Numerics and Control of PDEs

Lecture 2

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Stabilization with full information

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1. Finite Dimensional Linear Systems

1.1. Stability of Finite Dimensional Linear Systems

Theorem. Let A belong to $\mathbb{R}^{n\times n}$. The system

$$z' = Az, \quad z(0) = z_0,$$

is exponentially stable if and only if

$$\mathrm{Re}\,\sigma(A) < 0.$$

Lyapunov equation. Let A belong to $\mathbb{R}^{n\times n}$ and $Q = Q^* \geq 0$ belong to $\mathbb{R}^{n\times n}$. We consider the so-called Lyapunov equation

$$(LE) \qquad P = P^* \geq 0, \quad P \in \mathbb{R}^{n\times n}, \quad A^*P + PA + Q = 0.$$

Theorem. If A is exponentially stable, then (LE) admits a unique solution defined by

$$P = \int_0^\infty e^{A^*t} Q e^{At}\, dt.$$

Stability Theorem. The operator A is exponentially stable if and only if, for all $Q \in \mathbb{R}^{n \times n}$ satisfying $Q = Q^* > 0$, the Lyapunov equation (LE) admits a solution.

1.2. The Duhamel formula

When $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the solution to system

$$z' = Az + Bu, \quad z(0) = z_0,$$

is defined by

$$(E) \qquad z_{z_0,u}(t) = z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds.$$

The same formula holds true in $\mathbb{C}^n$. If $(\lambda_i)_{1\le i\le r}$ are the complex eigenvalues of A, we can define

$$E_{\mathbb{C}}(\lambda_j) = \mathrm{Ker}(A - \lambda_j I), \quad \dim E_{\mathbb{C}}(\lambda_j) = \ell_j = \text{geometric multiplicity of } \lambda_j,$$

$$G_{\mathbb{C}}(\lambda_j) = \mathrm{Ker}((\lambda_j I - A)^{m(\lambda_j)}), \quad \text{the generalized eigenspace ass. to } \lambda_j,$$

$$\dim G_{\mathbb{C}}(\lambda_j) = N(\lambda_j) = \text{algebraic multiplicity of } \lambda_j.$$

We have

$$\mathbb{C}^n = \oplus_{j=1}^r G_{\mathbb{C}}(\lambda_j), \quad A = \Sigma\Lambda\Sigma^{-1}, \quad AG_{\mathbb{C}}(\lambda_j) \subset G_{\mathbb{C}}(\lambda_j),$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \Lambda_r \end{pmatrix}, \quad \Lambda_j = \begin{pmatrix} J_j^1 & & & \\ & J_j^2 & & 0 \\ & 0 & \ddots & \\ & & & J_j^{\ell_j} \end{pmatrix},$$

$\Lambda_j \in \mathbb{C}^{N(\lambda_j)\times N(\lambda_j)}$, and

$$e^{tA} = \Sigma e^{t\Lambda}\Sigma^{-1} = \Sigma \begin{pmatrix} e^{t\Lambda_1} & & & \\ & e^{t\Lambda_2} & & 0 \\ & 0 & \ddots & \\ & & & e^{t\Lambda_r} \end{pmatrix} \Sigma^{-1}.$$

In particular

$$e^{tA} G_{\mathbb{C}}(\lambda_j) \subset G_{\mathbb{C}}(\lambda_j).$$

We can rewrite the equation

$$z' = Az + Bu, \quad z(0) = z_0,$$

in the form

$$\Sigma^{-1} z' = \Sigma^{-1} A \Sigma \Sigma^{-1} z + \Sigma^{-1} Bu, \quad \Sigma^{-1} z(0) = \Sigma^{-1} z_0,$$

that is

$$\zeta' = \Lambda \zeta + \mathbb{B} u, \quad \zeta(0) = \zeta_0,$$

where $\zeta = \Sigma^{-1} z$, $\mathbb{B} = \Sigma^{-1} B$, $\zeta_0 = \Sigma^{-1} z_0$.

The vector ζ may be written as

$$\zeta = \oplus_{j=1}^{r} \zeta^i = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_r \end{pmatrix} \quad \text{with} \quad \dim(\zeta_i) = N(\lambda_i) \quad \text{and } \zeta^i = \begin{pmatrix} \vdots \\ 0 \\ \zeta_i \\ 0 \\ \vdots \end{pmatrix}.$$

Setting

$$z = \oplus_{j=1,\, j\neq i}^{r} \Sigma\, \zeta^{i} = \oplus_{j=1,\, j\neq i}^{r} z^{i},$$

z^i is the projection of z onto $G_{\mathbb{C}}(\lambda_i)$ along $\oplus_{j=1,\, j\neq i}^{r} G_{\mathbb{C}}(\lambda_j)$.

We can also decompose $\mathbb{R}^n$ into *real generalized eigenspaces*

$$\mathbb{R}^n = \oplus_{j=1}^{r} G_{\mathbb{R}}(\lambda_j), \quad G_{\mathbb{R}}(\lambda_j) = G_{\mathbb{R}}(\bar{\lambda}_j) = \mathrm{vec}\{\mathrm{Re}G_{\mathbb{C}}(\lambda_j), \mathrm{Im}G_{\mathbb{C}}(\lambda_j)\},$$

$$AG_{\mathbb{R}}(\lambda_j) \subset G_{\mathbb{R}}(\lambda_j).$$

The projection onto $G_{\mathbb{R}}(\lambda_i)$ along $\oplus_{j=1,\, j\neq i}^{r} G_{\mathbb{R}}(\lambda_j)$ can be defined accordingly.

The same analysis can be done for parabolic partial differential equations.

