Numerics and Control of PDEs

Lecture 1

IFCAM – IISc Bangalore

July 22 – August 2, 2013

Introduction to feedback stabilization – Stabilizability of F.D.S.

Mythily R., Praveen C., Jean-Pierre R.

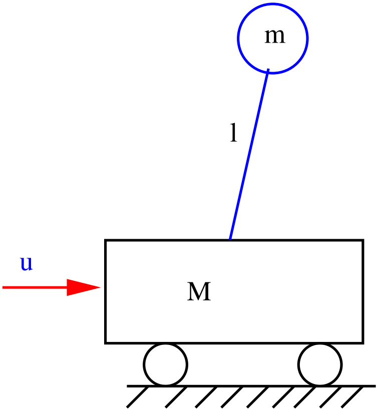
Q1. Controllability. How to drive a nonlinear system from a given initial state to another prescribed state by the action of some control during the time interval $(0, T)$?

Q2. Stabilizability. How to maintain a system at an unstable position, in the presence of perturbation by using some measurements to estimate the state at each time t, and by using the estimated state in a control law ?

Plan of Lecture 1

1. Models

1.1. The inverted pendulum



The nonlinear system

$$(M+m)x'' + m\,\ell\,\theta''\cos\theta - m\,\ell|\theta'|^2\sin\theta = u,$$
$$m\,\ell\,\theta'' + m\,x''\,\cos\theta = m\,g\,\sin\theta.$$

M is the mass of the car,

m is the mass of the pendulum,

$x(t)$ is the position of the car at time t,

$\theta(t)$ is the angle between the unstable position and the pendulum measured clockwise,

u is a force applied to the car. The Lagrange equations read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x'}\right) - \frac{\partial L}{\partial x} = u,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \theta'}\right) - \frac{\partial L}{\partial \theta} = 0,$$

where the Lagrangian $L = T - V$, T is the kinetic energy and V is the potential energy.

$$L = \frac{1}{2}(M+m)|x'|^2 + m\,\ell\, x'\theta' \cos\theta + \frac{1}{2} m\,\ell^2\,|\theta'|^2 - mg\ell\cos\theta.$$

$$x'' = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} + \frac{m\ell}{M + m\sin^2\theta}|\theta'|^2\sin\theta + \frac{1}{M + m\sin^2\theta}u,$$

$$\theta'' = -\frac{m}{M + m\sin^2\theta}|\theta'|^2\sin\theta\cos\theta + g\frac{M+m}{M\ell + m\ell\sin^2\theta}\sin\theta$$

$$-\frac{\cos\theta}{M\ell + m\ell\sin^2\theta}\,u.$$

As a first order system, we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ x' \\ \theta \\ \theta' \end{pmatrix} = F(x, x', \theta, \theta') + \begin{pmatrix} 0 \\ \dfrac{1}{M + m\sin^2\theta} \\ 0 \\ \dfrac{-\cos\theta}{M\ell + m\ell\sin^2\theta} \end{pmatrix} u = F(x, x', \theta, \theta') + G(\theta)\,u,$$

with

$$F(x, x', \theta, \theta') =$$

$$\begin{pmatrix} x' \\ -\dfrac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} + \dfrac{m\ell}{M+m}|\theta'|^2\sin\theta \\ \theta' \\ -\dfrac{m}{M + m\sin^2\theta}|\theta'|^2\sin\theta\cos\theta + g\dfrac{M+m}{M\ell + m\ell\sin^2\theta}\sin\theta \end{pmatrix}.$$

We linearize F about $(0,0,0,0)$, we write

$$F(x,x',\theta,\theta') = F(0,0,0,0) + DF(0,0,0,0)\begin{pmatrix} x \\ x' \\ \theta \\ \theta' \end{pmatrix} + \widetilde{N}(\theta,\theta'),$$

with $F(0,0,0,0) = (0,0,0,0)^T$ and

$$DF(0,0,0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix}.$$

Similarly

$$G(\theta)u = G(0)u + \widetilde{R}(\theta)u,$$

with

$$G(0) = \begin{pmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{M\ell} \end{pmatrix}.$$

