Numerics and Control of PDEs

Lecture 3

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Introduction to estimation of F.D.S. Coupling between estimation and control

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Stabilization by feedback with full information

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Estimation with partial information

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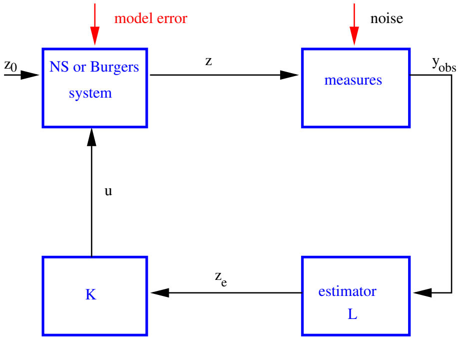
Stabilization by feedback with partial information



1. Estimation problem

We consider

$$z' = Az + f + \mu, \quad z(0) = z_0 + \mu_0.$$

The data f and z_0 are assumed to be known, while μ and η are model errors. We would like to estimate z thanks to some measurements

$$y_{obs}(t) = Hz(t) + \eta(t) \in Y_o.$$

Here, Y_o is the space of observations, $\eta(t)$ is a measure error and $y_{obs}(t)$ is the noisy observation.

Without measurements the only estimation of the state is made by solving the equation

$$z_e' = Az_e + f, \quad z_e(0) = z_0.$$

The error $e = z - z_e$ is

$$e(t) = e^{tA}\mu_0 + \int_0^t e^{(t-s)A}\mu(s)\, ds.$$

The goal is to use the measure $y_{obs}(t)$ to improve the estimation of z.

Example – The simplified linearized inverted pendulum

As in lecture 2, we consider the 'theoretical model'

$$\theta'' = \theta + u, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1,$$

and the 'noisy model'

$$\theta'' = \theta + u + \mu, \quad \theta(0) = \theta_0 + \mu_0, \quad \theta'(0) = \theta_1 + \mu_1.$$

Setting $\rho = \theta'$, we rewrite this system in the form

$$\begin{pmatrix} \theta \\ \rho \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \rho \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume that u is given by the feedback law determined in lecture 2

$$u = -2(\theta + \rho).$$

The next step consists in using an estimation (θ_e, ρ_e) of (θ, ρ) in the feedback law.

We measure θ

$$H\begin{pmatrix}\theta\\ \rho\end{pmatrix} = \theta.$$

Case 1. If the measure of θ is exact, we can evaluate θ' by taking the derivative of the measure, and we do not need the model to estimate the state. We are in the case of a full information coming from the measure.

Case 2. If the model

$$\theta'' = \theta + u, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1,$$

is exact, we can use the closed loop system

$$\theta'' = \theta - 2(\theta + \theta'), \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1,$$

to determine the control. In that case we do not use the measure to estimate the state.

Case 3. If the observation is $\theta_{obs} = \theta + \eta$, where η is a noise. We measure θ but the feedback law depends on θ and $\rho = \theta'$. If we apply the same strategy as in case 1, we are going to use θ_{obs} and θ'_{obs} in the feedback law. The noise is not necessarily differentiable and this naive approach introduces huge errors.

Another approach consists in using an asymptotic state estimation of the form

$$z_e' = Az_e + f + L(Hz_e - y_{obs}), \quad z_e(0) = z_0.$$

The term $L(Hz_e - y_{obs})$ is called a filtering gain and $L \in \mathcal{L}(Y_o, Z)$. The filtering gain is a corrector taking into account the measures.

We look for $L \in (Y_o, Z)$ such that

$$\left(e^{t(A+LH)}\right)_{t\geq 0} \quad \text{is exponentially stable on } Z.$$

When the semigroup $\left(e^{t(A+LH)}\right)_{t\geq 0}$ is exponentially stable on Z, a dynamical system of the form

$$z_e' = Az_e + f + L(Hz_e - y_{obs}), \quad z_e(0) = z_0, \quad \text{with } L \in (Y_o, Z),$$

is called a Luenberger observer.