2. Controllability and Reachability of Finite Dimensional Systems

In this part, $Z = \mathbb{R}^n$ or $Z = \mathbb{C}^n$ and $U = \mathbb{R}^m$ or $U = \mathbb{C}^m$. We make the identifications $Z = Z^*$ and $U = U^*$.

The operator $L_T \ : \ L^2(0, T; U) \longmapsto Z$

$$L_T u = \int_0^T e^{(T-s)A} B u(s)\, ds.$$

The reachable set from z_0 at time T

$$R_T(z_0) = e^{TA} z_0 + \operatorname{Im} L_T.$$

Exact controllability. The pair (A, B) is exactly controllable at time T if $R_T(z_0) = Z$ for all $z_0 \in Z$.

Reachability. A state z_f is reachable from $z_0 = 0$ at time $T < \infty$ if there exists $u \in L^2(0, T; U)$ such that

$$L_T u = z_{0,u}(T) = \int_0^T e^{(T-s)A} B u(s)\, ds = z_f.$$

The system (A, B) is reachable at time $T < \infty$ iff

$$\operatorname{Im} L_T = Z.$$

A finite dimensional system is reachable at time T iff it is controllable at time T.

The system is reachable (or exactly controllable) at time $T < \infty$ iff the matrix (called 'controllability Gramian')

$$W_{A,B}^T = \int_0^T e^{tA} B B^* e^{tA^*} \, dt$$

is invertible.

Idea of the proof. L_T is surjective iff its adjoint operator $L_T^* \in \mathcal{L}(Y, L^2(0, T; U))$,

$$(L_T^* \phi)(\cdot) = B^* e^{(T-\cdot)A^*} \phi,$$

is injective. This last condition is equivalent to the existence of $\alpha > 0$ such that

$$\int_0^T \|B^* e^{sA^*} \phi\|_U^2 \, ds = \|L_T^* \phi\|_{L^2(0,T;U)}^2 \geq \alpha \|\phi\|_Z^2, \quad \forall \phi \in Z.$$

Assume that the system (E) is reachable at time T. For a given $z_f \in Z$, the control

$$\bar{u}(t) = -B^* e^{(T-t)A^*} (W_{A,B}^T)^{-1} e^{TA} z_f$$

is such that

$$z_{0,\bar{u}}(T) = \int_0^T e^{(T-s)A} B\bar{u}(s)\, ds = z_f.$$

Moreover

$$\|\bar{u}\|^2_{L^2(0,T;U)} = ((W_{A,B}^T)^{-1} z_f, z_f)_Z$$

and

$$\|\bar{u}\|_{L^2(0,T;U)} \leq \|u\|_{L^2(0,T;U)},$$

for all u such that

$$z_{0,u}(T) = \int_0^T e^{(T-s)A} Bu(s)\, ds = z_f.$$

Applications. Let us calculate the eigenvalues and eigenvectors for the linearized inverted pendulum.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix}, \quad A^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\dfrac{mg}{M} & 0 & \dfrac{g(M+m)}{M\ell} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_1 = \sqrt{\frac{g(M+m)}{M\ell}}, \quad \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = -\sqrt{\frac{g(M+m)}{M\ell}}.$$

The associated eigenvectors and generalized eigenvectors for A are

$$\xi_1 = \left(-\frac{mg}{M}\frac{M\ell}{g(M+m)}, -\frac{mg}{M}\sqrt{\frac{g(M+m)}{M\ell}}, 1, \sqrt{\frac{g(M+m)}{M\ell}} \right)^T$$

$$A\xi_2 = 0, \quad \xi_2 = (1,0,0,0)^T, \quad A\xi_3 = \xi_2, \quad \xi_3 = (0,1,0,0)^T.$$

$$\xi_4 = \left(-\frac{mg}{M}\frac{M\ell}{g(M+m)}, \frac{mg}{M}\sqrt{\frac{g(M+m)}{M\ell}}, 1, -\sqrt{\frac{g(M+m)}{M\ell}}\right)^T.$$

We can also calculate the eigenvectors and generalized eigenvectors of A^T:

$$\varepsilon_1 = \left(0, 0, \frac{1}{2}, \frac{1}{2}\sqrt{\frac{M\ell}{g(M+m)}}\right)^T,$$

$$A^T\varepsilon_3 = 0, \quad \varepsilon_3 = \left(0, 1, 0, \frac{mg}{M}\frac{M\ell}{g(M+m)}\right)^T,$$

$$A\varepsilon_2 = \varepsilon_3, \quad \varepsilon_2 = (1, 0, \frac{mg}{M}\frac{M\ell}{g(M+m)}, 0)^T,$$

$$\varepsilon_4 = \left(0, 0, \frac{1}{2}, -\frac{1}{2}\sqrt{\frac{M\ell}{g(M+m)}}\right)^T.$$

We have chosen $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ to have

$$\varepsilon_i^T \xi_j = \delta_{i,j}.$$

Thanks to that bi-orthogonality property, we can define the projectors on each generarailized eigenspace parallel to the sum of the other ones in the following way

$$\pi_1 z = \left(\varepsilon_1^T z\right) \xi_1, \quad \pi_4 z = \left(\varepsilon_4^T z\right) \xi_4,$$

and the projection π_0 on the generalized eigenspace $G(0)$, associated with $\lambda_2 = \lambda_3 = 0$, parallel to the sum $\mathrm{Ker}(\lambda_1 I - A) \oplus \mathrm{Ker}(\lambda_4 I - A)$ is

$$\pi_0 z = \left(\varepsilon_2^T z\right) \xi_2 + \left(\varepsilon_3^T z\right) \xi_3.$$