Setting $z = (x, x', \theta, \theta')^T$, we have

$$z' = Az + Bu + N(z) + R(z)u, \quad z(0) = z_0,$$

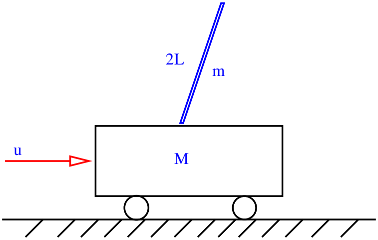
where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{M\ell} \end{pmatrix},$$

$$z_0 = (x_0, x_1, \theta_0, \theta_1)^T$$

and the two nonlinear terms N and R obey $N(0) = 0$ and $R(0) = 0$.

1.2. Simple rod inverted pendulum



The nonlinear system

$$(M+m)x'' + m\,\ell\,\theta''\cos\theta - m\,\ell|\theta'|^2\sin\theta = u,$$
$$m\,\ell^2\,\theta'' + m\ell\,x''\,\cos\theta = m\,\ell\,g\,\sin\theta,$$

is replaced by

$$(M+m)x'' + m\,\ell\,\theta''\cos\theta - m\,\ell|\theta'|^2\sin\theta = u - kx',$$
$$(I + m\ell^2)\theta'' + m\,\ell\,x''\,\cos\theta = m\,\ell\,g\,\sin\theta - d\theta',$$

where I is the moment of inertia of the pendulum about the COG, k is the cart viscous friction coefficient, d is the pendulum viscous friction coefficient.

1.3. Fluid flows in a neighbourhood of unstable solutions

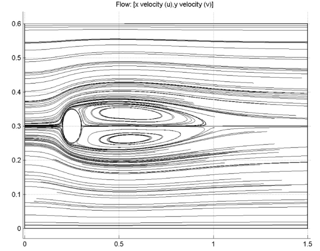
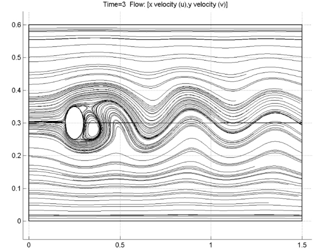
• We consider a fluid flow governed by the N.S.E. (w and q are the volocity and the pressure)

• Given an unstable stationary solution w_s.

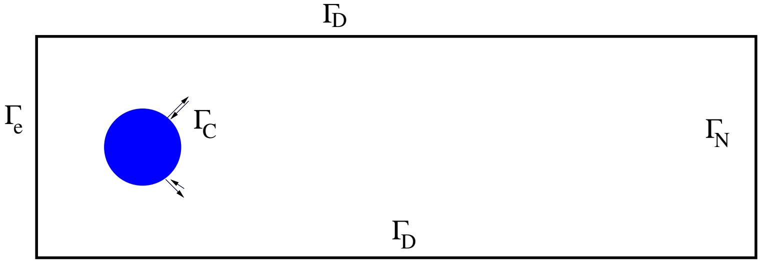
• Find a boundary control u able to stabilize w exponentially when $w(0) = w_s + z_0$.

We consider two different problems:
- Control of the wake behind a circular cylinder,
- The control of a flow in an open cavity.

Control of the wake behind an obstacle $Re = u_s D/\nu$

	$5 < Re < 50$	A fixed pair of vortices
	$50 < Re < 150$	Vortex street

Flow around a cylinder with an outflow boundary condition



$$\nu \frac{\partial z}{\partial n} - pn = 0 \quad \text{on} \quad \Gamma_N$$

or

$$\sigma(z,p)n = \nu\left(\nabla z + (\nabla z)^T\right)n - pn = 0 \quad \text{on} \quad \Gamma_N.$$

The unstable stationary solution w_s of the N.S.E.

$$-\nu\Delta w_s + (w_s \cdot \nabla) w_s + \nabla q_s = 0, \quad \text{in } \Omega,$$

$$\text{div } w_s = 0 \quad \text{in } \Omega, \quad w_s = u_s \text{ on } \Gamma_e, \quad + \text{Other B.C. on } \Gamma \setminus \Gamma_e.$$