The equation for the error $e = z - z_e$ is

$$e' = (A + LH)e + \mu - L\eta, \quad e(0) = \mu_0.$$

Theorem. Assume that

$$\|e^{t(A+LH)}\|_{\mathcal{L}(Z)} \leq Ce^{-t\omega}, \quad \text{with } \omega > 0.$$

If $e^{t\omega}\,\eta \in L^2(0,\infty; Y_o)$ and $e^{t\omega}\,\mu \in L^2(0,\infty; Z)$, then

$$\|e\|_{L^2(0,\infty;Z)} \leq Ce^{-t\omega_1}\left(\|\mu_0\|_Z + \|e^{t\omega}\,\eta\|_{L^2(0,\infty;Y_o)} + \|e^{t\omega}\,\mu\|_{L^2(0,\infty;Z)}\right),$$

with $0 < \omega_1 < \omega$.

2. Observability and Detectability of finite dimensional systems.

Observability.

An initial condition $z_0 \in Z$ is unobservable for the pair (A, H) when

$$H\, e^{tA}\, z_0 = 0 \quad \text{for all } t \geq 0.$$

For finite dimensional systems, an initial condition $z_0 \in Z$ is unobservable for the pair (A, H) if and only if it is not reachable for the pair (A^*, H^*). Indeed, if $z_0 \in Z$ is unobservable, then

$$0 = \int_0^T \big(y(t), H\, e^{tA}\, z_0\big)_{Y_o}\, dt = \int_0^T \Big(e^{tA^*}\, H^* y(t), z_0\Big)_Z\, dt$$

for all $y \in L^2(0, T; Y_o)$. The converse is obvious.

A system (A, H) of finite dimension is observable when the set of unobservable state is reduced to $\{0\}$.

The pair (A, H) is observable iff the pair (A^*, H^*) is controllable.

Detectability.

The dual notion of stabilizability is the notion of detectability. We say that the pair (A, H) is detectable iff there exists $L \in (Y_o, Z)$ such that

$$\left(e^{t(A+LH)}\right)_{t\geq 0} \quad \text{is exponentially stable on } Z.$$

The semigroup generated by $A + LH$ is exponentially stable on Z iff the semigroup generated by $A^* + H^*L^*$ is exponentially stable on $Z^* \equiv Z$.

This means that the pair (A, H) is detectable iff the pair (A^*, H^*) is stabilizable. One way to find L^* such that $A^* + H^*L^*$ is exponentially stable consists in using a stabilizing feedback control by solving the Riccati equation

$$P_e = P_e^* \geq 0, \quad P_eA^* + AP_e - P_eH^*HP_e + I = 0.$$

Taking into account the knowledge of covariance noises, we can solve a Riccati equation of the form

$$P_e = P_e^* \geq 0, \quad P_eA^* + AP_e - P_eH^*\, R_\eta^{-1}\, HP_e + Q_\mu = 0,$$

where $R_\eta \in \mathcal{L}(Y_o)$ and $Q_\mu \in \mathcal{L}(Z)$ are two symmetric and semidefinite positive operators (the covariance operators of the noises).

Detectability of finite dimensional systems

Example 1

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + f + \mu, \quad z(0) = z_0 + \mu_0.$$

Let us take

$$H_1 z = z_1 + z_2.$$

The state $z_0 = (1, -1)^T$ is not observable. But if $\lambda < 0$, then (A, H_1) is detectable. If $\lambda > 0$, the pair (A, H_1) is not detectable.

If $H_2 z = (z_1 + z_2, z_1 - z_2)$, then (A, H_2) is detectable. (We have a full 'noisy' information.)

Example 2

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + f + \mu, \quad z(0) = z_0 + \mu_0.$$

Let us take

$$H_1 z = z_1 + z_2.$$

If $\lambda_1 \neq \lambda_2$, the pair (A, H_1) is detectable and observable.

- The pair (A, H) is detectable iff there exists $\alpha > 0$ such that

$$\int_0^\infty \left\| H\,\pi_u\, e^{-tA_u} z \right\|_{Y_o}^2 \, dt \geq \alpha \|z\|_Z^2.$$

- The best constant $\alpha > 0$ can be taken as an evaluation of *a degree of detectability*.