Controllability of Finite Dimensional Systems. The system (A, B) is controllable at time T if and only if one of the following conditions is satisfied.

$$W^*_{A,B} = \int_0^T e^{tA} BB^* e^{tA^*} \, dt > 0, \quad \forall T > 0,$$

$$\text{rank} \left[B \,|\, AB \,|\, \ldots \,|\, A^{n-1} B \right] = n,$$

$$\forall \lambda \in \mathbb{C}, \quad \text{rank} \, [A - \lambda I \,|\, B] = n,$$

$$\forall \lambda \in \mathbb{C}, \quad A^* \varepsilon = \lambda \varepsilon \quad \text{and} \quad B^* \varepsilon = 0 \Rightarrow \varepsilon = 0,$$

$$\exists K \in \mathcal{L}(Z, U), \quad \sigma(A + BK) \text{ can be freely assign}$$

(with the cond. that complex eigenvalues are in conjugate pairs).

The controllability of the linearized inverted pendulum.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(M+m)}{M\ell} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{pmatrix},$$

The controllability matrix is

$$[B \mid AB \mid A^2B \mid A^3B] = \begin{pmatrix} 0 & \frac{1}{M} & 0 & \frac{mg}{M^2\ell} \\ \frac{1}{M} & 0 & \frac{mg}{M^2\ell} & 0 \\ 0 & -\frac{1}{M} & 0 & -\frac{g(M+m)}{M^2\ell^2} \\ -\frac{1}{M} & 0 & -\frac{g(M+m)}{M^2\ell^2} & 0 \end{pmatrix}.$$

Its determinent is

$$\begin{aligned}
&\det[B \mid AB \mid A^2B \mid A^3B] \\
&= -\frac{g(M+m)}{M^3\ell^2}\left(-\frac{g(M+m)}{M^3\ell^2} + \frac{mg}{M^3\ell^2}\right) \\
&\quad + \frac{mg}{M^3\ell^2}\left(-\frac{g(M+m)}{M^3\ell^2} + \frac{mg}{M^3\ell^2}\right) = \frac{g^2}{M^4\ell^4}.
\end{aligned}$$

The linearized inverted pendulum is controllable at any time $T > 0$.

Verification of the Hautus criterion.

We have

$$B^* = \begin{pmatrix} 0 & \frac{1}{M} & 0 & -\frac{1}{M\ell} \end{pmatrix},$$

and

$$B^*(\alpha\varepsilon_1) = -\frac{\alpha}{2}\frac{1}{M\ell}\sqrt{\frac{M\ell}{g(M+m)}} = 0 \Rightarrow \alpha = 0,$$

$$B^*(\alpha\varepsilon_3) = \frac{\alpha}{M} - \frac{\alpha}{M\ell}\frac{mg}{M}\frac{M\ell}{g(M+m)} = 0 \Rightarrow \alpha = 0,$$

$$B^*(\alpha\varepsilon_4) = \frac{\alpha}{2}\frac{1}{M\ell}\sqrt{\frac{M\ell}{g(M+m)}} = 0 \Rightarrow \alpha = 0.$$

Thus, we recover that the pair (A, B) is controllable.

3. Stabilizability of F.D.S.

3.i. Open loop stabilizability. System (A, B) is *open loop stabilizable in Z* when for any initial condition $z_0 \in Z$, there exists a control $u \in L^2(0, \infty; U)$ s.t.

$$\int_0^\infty \|z_{z_0,u}(t)\|_Z^2 \, dt < \infty.$$

Stabilizability by feedback. System (A, B) is *stabilizable by feedback* when there exists an operator $K \in \mathcal{L}(Z, U)$ s. t. $A + BK$ is exponentially stable in Z.

Open loop stabilizability is equivalent to *stabilizability by feedback* for finite dimensional systems or for parabolic systems with Dirichlet boundary conditions like the heat equation, the linearized Burgers equation, the Stokes equation, the linearized Navier-Stokes equations.

3.ii. Characterization of the stabilizability of F.D.S.

The finite dimensional case. System (E) is stabilizable if and only if one of the following conditions is satisfied.

$$(i)\ \forall\lambda,\ \mathrm{Re}\lambda \geq 0, \quad \mathrm{rank}\,[A-\lambda I \,|\, B] = n,$$

$$(ii)\ \forall\lambda,\ \mathrm{Re}\lambda \geq 0, \quad A^*\varepsilon = \lambda\varepsilon \quad \text{and} \quad B^*\varepsilon = 0 \Rightarrow \varepsilon = 0,$$

$$(iii)\ \forall\lambda,\ \mathrm{Re}\lambda \geq 0, \quad \mathrm{Ker}(\lambda I - A^*) \cap \mathrm{Ker}(B^*) = \{0\},$$

$$(iv)\ \exists K \in \mathcal{L}(Z,U), \quad \sigma(A+BK) \text{ is stable.}$$

Conditions (ii), (iii), (iv) are equivalent to the stabilizability of (E) for Infinite Dimensional Systems under the previous conditions on (A, B) (analyticity, compactness, degree of unboundness of B).

FDS Example of a stabilizing feedback. The system (A, B) is controllable iff

$$W^T_{-A,B} = \int_0^T e^{-tA} BB^* e^{-tA^*} \, dt,$$

is invertible for all $T > 0$. Indeed

$$e^{TA} W^T_{-A,B} e^{TA^*} = \int_0^T e^{(T-t)A} BB^* e^{(T-t)A^*} \, dt = \int_0^T e^{\tau A} BB^* e^{\tau A^*} \, d\tau = W^T_{A,B}.$$

Assume that (A, B) is controllable, then

$$K = -B^* (W^T_{-A,B})^{-1}$$

is a stabilizing feedback.