The stabilization problem

$$\text{Find } u \text{ s.t. } \|w(t) - w_s\|_Z \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty,$$

$$\frac{\partial w}{\partial t} - \nu\Delta w + (w \cdot \nabla) w + \nabla q = 0, \qquad \text{div } w = 0 \quad \text{in } Q,$$

$$w = u_s \text{ on } \Sigma_e = \Gamma_e \times (0,\infty), \quad w = Mu \text{ on } \Sigma_c = \Gamma_c \times (0,\infty),$$

$$+ \text{Other B.C. on } \Sigma \setminus (\Sigma_e \cup \Sigma_c), \qquad w(0) = w_0 \text{ in } \Omega.$$

Set $z = w - w_s$, $p = q - q_s$. The linearized (resp. nonlinear) equation is

$$\frac{\partial z}{\partial t} - \nu \Delta z + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + (z \cdot \nabla) z + \nabla p = 0,$$

$$\operatorname{div} z = 0 \quad \text{in } Q, \quad z = Mu \quad \text{on } \Sigma_D,$$

$$+ \text{ Other B.C. on } \Sigma \setminus (\Sigma_D), \qquad z(0) = z_0 \quad \text{in } \Omega\,.$$

with

$$\operatorname{supp} M \subset \Gamma_c.$$

1.4.a A simplified model – The 1D Burgers equation

$$
\begin{aligned}
&\frac{\partial w}{\partial t} - \nu \frac{\partial^2 w}{\partial x^2} + w \frac{\partial w}{\partial x} = f_s \quad \text{in } (0,1) \times (0,\infty), \\
&w(0,t) = u_s + u(t) \quad \text{and} \quad \nu \frac{\partial w}{\partial x}(1,t) = g_s \quad \text{for } t \in (0,\infty), \\
&w(\cdot,0) = w_0 \quad \text{in } (0,1).
\end{aligned}
$$

We assume that f_s, g_s and u_s are stationary data, and that w_s is the solution to the equation

$$
\begin{aligned}
&-\nu\, w_{s,xx} + w_s\, w_{s,x} = f_s \quad \text{in } (0,1), \\
&w_s(0) = u_s \quad \text{and} \quad \nu\, w_{s,x}(1) = g_s.
\end{aligned}
$$

This model is *from the mathematical analysis viewpoint* very similar to the N.S.E., because the nonlinear term has the same type of properties.

The function u is a control variable.

First notice that $w = w_s$ is the unique solution to the dynamical system

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \nu \frac{\partial^2 w}{\partial x^2} + w\frac{\partial w}{\partial x} = f_s \quad \text{in } (0,1)\times(0,\infty),\\
&w(0,t) = u_s \quad \text{and} \quad \nu \frac{\partial w}{\partial x}(1,t) = g_s \quad \text{for } t \in (0,\infty),\\
&w(\cdot,0) = w_s \quad \text{in } (0,1).
\end{aligned}$$

We assume that w_s is an unstable solution of this dynamical system.

Starting with the initial condition

$$w(\cdot,0) = w_s + z_0 = w_0 \quad \text{in } (0,1),$$

the goal is to find a boundary control u such that

$$\|e^{\omega t}(w_{w_0,u} - w_s)\| \le C,$$

provided that $w_0 - w_s = z_0$ is small enough.