- The pair (A, H) is detectable iff, for each 'unstable' eigenvalue λ_j of A, the corresponding family of eigenvectors (eigenfunctions) $(e_j^k)_{1\leq k\leq \ell_j}$ is such that the family

$$(He_j^1, He_j^2, \cdots, He_j^{\ell_j})$$

is linearly independent.

Example 3

$$z' = Az + f + \mu, \quad z(0) = z_0 + \mu_0, \quad z(t) \in \mathbb{R}^N, \quad A = \mathrm{diag}(\lambda_1, \cdots, \lambda_N),$$

$$Hz(t) = \textstyle\sum_{i=1}^N z_i(t) \in \mathbb{R} \quad \text{and} \quad y_{obs}(t) = Hz(t) + \eta(t).$$

We are going to see that if the eigenvalues are two by two distinct, then the measure Hz is enough to estimate z.

Assume that all the eigenvalues are unstable and of multiplicity equal to 1. We have

$$H = (1, \cdots, 1).$$

The eigenvectors are the basis vectors e_i, $1 \leq i \leq N$. We have

$$He_i = 1.$$

Thus the detectability condition is trivially satisfied. However, we have

$$e^{-tA_u^*} H^* H e^{-tA_u} = \mathrm{diag}(e^{-\lambda_1 t}, \cdots, e^{-\lambda_N t})\, H^* H\, \mathrm{diag}(e^{-\lambda_1 t}, \cdots, e^{-\lambda_N t}),$$

$$H^* H = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

and

$$\int_0^\infty e^{-tA_u^*} H^* H e^{-tA_u} dt =$$

$$= \begin{pmatrix} \frac{1}{2\lambda_1} & \cdots & \frac{1}{\lambda_1+\lambda_j} & \cdots & \frac{1}{\lambda_1+\lambda_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_i+\lambda_1} & \cdots & \frac{1}{2\lambda_i} & \cdots & \frac{1}{\lambda_i+\lambda_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_N+\lambda_1} & \cdots & \frac{1}{\lambda_N+\lambda_j} & \cdots & \frac{1}{2\lambda_N} \end{pmatrix}.$$

In the case when $N = 2$,

$$\det \int_0^\infty e^{-tA_u^*} H^* H e^{-tA_u} dt = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right)^2.$$

Thus if two eigenvalues are very close, the pair (A, H) is 'weakly' detectable.

Example 4 – The linearized inverted pendulum

We measure θ and x. We have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The controllability matrix of (A^*, H_1^*) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{mg}{M} & \frac{g(M+m)}{M\ell} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{mg}{M} & \frac{g(M+m)}{M\ell} \end{pmatrix}.$$

Thus (A^*, H_1^*) is stabilizable and (A, H_1) is detectable.

If we choose the measure operator

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

the controllability matrix of (A^*, H_2^*) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & -\frac{mg}{M} \end{pmatrix}.$$

Thus (A^*, H_2^*) is also stabilizable and (A, H_2) is detectable.

3. Calculation of filtering gains L

The deterministic approach to find L

Assume that $\mu_0 = 0$. When μ is a white Gaussian noise with mean value zero and with covariance Q_μ ($Q_\mu = Q_\mu^* \geq 0$), and when η is a white Gaussian noise with mean value zero and with covariance R_η ($R_\eta = R_\eta^* > 0$), it can be shown that the best linear estimator of z, without bias, knowing y_{obs}, is the solution to the problem

$$(\mathcal{EP}) \qquad \begin{aligned} &\inf J(z,\mu,\eta), \quad (z,\mu,\eta) \text{ obeys} \\ &z' = Az + f + \mu, \quad z(0) = z_0, \\ &y_{obs}(t) = Hz(t) + \eta(t) \in Y_o, \end{aligned}$$

and

$$J(z,\mu,\eta) = \frac{1}{2}\int_0^\infty \Big(R_\eta^{-1}(Hz - y_{obs}), Hz - y_{obs}\Big)_{Y_o} + \frac{1}{2}\int_0^\infty (Q_\mu^{-1}\mu,\mu)_Z.$$

The weights R_η^{-1} and Q_μ^{-1} are used to obtain the best ponderation between the model error and the measure error.