Idea of the proof. The mapping

$$z \longmapsto ((W^T_{-A,B})^{-1} z, z)_Z$$

is a Lyapunov function of the closed loop linear system.

Proof. The closed loop linear system is

$$z' = (A + BK)z, \quad z(0) = z_0.$$

We have

$$((W^T_{-A,B})^{-1} z(t), z(t))_Z \geq 0 \quad \text{if } z(t) \neq 0,$$

$$\frac{d}{dt}((W^T_{-A,B})^{-1} z(t), z(t))_Z = 2((W^T_{-A,B})^{-1} z'(t), z(t))_Z$$

$$= 2((W^T_{-A,B})^{-1}(A + BK)z(t), z(t))_Y < 0.$$

FDS – Characterization of the stabilizability in terms of Gramians.

We assume that

$$Z = Z_s \oplus Z_u, \quad Z_u = \oplus_{j=1}^{N_u} G_{\mathbb{R}}(\lambda_j), \quad Z_s = \oplus_{j=N_u+1}^{r} G_{\mathbb{R}}(\lambda_j),$$

$$\mathrm{Re}\lambda_j > 0 \quad \text{if } 1 \le j \le N_u,$$

$$\mathrm{Re}\lambda_j < 0 \quad \text{if } N_u + 1 \le j \le r.$$

Recall that

$$e^{tA} Z_u \subset Z_u \quad \text{and} \quad e^{tA} Z_s \subset Z_s.$$

We also have

$$Z = Z^* = Z_s^* \oplus Z_u^*, \quad Z_u^* = \oplus_{j=1}^{N_u} G_{\mathbb{R}}^*(\lambda_j), \quad Z_s^* = \oplus_{j=N_u+1}^{r} G_{\mathbb{R}}^*(\lambda_j),$$

$$e^{tA^*} Z_u^* \subset Z_u^* \quad \text{and} \quad e^{tA^*} Z_s^* \subset Z_s^*.$$

Let π_u the projection onto Z_u along Z_s and set $\pi_s = I - \pi_u$. The system

$$\pi_u z' = z_u' = A_u z_u + \pi_u B u, \quad \pi_s z' = z_s' = A_s z_s + \pi_s B u,$$

can be written as a system in $Z \times Z$ of the form (using matrix notation)

$$\begin{pmatrix} z_u' \\ z_s' \end{pmatrix} = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix} + \begin{pmatrix} \pi_u B u \\ \pi_s B u \end{pmatrix}, \quad \begin{pmatrix} z_u(0) \\ z_s(0) \end{pmatrix} = \begin{pmatrix} \pi_u z_0 \\ \pi_s z_0 \end{pmatrix}.$$

If the system (A, B) is stabilizable then the system $(A_u, B_u) = (\pi_u A, \pi_u B)$ is also stabilizable. The converse is true. Assume that $(A_u, B_u) = (\pi_u A, \pi_u B)$ is stabilizable and let $K \in \mathcal{L}(Z_u, U)$ be a stabilizing feedback. Then the system

$$\begin{pmatrix} z_u' \\ z_s' \end{pmatrix} = \begin{pmatrix} A_u + B_u K & 0 \\ B_s K & A_s \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix}$$

is also stable.

4. Characterization of the stabilizability of F.D.S.

The following conditions are equivalent

(a) (A, B) is stabilizable,

(b) $(A_u, B_u) = (\pi_u A, \pi_u B)$ is stabilizable,

(c) The Gramian

$$W^{\infty}_{-A_u,B_u} = \int_0^{\infty} e^{-tA_u} B_u B_u^* e^{-tA_u^*}\, dt$$

is invertible.

(d) There exists $\alpha > 0$ such that for all $\phi \in Z_u^*$,

$$(O.I.) \qquad (W^{\infty}_{-A_u,B_u}\phi, \phi)_Z = \int_0^{\infty} \left\| B_u^* e^{-tA_u^*}\phi \right\|_U^2 dt \ \geq \ \alpha \|\phi\|_Z^2.$$

The operator

$$P_u = (W^{\infty}_{-A_u,B_u})^{-1} \in \mathcal{L}(Z_u, Z_u^*), \quad P_u = P_u^* \geq 0,$$

provides a stabilizing feedback

$$A_u - B_u B_u^* P_u \quad \text{is exponentially stable.}$$

Recall that

$$B_u = \pi_u B \quad \text{and} \quad B_u^* = B^* \pi_u^*,$$

where π_u^* is the projection onto Z_u^* along Z_s^*.

The operator P_u satisfies the following Algebraic Bernoulli equation (a degenerate Algebraic Riccati equation)

$$P_u A_u + A_u^* P_u - P_u B_u B_u^* P_u = 0.$$

If we set

$$P = \pi_u^* P_u \pi_u.$$

Then $P \in \mathcal{L}(Z)$ is such that $P = P^* \geq 0$ and solves the following $(A.B.E.)$

$$A.B.E. \qquad \begin{aligned} & P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \\ & A^* P + PA - PBB^* P = 0, \\ & A - BB^* P \text{ generates} \\ & \text{an exponentially stable semigroup.} \end{aligned}$$

Moreover, the feedback $-B^*P$ provides the control of minimal norm in $L^2(0,\infty;U)$