We set $z = w - w_s$. The equation satisfied by z is

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \nu\,\frac{\partial^2 z}{\partial x^2} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x} + z\frac{\partial z}{\partial x} = 0 \quad \text{in } (0,1)\times(0,\infty),\\
&z(0,t) = u(t) \quad \text{and} \quad \nu\,\frac{\partial z}{\partial x}(1,t) = 0 \quad \text{for } t\in(0,T),\\
&z(\cdot,0) = w_0 - w_s = z_0 \quad \text{in } (0,1).
\end{aligned}$$

The linearized model is

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \nu\,\frac{\partial^2 z}{\partial x^2} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x} = 0 \quad \text{in } (0,1)\times(0,\infty),\\
&z(0,t) = u(t) \quad \text{and} \quad \nu\,\frac{\partial z}{\partial x}(1,t) = 0 \quad \text{for } t\in(0,T),\\
&z(\cdot,0) = z_0 \quad \text{in } (0,1).
\end{aligned}$$

As a preliminary step, we are going to study the exponential stabilization of the heat equation with mixed boundary conditions

$$
\begin{aligned}
&\frac{\partial z}{\partial t} - \nu \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1) \times (0,\infty),\\
&z(0,t) = u(t) \quad \text{and} \quad \nu \frac{\partial z}{\partial x}(1,t) = 0 \quad \text{for } t \in (0,T),\\
&z(\cdot,0) = z_0 \quad \text{in } (0,1),
\end{aligned}
$$

or the exponential stabilization of the heat equation with Dirichlet boundary conditions

$$
\begin{aligned}
&\frac{\partial z}{\partial t} - \nu \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1) \times (0,\infty),\\
&z(0,t) = u(t) \quad \text{and} \quad z(1,t) = 0 \quad \text{for } t \in (0,T),\\
&z(\cdot,0) = z_0 \quad \text{in } (0,1).
\end{aligned}
$$

1.4.b Other models in 2D – A Burgers type equation with D.B.C.

$$\frac{\partial w}{\partial t} - \Delta w + \partial_i(w^2) = f_s \quad \text{in } Q,$$

$$w = \sum_{i=1}^{N_c} v_i\, g_i + g_s \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega\,.$$

Let w_s be the stationnary solution to

$$-\Delta w_s + \partial_i(w_s^2) = f_s \quad \text{in } \Omega, \qquad w_s = g_s \quad \text{on } \Gamma\,.$$

The L. E. and the nonlinear equation satisfied by $z = w - w_s$ are

$$\frac{\partial z}{\partial t} - \Delta z + 2\,\partial_i w_s\, z + 2\,\partial_i z\, w_s \; + \; \partial_i(z^2) = 0,$$

$$z = \sum_{i=1}^{N_c} v_i\, g_i \quad \text{on } \Sigma, \quad z(0) = z_0 = w_0 - w_s \quad \text{in } \Omega\,.$$

We can study the corresponding model with mixed boundary conditions. We shall also sudy the feedback stabilization of the 2D heat equation.

2. Feedback stabilization

2.i Representation of systems

- The model of the inverted pendulum can be written in the form

$$z' = Az + Bu + N(z) + R(z)u, \quad z(0) = z_0,$$

where $z = (x, x', \theta, \theta')$, the two nonlinear terms N and R obey $N(0) = 0$, $N'(0) = 0$, and $R(0) = 0$.

- The Burgers eq. can be written in the form

$$z' = Az + Bu + F(z), \quad z(0) = z_0,$$

where $z = w - w_s$, w is the velocity of the instationary flow, while w_s is the stationary solution.

2.ii. The open loop stabilization problem

For these models, the problem consists in finding a control $u \in L^2(0,\infty;U)$ such that

$$\|z_{z_0,u}(t)\|_Z \le Ce^{-\omega t}\|z_0\|_Z, \quad C>0, \quad \omega>0,$$

provided that $\|z_0\|_Z$ is small enough.

Existence of a stabilizing control. Except for finite dimensional systems, there is no general result for the stabilizability of nonlinear systems.

Stabilization of the linearized model. Similar problems can be studied for the linearized model

$$z' = Az + Bu, \quad z(0) = z_0.$$

If we find a control $u \in L^2(0,\infty;U)$ able to stabilize the linearized system (e.g. by solving an optimal control problem), there is no reason that u also stabilizes the nonlinear system.

