mathbb{D}^2)$ which vanish to order 2 along the analytic hyper-surface $\{(w,z)|w - z = u = 0\}$. The generalization of the Aronszajn theorem states that

$$H^{(\lambda,\mu)}(\mathbb{D}^2)/[(w-z)^2] \cong JH^{(\lambda,\mu)}(\mathbb{D}^2)/(JH^{(\lambda,\mu)}(\mathbb{D}^2))_0 \cong JH^{(\lambda,\mu)}(\mathbb{D}^2)|_{res(u=0)}.$$

A calculation shows that

$$JH^{(\lambda,\mu)}(\mathbb{D}^2)|_{res(u=0)} = H^{(\lambda,\mu)}(\mathbb{D}^2)|_{res\ u=0}\dot{+}\frac{\partial}{\partial v}H^{(\lambda,\mu)}(\mathbb{D}^2)|_{res\ u=0}.$$

Hence we see that the quotient module $H^{(\lambda,\mu)}(\mathbb{D}^2)/[(w-z)^2]$ is isomorphic to the direct sum of the modules $H^{(\lambda+\mu)}(\mathbb{D})$ and $H^{(\lambda+\mu+2)}(\mathbb{D})$ in one-variable in which the u action is the direct sum of weighted shifts and the v action is the shift of summands followed by $\frac{\partial}{\partial v}$. One can calculate the reproducing kernel for the vector-valued kernel Hilbert space using the formula above.

Again it is possible to use the work of [4] to attempt to characterize this quotient module $H^2(\mathbb{D}^2)/[(w-z)^2]$ as the case considered in which the vector bundle has rank two. One would need to generalize that work, however, since it does not take into account the nilpotent v-action. In work currently in progress, we believe, we know how to do this and plan to present it in a subsequent paper. Along similar lines, but in a different language, these questions or closely related ones were considered by Martin and Salinas in [10].

In the preceding paragraphs, we have described how to characterize quotient modules for submodules obtained as the closure of prime ideals in $A(\Omega)$ or powers of prime ideals. If one considers other principal ideals, the same approach applies but $\mathcal{Z}$ is a reducible hypersurface. An illustrative example is obtained by considering the submodule $[wz]$ in $H^2(\mathbb{D}^2)$. In this case the quotient module $H^2(\mathbb{D}^2)/[wz]$ is isometrically isomorphic to the direct sum of two copies of the one-variable Hardy module $H^2(\mathbb{D})$ with w acting on one and z acting on the other both as unilateral shifts, *but* in which one takes the quotient by the vector which is the difference of the two constant functions. In particular, the quotient module is a kernel Hilbert module over the zero set $\{zw = 0\} = \{w = 0\} \cup \{z = 0\}$ with the kernel function given by restriction.

Finally, characterizing quotient modules by submodules obtained as the closure of non-principal ideals poses more serious issues although for those for which the zero set is a complete intersection the issue is largely one of notation and providing an effective description rather than conceptual. However, the general case seems to pose some interesting challenges. Finally, the question of quotients by more general submodules, especially those defined by boundary behavior, is particularly daunting. In particular, consider the quotient of $H^2(\mathbb{D})$ by $\theta H^2(\mathbb{D})$ for a simple singular inner function. One knows multiplication by z is nearly unitary in various senses but whether the generalization to several variables yields Hilbert modules supported on the boundary in some sense remains to be seen.

We now give an application of these ideas to produce an alternative description of a class of operators first studied by Wilkins [12].

Let Möb denote the group of bi-holomorphic automorphisms of the unit disc $\mathbb{D}$. Then it is easy to see that Möb $= \{\varphi_{\alpha,\beta} : \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where $\varphi_{\alpha,\beta}(z) = \alpha \frac{z-\beta}{1-\bar{\beta}z}$, $z \in \mathbb{D}$.

An operator T is called homogeneous if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T.

Most of the results on homogeneous operators are discussed in a recent survey article [2].

The operator $M^{(\lambda)}$, $\lambda \geq 0$ of multiplication by the coordinate function on $H^{(\lambda)}(\mathbb{D})$ are examples of homogeneous operators. The adjoints of these operators are the only homogeneous operators of rank 1 in the Cowen-Douglas class of $\mathbb{D}$. Let $W^{(\lambda,\mu)}$ be the compression of the operator $M \otimes I$ to the quotient $H^{(\lambda,\mu)}(\mathbb{D}^2)/[(w - z)^2]$. Recalling the identification $H^{(\lambda,\mu)}(\mathbb{D}^2)/[(w - z)^2] \cong JH^{(\lambda,\mu)}(\mathbb{D}^2)|_{res(u=0)}$ and the explicit calculation of the reproducing

kernel for $JH^{(\lambda,\mu)}(\mathbb{D}^2)|_{res(u=0)}$, we find that the adjoint operators $W^{(\lambda,\mu)*}$ match the description of homogeneous operators of rank 2 in the Cowen-Douglas class given in [12]. However, this method also produces examples of homogeneous operators of rank k by simply compressing $M \otimes I$ to the quotient $H^{(\lambda,\mu)}(\mathbb{D}^2)/[(w-z)^k]$. The details are given in [2].

References

[1] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337–404.

[2] B. Bagchi and G. Misra, Homogeneous operators and projective representations of the Möbius group, *Proc. Indian Acad. Sci. (Math. Sci.)* **111** (2001), 415–437.

[3] X. Chen and R. G. Douglas, Localization of Hilbert Modules, *Mich Math. J.*, **39** (1992), 443 - 454.

[4] M. Cowen and R. G. Douglas, Complex geometry and operator theory, *Acta Math.* **141** (1978), 187–261.

[5] R. G. Douglas and G. Misra, Some calculations for Hilbert modules, *J. Orissa Math. Soc.* **12–16** (1993–96), pp. 76–85.

[6] R. G. Douglas and G. Misra, Geometric invariants for resolutions of Hilbert modules, *in* "Operator Theory: Advances and Applications," Vol. 104, pp. 83–112, Birkhäuser, Basel, 1998.

[7] R. G. Douglas, G. Misra and C. Varughese, On quotient modules—The case of arbitrary multiplicity, *J. Funct. Anal.* **174** (2000), 364–398.

[8] R. G. Douglas and V. I. Paulsen, "Hilbert Modules over Function Algebras," Longman Research Notes, Vol. 217, Harlow, New York, 1989.

[9] R. G. Douglas, V. I. Paulsen, C.-H. Sah and K. Yan, Algebraic reduction and rigidity for Hilbert modules, *Amer. J. Math.* **117** (1995), no. 1, 75–92.

[10] M. Martin and N. Salinas, Flag manifolds and the Cowen–Douglas theory, *J. Operator Theory* **38** (1997), 329–365.

[11] W. Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, Inc., New York–Amsterdam, 1969.

[12] D. R. Wilkins, Homogeneous vector bundles and Cowen-Douglas operators, *Int. J. Math.* **4** (1993), 503–520.

Texas A&M University, College Station, Texas 77843
E-mail address: rgd@tamu.edu

Indian Statistical Institute, RV College Post, Bangalore 560 059
E-mail address: gm@isibang.ac.in