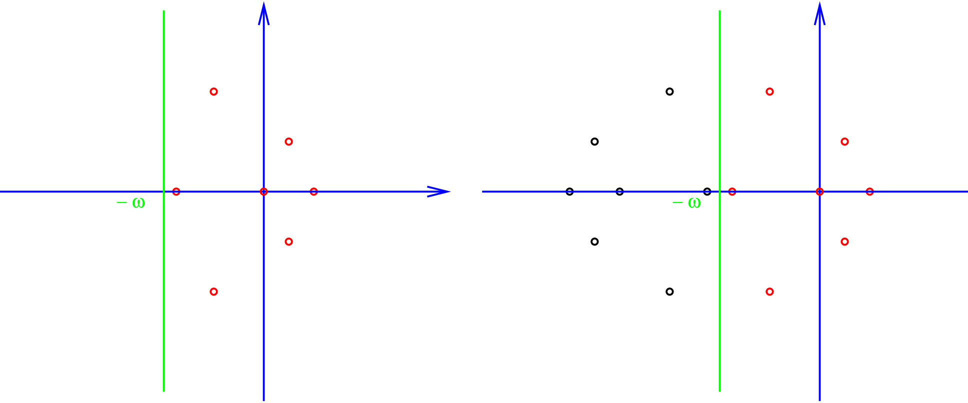
$$u(t) = -B^*Pe^{t(A-BB^*P)}z_0,$$

and the spectrum of $A - BB^*P$ is

$$\sigma(A - BB^*P) = \{-\mathrm{Re}\lambda + i\,\mathrm{Im}\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\lambda > 0\}$$
$$\cup\{\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\lambda < 0\}.$$

To obtain a better exponential decay we can replace A by $A + \omega I$ and determine the corresponding feedback $-B^*P_\omega$. In that case the spectrum of $A - BB^*P_\omega$ is as follows

Spectrum of A and of $A - BB^* P_\omega$



Example 1.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with $\lambda_1 > \lambda_2 > 0$. Thus $A = A_u$, and

$$W^{\infty}_{-A_u,B_u} = \int_0^{\infty} e^{-tA_u} B_u B_u^* e^{-tA_u^*}\ dt = \int_0^{\infty} \begin{pmatrix} e^{-2t\lambda_1} & e^{-t(\lambda_1+\lambda_2)} \\ e^{-t(\lambda_1+\lambda_2)} & e^{-2t\lambda_2} \end{pmatrix} dt$$

$$= \begin{pmatrix} \frac{1}{2\lambda_1} & \frac{1}{\lambda_1+\lambda_2} \\ \frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_2} \end{pmatrix}.$$

$$(W^{\infty}_{-A_u,B_u})^{-1} = \frac{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2} \begin{pmatrix} \frac{1}{2\lambda_2} & -\frac{1}{\lambda_1+\lambda_2} \\ -\frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_1} \end{pmatrix}.$$

$$BB^*P = \frac{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2\lambda_2} & -\frac{1}{\lambda_1+\lambda_2} \\ -\frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\lambda_1(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} & -\frac{2\lambda_2(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} \\ \frac{2\lambda_1(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} & -\frac{2\lambda_2(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} \end{pmatrix}.$$

$$P^{-1}A^*P = \frac{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}\begin{pmatrix}\frac{1}{2\lambda_1} & \frac{1}{\lambda_1+\lambda_2}\\ \frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_2}\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\frac{1}{2\lambda_2} & -\frac{1}{\lambda_1+\lambda_2}\\ -\frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_1}\end{pmatrix}$$

$$= \frac{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}\begin{pmatrix}\frac{1}{2\lambda_1} & \frac{1}{\lambda_1+\lambda_2}\\ \frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_2}\end{pmatrix}\begin{pmatrix}\frac{\lambda_1}{2\lambda_2} & -\frac{\lambda_1}{\lambda_1+\lambda_2}\\ -\frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_2}{2\lambda_1}\end{pmatrix}$$

$$= \frac{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}\begin{pmatrix}\frac{1}{4\lambda_2} - \frac{\lambda_2}{(\lambda_1+\lambda_2)^2} & -\frac{1}{2(\lambda_1+\lambda_2)} + \frac{\lambda_2}{2\lambda_1(\lambda_1+\lambda_2)}\\ \frac{\lambda_1}{2\lambda_2(\lambda_1+\lambda_2)} - \frac{1}{2(\lambda_1+\lambda_2)} & -\frac{\lambda_1}{(\lambda_1+\lambda_2)^2} + \frac{1}{4\lambda_1}\end{pmatrix}$$

$$= \frac{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}\begin{pmatrix}\frac{(\lambda_1-\lambda_2)(\lambda_1+3\lambda_2)}{4\lambda_2(\lambda_1+\lambda_2)^2} & \frac{\lambda_2-\lambda_1}{2\lambda_1(\lambda_1+\lambda_2)}\\ \frac{\lambda_1-\lambda_2}{2\lambda_2(\lambda_1+\lambda_2)} & \frac{(\lambda_2-\lambda_1)(3\lambda_1+\lambda_2)\lambda_1}{4\lambda_1(\lambda_1+\lambda_2)^2}\end{pmatrix}$$

$$= \frac{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}\begin{pmatrix}\frac{(\lambda_1-\lambda_2)(\lambda_1+3\lambda_2)}{4\lambda_2(\lambda_1+\lambda_2)^2} & \frac{\lambda_2-\lambda_1}{2\lambda_1(\lambda_1+\lambda_2)}\\ \frac{\lambda_1-\lambda_2}{2\lambda_2(\lambda_1+\lambda_2)} & \frac{(\lambda_2-\lambda_1)(3\lambda_1+\lambda_2)}{4\lambda_1(\lambda_1+\lambda_2)^2}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\lambda_1(\lambda_1+3\lambda_2)}{(\lambda_1-\lambda_2)} & \frac{2\lambda_2(\lambda_1+\lambda_2)}{(\lambda_2-\lambda_1)}\\ \frac{2\lambda_1(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)} & \frac{\lambda_2(3\lambda_1+\lambda_2)}{(\lambda_2-\lambda_1)}\end{pmatrix}$$

and

$$A - BB^*P = \begin{pmatrix}\lambda_1 - \frac{2\lambda_1(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} & -\frac{2\lambda_2(\lambda_1+\lambda_2)}{\lambda_2-\lambda_1}\\ -\frac{2\lambda_1(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} & \lambda_2 - \frac{2\lambda_2(\lambda_1+\lambda_2)}{\lambda_2-\lambda_1}\end{pmatrix}.$$