2.iii. Stabilization by feedback with full information

One way for finding a control able to stabilize the nonlinear system is to look for a control stabilizing the linearized system in feedback form. We look for $K \in \mathcal{L}(Z, U)$ such that the system

$$z' = (A + BK)z, \quad z(0) = z_0,$$

is exponentially stable in Z. If we denote by $z_{z_0,u}$ the solution to this closed loop system, and if we set

$$u(t) = K\, z_{z_0,u}(t),$$

then u stabilizes the system $z' = Az + Bu$, $z(0) = z_0$. We can expect that the closed loop nonlinear system

$$z' = Az + BKz + F(z), \quad z(0) = z_0,$$

is locally asymptotically stable around 0.

The issues

- What are the conditions on (A, B) so that there exists $K \in \mathcal{L}(Z, U)$ for which $A + BK$ is stable ?

- How to determine such operators K ?

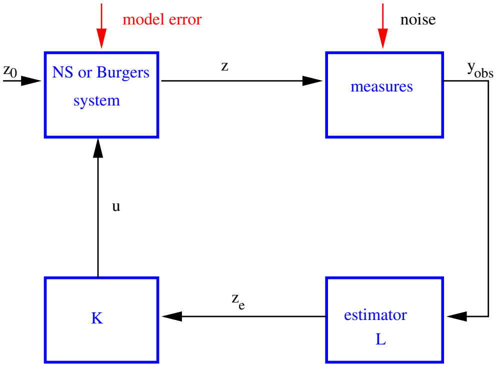
- Is this programme useful in pratical applications ?

2.iv. Stabilization by feedback with partial information

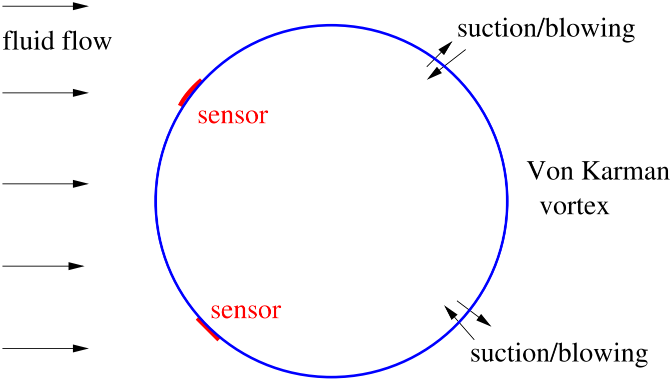
If the feedback K also stabilizes the nonlinear system, by solving

$$z' = Az + BKz + F(z) + \mu, \quad z(0) = z_0 + \mu_0,$$

we know a control $u(t) = Kz_{c\ell n, z_0}(t)$ stabilizing the nonlinear system. But in pratice μ_0 is not necessarily known, we can also have other types of uncertainties μ. This is why we have to look for an estimation z_e of the state z in function of measurements, and use the control law $u(t) = K\, z_e(t)$.



The complete problem: estimation + feedback control



The decoupling principle for linear systems

Step 1. Feedback gain. Find $K \in \mathcal{L}(Z, U)$ such that $A + BK$ is stable.

Step 2. From the linear noisy model

$$z' = Az + Bu + \mu, \quad z(0) = z_0 + \mu_0,$$

and noisy measurements

$$y_{obs}(t) = Hz(t) + \eta(t),$$

find an estimation z_e of z. We shall choose and estimator of the form

$$z_e' = Az + Bu + L(Hz_e - y_{obs}), \quad z(0) = z_0,$$

where $L \in \mathcal{L}(Y, Z)$ is such that $A + LH$ is stable.

Step 3. Choose $u(t) = Kz_e(t)$ as control.

This leads to the following system

$$\begin{aligned}
z' &= Az + BKz_e + \mu, \quad z(0) = z_0 + \mu_0, \\
z_e' &= Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_0, \\
y_{obs}(t) &= Hz(t) + \eta(t).
\end{aligned}$$

We are going to see that the feedback gain K and the filtering gain L may be obtained by solving two Algebraic Riccati equations.