When $Q_\mu = 0$, then $\mu = 0$. The model error is equal to zero and the best estimation (which is also the exact one) is given by

$$z' = Az + f, \quad z(0) = z_0.$$

At the opposite, if $R_\eta = 0$, then $\eta = 0$ and we have to use the equations $y_{obs}(t) = Hz(t)$ and $z' = Az + f + \mu$, $z(0) = z_0$ to identify μ.

Does $(\mathcal{EP})$ admit a solution ? How to characterize it ?

The existence of solutions to $(\mathcal{EP})$ is related to what is called the detectability of the pair (A, H). Let us explain why.

Let us denote by $z_{z_0,f}$ the solution to

$$z'_{z_0,f} = Az_{z_0,f} + f, \quad z_{z_0,f}(0) = z_0,$$

set $\zeta = z - z_{z_0,f}$, $\bar{y}_{obs} = y_{obs} - Hz_{z_0,f}$ and

$$I(\zeta, \mu, \eta) = \frac{1}{2}\int_0^\infty \Big(R_\eta^{-1}(H\zeta - \bar{y}_{obs}), H\zeta - \bar{y}_{obs}\Big)_{Y_o} + \frac{1}{2}\int_0^\infty (Q_\mu^{-1}\mu, \mu)_Z.$$

Problem $(\mathcal{EP})$ is transformed as follows

$$(\mathcal{NEP}) \qquad \begin{aligned} & \inf I(\zeta,\mu,\eta), \quad (\zeta,\mu,\eta) \text{ obeys} \\ & \zeta' = A\zeta + \mu, \quad \zeta(0) = 0, \\ & \bar{y}_{obs}(t) = H\zeta(t) + \eta(t) \in Y_o. \end{aligned}$$

Assume that this problem admits a unique solution and let us write *formally* the optimality system

$$\begin{aligned} & \zeta' = A\zeta - Q_\mu \phi, \quad \zeta(0) = 0, \\ & -\phi' = A^*\phi + H^* R_\eta^{-1}(H\zeta - \bar{y}_{obs}), \quad \phi(\infty) = 0. \end{aligned}$$

In this *primal formulation*, ζ is the state variable and ϕ the adjoint state. We can look for a *dual problem* in which ϕ will be the state variable and ζ the adjoint state. For that, we set $\psi = R_\eta^{-1/2} H\zeta$ and we rewrite the above system as follows

$$\begin{aligned} & \zeta' = A\zeta - Q_\mu^{1/2} Q_\mu^{1/2} \phi, \quad \zeta(0) = 0, \\ & -\phi' = A^*\phi + H^* R_\eta^{-1/2}\psi - H^* R_\eta^{-1}\bar{y}_{obs}, \quad \phi(\infty) = 0. \end{aligned}$$

We can verify that this system is the O.S. of the dual problem

$$(\mathcal{DP})\qquad \begin{array}{l}\inf F(\phi,\psi), \quad (\phi,\psi) \text{ obeys} \\ -\phi' = A^*\phi + H^* R_\eta^{-1/2}\psi - H^* R_\eta^{-1}\bar{y}_{obs}, \quad \phi(\infty) = 0,\end{array}$$

where

$$F(\phi,\psi) = \frac{1}{2}\int_0^\infty \Big(Q_\mu\phi,\phi\Big)_Z + \frac{1}{2}\int_0^\infty \|\psi\|^2_{Y_o}.$$

For the well posedness of the state equation of $(\mathcal{DP})$, we need that (A^*, H^*) is stabilizable. This is exactly equivalent to the definition of the detectability of the pair (A, H). The Riccati equation for $(\mathcal{DP})$ is

$$P_e = P_e^* \geq 0, \quad P_e A^* + AP_e - P_e H^* R_\eta^{-1} HP_e + Q_\mu = 0.$$

This Riccati equation enable us to define a Luenberger observer by setting

$$L = -P_e H^* R_\eta^{-1}.$$

Then $A^* + H^* L^*$ and $A + LH$ are exponentially stable.

The stochastic approach to find L

We assume that μ is a Gaussian stationary white noise with mean value zero and covarinace Q_μ, η is a Gaussian stationary white noise with mean value zero and covariance R_η. Moreover we assume that μ and η are independent.