Since

$$A - BB^* P = -P^{-1} A^* P,$$

the matrix P solves the Bernoulli equation and

$$\text{spec}(A - BB^* P) = \{-\lambda_1, -\lambda_2\}.$$

As expected the symmetry property is satisfied between the spectra of $A - BB^* P$ and A.

Example 2.

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $\lambda > 0$. Thus $A = A_u$, and

$$W^{\infty}_{-A_u,B_u} = \int_0^{\infty} e^{-tA_u} B_u B_u^* e^{-tA_u^*}\, dt$$

$$= \int_0^{\infty} \begin{pmatrix} e^{-\lambda t} & -te^{-\lambda t} \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\lambda t} & 0 \\ -te^{-\lambda t} & e^{-\lambda t} \end{pmatrix} dt$$

$$= \int_0^{\infty} \begin{pmatrix} t^2\, e^{-2\lambda t} & -t\, e^{-2\lambda t} \\ -t\, e^{-2\lambda t} & e^{-2\lambda t} \end{pmatrix} dt = \begin{pmatrix} \frac{1}{4\lambda^3} & -\frac{1}{4\lambda^2} \\ -\frac{1}{4\lambda^2} & \frac{1}{2\lambda} \end{pmatrix}$$

$$P = (W^{\infty}_{-A_u,B_u})^{-1} = \begin{pmatrix} 8\lambda^3 & 4\lambda^2 \\ 4\lambda^2 & 4\lambda \end{pmatrix},$$

$$BB^*P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 8\lambda^3 & 4\lambda^2 \\ 4\lambda^2 & 4\lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4\lambda^2 & 4\lambda \end{pmatrix},$$

$$A - BB^*P = \begin{pmatrix} \lambda & 1 \\ -4\lambda^2 & -3\lambda \end{pmatrix}.$$

$$P^{-1}A^*P = \begin{pmatrix} \frac{1}{4\lambda^3} & -\frac{1}{4\lambda^2} \\ -\frac{1}{4\lambda^2} & \frac{1}{2\lambda} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} 8\lambda^3 & 4\lambda^2 \\ 4\lambda^2 & 4\lambda \end{pmatrix} = \begin{pmatrix} 2\lambda - 3\lambda & 1 - 2 \\ -2\lambda^2 + 6\lambda^2 & -\lambda + 4\lambda \end{pmatrix}$$

Since

$$A - BB^*P = -P^{-1}A^*P,$$

the matrix P solves the Bernoulli equation and the symmetry property is satisfied between the spectra of $A - BB^*P$ and A.

Example 3. The reduced linearized inverted pendulum.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This system comes from the linearization of the controlled pendulum equation

$$\theta'' - \sin\theta = u.$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. In order to determine the projection onto the unstable subspace, we have to find the eignevectors of $A = A^*$.

$$E(\lambda_1) = \{(\theta,\rho) \in \mathbb{R}^2 \mid \theta = \rho\} \quad \text{and} \quad E(\lambda_2) = \{(\theta,\rho) \in \mathbb{R}^2 \mid \theta = -\rho\}.$$

We set

$$\xi_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad \xi_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Notice that $\xi_1 \in E(\lambda_1)$, $\xi_2 \in E(\lambda_2)$, and (ξ_1, ξ_2) is an orthonormal basis of $\mathbb{R}^2$. Thus, the projection π_u on $E(\lambda_1)$ along $E(\lambda_2)$ is defined by

$$\pi_u z = (\xi_1^T z)\xi_1.$$

In order to find $A_u \in \mathcal{L}(Z_u)$ and $B_u \in \mathcal{L}(U, Z_u)$, it is convenient to introduce the change of variables

$$\begin{pmatrix} z_u \\ z_s \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \text{with} \quad \Sigma = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The system

$$\Sigma^{-1} z' = \Sigma^{-1} A \Sigma \Sigma^{-1} z + \Sigma^{-1} B u, \quad \Sigma^{-1} z(0) = \Sigma^{-1} z_0,$$

is

$$\begin{pmatrix} z_u \\ z_s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} u, \quad \begin{pmatrix} z_u(0) \\ z_s(0) \end{pmatrix} = \begin{pmatrix} z_{0,u} \\ z_{0,s} \end{pmatrix}.$$

Thus

$$z_u' = z_u + \frac{1}{\sqrt{2}} u, \quad z_u(0) = z_{0,u}.$$

The Bernoulli equation for this controlled system is

$$p > 0, \quad 2p - \frac{p^2}{2} = 0.$$

Thus $p = 4$, and the controlled system with feedback is

$$z_u' = z_u - \left(\frac{1}{\sqrt{2}}\right)^2 4z_u = -z_u, \quad z_u(0) = z_{0,u}.$$