Theorem. If the pair (A, B) is stabilizable, and if there is no eigenvalue of A on the imaginary axis, and $C \in \mathcal{L}(Z, Y)$, then the following (ARE) admits a unique solution

$$(ARE) \qquad \begin{aligned} & P \in \mathcal{L}(Z), \quad A^*P + PA - PBB^*P + C^*C = 0, \\ & P = P^* \geq 0, \quad A - BB^*P \text{ is stable.} \end{aligned}$$

In particular, setting $K = -B^*P$, then $A + BK$ is stable.

Theorem. Let $Q_\mu \in \mathcal{L}(Z)$, $Q_\mu = Q_\mu^* \geq 0$, $R_\eta \in \mathcal{L}(Y_o)$, $R_\eta = R_\eta^* \geq 0$. If the pair (A, H) is detectable, if there is no eigenvalue of A on the imaginary axis, then the following $(ARE)_e$ admits a unique solution

$$(ARE)_e \qquad \begin{aligned} & P_e \in \mathcal{L}(Z), \quad P_e = P_e^* \geq 0, \\ & P_eA^* + AP_e - P_eH^*R_\eta^{-1}HP_e + Q_\mu = 0, \\ & A - P_eH^*R_\eta^{-1}H \quad \text{is stable.} \end{aligned}$$

In particular, setting $L = -P_eH^*R_\eta^{-1}$, $A + LH$ is stable.

Moreover the system

$$z' = Az + BKz_e + \mu, \quad z(0) = z_0 + \mu_0,$$
$$z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_0,$$
$$y_{obs}(t) = Hz(t) + \eta(t),$$

can be written in the form

$$\begin{pmatrix} z' \\ z_e' \end{pmatrix} = \begin{pmatrix} A & BK \\ -LH & A + BK + LH \end{pmatrix} \begin{pmatrix} z \\ z_e \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \quad \begin{pmatrix} z(0) \\ z_e(0) \end{pmatrix} = \begin{pmatrix} z_0 + \mu_0 \\ z_0 \end{pmatrix},$$

and the operator

$$\mathcal{A} = \begin{pmatrix} A & BK \\ -LH & A + BK + LH \end{pmatrix}$$

is stable.

3. Issues to be solved

3.1. Issues to be solved for F.D.S.

- To find when a linear finite dimensional system is stabilizable, when it is detectable.

- To find feedback gains for stabilizable linear finite dimensional systems.

- To find filtering gains for detectable linear finite dimensional systems.

- To couple the feedback and filtering gains for linear finite dimensional systems which are both stabilizable and detectable.

- To find 'the best linear strategy' efficient enough to stabilize locally a nonlinear system in the case of partial information.

3.2. Issues to be solved for I.D.S.

- To write a linear PDE with a boundary (or a distributed) control and a boundary (or a distributed) observation as a linear controlled system in order to adapt the study of stabilizability and detectablity from FDS to IDS.

- To find feedback and filtering gains for infinite dimensional systems.

- To calculate numerically feedback and filtering gains for infinite dimensional systems
- either by approximating the PDE by a numerical scheme (e.g. a FEM for the space discretization) and by finding the feedback and filtering gains for the corresponding approximate finite dimensional system,
- or by approximating directly the feedback and filtering gains of the infinite dimensional system.

The first strategy 'first approximate and next control' is not always usable, because, after discretization, the size of the state variable may be between 10000 and more than 100000.
- The algorithms for finding feedback and filtering gains cannot be used in that case.
- Even if we can find feedback and filtering gains for large size problem, for a practical use the estimator must be of small size.

This is why we need to develop new reduction strategy for control problems.

4. Notation

4.1. Notation for F.D.S.

The controlled system

$$z' = Az + Bu, \quad z(0) = z_0,$$

with either $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $z(t) \in \mathbb{R}^n = Z$, $u(t) \in \mathbb{R}^m = U$,

or $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $z(t) \in \mathbb{C}^n = Z$, $u(t) \in \mathbb{C}^m = U$.