The full closed loop system is

$$\begin{pmatrix} z_u \\ z_s \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix}, \quad \begin{pmatrix} z_u(0) \\ z_s(0) \end{pmatrix} = \begin{pmatrix} z_{0,u} \\ z_{0,s} \end{pmatrix}.$$

Coming back to the system satisfied by z we have

$$z' = \Sigma \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \Sigma^{-1} z = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} z, \quad z(0) = z_0.$$

4. Feedback gain minimizing a cost functional

We can find a feedback gain by considering the following optimal control problem $(\mathcal{P})$

$$\text{Minimize} \;\; J(z,u) = \frac{1}{2}\int_0^\infty \left\|Cz(t)\right\|_Y^2 dt + \frac{R}{2}\int_0^\infty \left\|u(t)\right\|_U^2 dt$$

$$z' = Az + Bu, \qquad z(0) = z_0, \quad \lim_{t\to\infty} \|z(t)\|_Z = 0,$$

$$\text{where } C \in \mathcal{L}(Z,Y), \quad R > 0.$$

We assume that the pair (A, B) is stabilizable, and that there is no eigenvalue of A on the imaginary axis.
We know that this problem admits a unique solution. We would like to find it. We denote by $z_{z_0,u}$ the solution ot the state equation and we set $F(u) = J(z_{z_0,u}, u)$.
We would like to characterize the optimal control $\bar{u}$ by setting $F'(\bar{u}) = 0$. This is not possible because we do not know the subspace of controls $U_f \subset U$ for which F is finite.

Method 1. Approximate $(\mathcal{P})$ by a sequence of control problems stated over a finite time interval $(0, k)$. Write the optimality conditions for this sequence of approximate problems and pass to the limit in the optimality systems.

Method 2. Look for the solution to $(\mathcal{P})$ in feedback form, that is look for K such that $A + BK$ is stable. Characterize the feedback gain K which minimizes the cost functional.

Theorem. We assume that the pair (A, B) is stabilizable, and that there is no eigenvalue of A on the imaginary axis. Then problem $(\mathcal{P})$ admits a unique solution defined by

$$u(t) = -Ke^{t(A+BK)}z_0,$$

where $K \in \mathcal{L}(Z, U)$, $K = -B^*P$ and P is the solution to the Riccati equation

$$(A.R.E.) \qquad \begin{aligned} &P \in \mathcal{L}(Z), \quad A^*P + PA - PBB^*P + CC^* = 0, \\ &P = P^* \geq 0, \quad A - BB^*P \text{ is stable.} \end{aligned}$$

Proof. We replace $z(t)$ by $e^{t(A+BK)}z_0$ and $u(t)$ by $Ke^{t(A+BK)}z_0$ in the functional J, we get

$$J(z,u) = \frac{1}{2}\int_0^\infty (e^{t(A+BK)^*}C^*Ce^{t(A+BK)}z_0, z_0)_Z\,dt$$
$$+\frac{R}{2}\int_0^\infty (e^{t(A+BK)^*}K^*Ke^{t(A+BK)}z_0, z_0)_Z\,dt = \frac{1}{2}(Pz_0, z_0)_Z,$$

with

$$P = \int_0^\infty e^{t(A+BK)^*}(C^*C + K^*RK)e^{t(A+BK)}dt.$$

It is clear that $P = P^* \geq 0$. Since $A + BK$ is stable, from Lyapunov Stability Theorem, it follows that P is the solution to the following Lyapunov equation

$$(A+BK)^*P + P(A+BK) + C^*C + K^*RK = 0.$$

Now, we want to characterize the feedback gain K which minimizes the cost functional. For that, we assume that K is the minimizer, and we compare the costs obtained with K and $K+\Delta K$, under the conditions $A+BK$ and $A+B(K+\Delta K)$ are stable.
We denote by $P+\Delta P$ the operator

$$P+\Delta P=\int_0^\infty e^{t(A+B(K+\Delta K))^*}(C^*C+(K+\Delta K)^*R(K+\Delta K))e^{t(A+B(K+\Delta K))}dt.$$

We verify that ΔP is the solution to

$$\begin{aligned}(A+BK)^*\Delta P+\Delta P(A+BK)+(\Delta K)^*(RK+B^*P)\\ +(RK+B^*P)^*\Delta K+(\Delta K)^*R\Delta K=0.\end{aligned}$$

If K is optimal, then $\Delta P\geq 0$. Since $A+BK$ is stable, $\Delta P\geq 0$ if and only if

$$(\Delta K)^*(RK+B^*P)+(RK+B^*P)^*\Delta K+(\Delta K)^*R\Delta K\geq 0.$$

Since $(\Delta K)^* R \Delta K \geq 0$ for all ΔK (and $(\Delta K)^* R \Delta K > 0$ for some ΔK), we must have

$$RK + B^* P = 0, \quad \text{that is } K = -R^{-1} B^* P.$$

Replacing K by $-R^{-1} B^* P$ in the previous Lyapunov equation, we prove that P solves the Riccati equation

$$PA + A^* P - PBR^{-1} B^* P + C^* C = 0.$$

5. An algorithm for solving Riccati equations

We consider the A.R.E.

$$ARE \qquad \begin{array}{l} P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \\ A^*P + PA - PBR^{-1}B^*P + C^*C = 0, \\ A - BR^{-1}B^*P \text{ generates} \\ \text{an exponentially stable semigroup,} \end{array}$$

where the unknown P belongs to $\mathcal{L}(\mathbb{R}^n)$, $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, where $\mathbb{R}^m$ is the discrete control space, $C \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^\ell)$, where $\mathbb{R}^\ell$ is the observation space.