Diagonalisation and shift

$$\begin{array}{ccc}
z' = Az + Bu & \xrightarrow[\widehat{u}(t) = e^{\omega t}u(t)]{\widehat{z}(t) = e^{\omega t}z(t)} & \begin{array}{l} \widehat{z}' = A_\omega \widehat{z} + B\widehat{u} \\ A_\omega = A + \omega I \end{array} \\
\downarrow & & \downarrow \\
\text{diagonalisation} & & \text{diagonalisation} \\
z = \Sigma\zeta & & \widehat{z} = \Sigma\widehat{\zeta} \\
\downarrow & & \downarrow \\
\begin{array}{l} \zeta' = \Lambda\zeta + \mathbb{B}u \\ \Lambda = \Sigma^{-1}A\Sigma, \quad \mathbb{B} = \Sigma^{-1}B \end{array} & \longrightarrow & \begin{array}{l} \widehat{\zeta}' = \Lambda_\omega\widehat{\zeta} + \mathbb{B}\widehat{u} \\ \Lambda_\omega = \Sigma^{-1}A_\omega\Sigma \end{array}
\end{array}$$

Projection and shift

$$z' = Az + Bu \longrightarrow \begin{aligned} \widehat{z}' &= A_\omega \widehat{z} + B\widehat{u} \\ A_\omega &= A + \omega I \end{aligned}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

Projection via $Z = Z_u \oplus Z_s$ — Projection via $Z = Z_{\omega,u} \oplus Z_{\omega,s}$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{aligned} z_u' &= A_u z_u + B_u u \\ z_s' &= A_s z_s + B_s u \end{aligned} \longrightarrow \begin{aligned} z_{\omega,u}' &= A_{\omega,u} z_{\omega,u} + B_{\omega,u} u \\ z_{\omega,s}' &= A_{\omega,s} z_{\omega,s} + B_{\omega,s} u \end{aligned}$$

or

$$\zeta' = \Lambda\zeta + \mathbb{B}u \longrightarrow \begin{aligned} \widehat{\zeta}' &= \Lambda_\omega \widehat{\zeta} + \mathbb{B}\widehat{u} \\ \Lambda_\omega &= \Sigma^{-1} A_\omega \Sigma \end{aligned}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

Projection via $\Sigma^{-1}Z = (\Sigma^{-1}Z)_u \oplus (\Sigma^{-1}Z)_s$ — Projection via $\Sigma^{-1}Z = (\Sigma^{-1}Z)_{\omega,u} \oplus (\Sigma^{-1}Z)_{\omega,s}$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{aligned} \zeta_u' &= \Lambda_u \zeta_u + \mathbb{B}_u u \\ \zeta_s' &= \Lambda_s \zeta_s + \mathbb{B}_s u \end{aligned} \longrightarrow \begin{aligned} \zeta_{\omega,u}' &= \Lambda_{\omega,u} \zeta_{\omega,u} + \mathbb{B}_{\omega,u} u \\ \zeta_{\omega,s}' &= \Lambda_{\omega,s} \zeta_{\omega,s} + \mathbb{B}_{\omega,s} u \end{aligned}$$

4.2. Notation for I.D.S.

Diagonalisation and approximation

$$
\begin{array}{ccc}
z' = Az + Bu & \xrightarrow[\text{approximation}]{\text{F.E.}} & \begin{array}{c} \mathbf{E}\mathbf{z}' = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \\ \mathbf{A} \in \mathbb{R}^{\mathcal{N}_\Omega \times \mathcal{N}_\Omega},\ \mathbf{B} \in \mathbb{R}^{\mathcal{N}_\Omega \times \mathcal{N}_{\Gamma_c}} \end{array} \\
\downarrow & & \downarrow \\
\text{diagonalisation} & & \text{diagonalisation} \\
A\xi = \lambda\xi & & \mathbf{A}\boldsymbol{\xi} = \boldsymbol{\lambda}\mathbf{E}\boldsymbol{\xi} \\
\downarrow & & \downarrow \\
\zeta' = \Lambda\zeta + \mathbb{B}u & \longrightarrow & \boldsymbol{\zeta}' = \boldsymbol{\Lambda}\boldsymbol{\zeta} + \mathbb{B}\mathbf{u}
\end{array}
$$