We make the following assumption.

(H_1) The pair (A, B) is stabilizable

(H_2) The pair (A, C) is detectable or A has no eigenvalue on the imaginary axis.

The *ARE* admits a unique solution.

The matrix

$$\mathcal{H} = \begin{bmatrix} A & -BR^{-1}B^* \\ CC^* & -A^* \end{bmatrix}$$

is called the Hamiltonian matrix associated with the *ARE*.

It is a symplectic matrix

$$\left[\begin{array}{cc} 0 & I \\ -I & 0 \end{array}\right]^{-1} \mathcal{H} \left[\begin{array}{cc} 0 & I \\ -I & 0 \end{array}\right] = -\mathcal{H}^*.$$

Therefore the matrix $\mathcal{H}$ and $-\mathcal{H}^*$ are similar and they have the same eigenvalues. On the other hand $\mathcal{H}$ and $\mathcal{H}^*$ have also the same set of eigenvalues. Thus if λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of $\mathcal{H}$ with the same multiplicity.

Let us denote by $-\lambda_1, -\lambda_2, \ldots, -\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_n$, the eigenvalues of $\mathcal{H}$, where $\text{Re}(\lambda_i) \geq 0$ for $i = 1, \ldots, n$. Under assumptions (H_1) and (H_2), it can be shown that the matrix $\mathcal{H}$ has no pure imaginary eigenvalues, that is $\text{Re}(\lambda_i) > 0$ for $i = 1, \ldots, n$.

There exists a matrix V whose columns are eigenvectors, or generalized eigenvectors of $\mathcal{H}$, such that

$$V^{-1}\mathcal{H}V = \left[\begin{array}{cc} -J & 0 \\ 0 & J \end{array}\right],$$

where $-J$ is composed of Jordan blocks corresponding to eigenvalues with negative real part, and

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

is such that $V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}$ is a matrix whose columns are eigenvectors corresponding to eigenvalues with negative real parts. It may be proved that V_{11} is nonsingular and the unique positive semidefinite solution of equation $(A.R.E.)$ is given by

$$P = V_{21} V_{11}^{-1}.$$

A very simple example

Set

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $\lambda > 0$, $R^{-1} = I$, and $C = 0$. Then

$$\mathcal{H} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & -1 & -\lambda \end{pmatrix}.$$

In that case ($C = 0$), we replace the condition '(A, C) is detectable' by the condition '$A - BB^* P$ is stable'.

We have $\mathcal{H}V_1 = -\lambda V_1$ with

$$V_1 = (-1, 2\lambda, 0, 4\lambda^2)^T.$$

The solution to

$$(\mathcal{H} + \lambda I)V_2 = V_1,$$

is

$$V_2 = (-1/\lambda, 1, -4\lambda^2, 0)^T.$$

We obtain the solution to the $(A.B.E.)$ (the degenerate $(A.R.E.)$)

$$P = \begin{pmatrix} 0 & -4\lambda^2 \\ 4\lambda^2 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1/\lambda \\ 2\lambda & 1 \end{pmatrix}^{-1},$$

$$P = \begin{pmatrix} 0 & -4\lambda^2 \\ 4\lambda^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/\lambda \\ -2\lambda & -1 \end{pmatrix} = \begin{pmatrix} 8\lambda^3 & 4\lambda^2 \\ 4\lambda^2 & 4\lambda \end{pmatrix}.$$

Moreover

$$A - BB^* P = \begin{pmatrix} \lambda & 1 \\ -4\lambda^2 & -3\lambda \end{pmatrix}.$$

Another simple example – A reduced inverted pendulum

Instead of studying the equations of the inverted pendulum, we can consider the simple model

$$\theta'' - \sin\theta = u,$$

where θ is the angular displacement from the unstable vertical equilibrium, and u is taken as a control. The linearized system about 0 is

$$z' = Az + Bu, \quad z(0) = z_0,$$

where

$$z = (\theta, \rho)^T, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \rho = \theta'.$$

We choose $R^{-1} = I$ and $C = 0$. Then

$$\mathcal{H} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We can verify the $\lambda = 1$ and $\lambda = -1$ are the two eigenvalues of $\mathcal{H}$ of multiplicity 2.

Rather than computing the matrix Riccati equation, we can equivalently determine directly the control of minimal norm stabilizing the system.

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. They are both of multiplicity 1. We have

$$E(\lambda_1) = \{(\theta, \rho) \in \mathbb{R}^2 \mid \theta = \rho\} \quad \text{and} \quad E(\lambda_2) = \{(\theta, \rho) \in \mathbb{R}^2 \mid \theta = -\rho\}.$$

We can rewrite the system as follows

$$\left(\frac{\theta + \rho}{2}\right)' = \left(\frac{\theta + \rho}{2}\right) + \frac{1}{2}u,$$
$$\left(\frac{\rho - \theta}{2}\right)' = -\left(\frac{\rho - \theta}{2}\right) + \frac{1}{2}u.$$

The first equation corresponds to the projected system onto the unstable subspace. The feedback of minimal norm stabilizing the unstable system is obtained by solving the one dimensional Riccati equation

$$p > 0, \quad 2p - p^2/4 = 0.$$

Thus $p = 8$ and the feedback law is

$$u(t) = -\frac{1}{2}\, 8 \left(\frac{\theta + \rho}{2}\right) = -2(\theta + \rho).$$

Thus the closed loop linear system is

$$\begin{pmatrix} \theta \\ \rho \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \theta \\ \rho \end{pmatrix}.$$

We notice that the two eigenvalues of the generator of this system are $\lambda = -1$. We recover the result already obtained previously.