The Kalman filtering consists in choosing the estimation z_e of z in such a way that the covariance of the error $e = z - z_e$ is minimized.
Let us look for an estimation of the form

$$z_e' = Az_e + f + L(Hz_e - y_{obs}), \quad z_e(0) = z_0.$$

Let us look for L which minimzes the covariance of the error. The equation for the error e is

$$e' = (A + LH)e + \mu + L\eta, \quad e(0) = \mu_0.$$

The mean value of the error is

$$E[e(t)] = e^{t(A+LH)}E[\mu_0].$$

If $A + LH$ is stable then $\lim_{t\to\infty} E[e(t)] = 0$.
The covariance of estimation error is

$$E[e(t)e(t)^T] = P_e.$$

In the case when $E[\mu_0] = 0$ (or if we look for the asymptotic regime), the covariance P_e satisfies the Lyapunov equation

$$(A + LH)P_e + P_e(A + LH)^T + Q_\mu + LR_\eta L^T = 0.$$

As in the calculation of the minimizing feedback control, it can be shown that the filtering gain minimizing the covariance of the error is

$$L = -P_e\, H^T\, R_\eta^{-1}.$$

The corresponding covariance is the solution to the Riccati equation

$$P_e = P_e^* \geq 0, \quad P_e A^* + AP_e - P_e H^*\, R_\eta^{-1}\, HP_e + Q_\mu = 0.$$

Thus, we recover the same filtering gain as in the deterministic approach.

The simplified linearized inverted pendulum revisited

We have

$$\begin{pmatrix} \theta \\ \rho \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \rho \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\theta_{obs} = \theta + \eta.$$

We assume that a, the variance of η, is positive and b, the variance of μ is also positive. We have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = (1\ 0).$$

The pair (A^*, H^*) is stabilizable and the pair (A, H) is detectable. The minimization problem giving the best estimator is

$$\inf \frac{1}{2a} \int_0^\infty (\theta - \theta_{obs})^2 dt + \frac{1}{2b} \int_0^\infty \mu^2 dt,$$
$$\theta'' = \theta + u + \mu, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1.$$

The Riccati equation for the filtering operator is

$$P_e A^* + AP - \frac{1}{a} P_e H^* H P_e + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We obtain

$$p_{12} + p_{21} - \frac{1}{a} p_{11}^2 = 0, \quad p_{11} + p_{22} - \frac{1}{a} p_{11} p_{12} = 0,$$
$$p_{11} + p_{22} - \frac{1}{a} p_{11} p_{21} = 0, \quad p_{21} + p_{12} - \frac{1}{a} p_{21} p_{12} + b = 0.$$

Since $p_{12} = p_{21}$, we obtain

$$2p_{12} - \frac{1}{a} p_{11}^2 = 0, \quad p_{11} + p_{22} - \frac{1}{a} p_{11} p_{12} = 0,$$
$$-2a\, p_{12} + p_{12}^2 - ab = 0.$$

Which gives $p_{12} = a + \sqrt{a^2 + ab}$ and

$$p_{11} = \sqrt{2a^2 + 2a\sqrt{a^2 + ab}}, \quad p_{22} = \sqrt{2a + 2\sqrt{a^2 + ab}}\,\sqrt{a^2 + b}.$$

4. Coupling between control and estimation

The stabilization problem with full information. Consider a noisy control system

$$z' = Az + Bu + \mu, \quad z(0) = z_0 + \mu_0.$$

Assume that (A, B) is stabilizable, we can find $K \in \mathcal{L}(Z, U)$, such that $A + BK$ is exponentially stable on Z, by solving an Algebraic Riccati Equation of the form

$$P = P^* \geq 0, \quad A^*P + PA - PBR^{-1}B^*P + Q = 0,$$

with $R = R^* > 0$ and $Q = Q^* \geq 0$, and by choosing

$$K = -R^{-1}B^*P.$$

The estimation problem. We solve the following A.R.E.

$$P_e = P_e^* \geq 0, \quad AP_e + P_eA^* - P_eH^*R_\eta^{-1}HP_e + Q_\mu = 0,$$

and we choose $L = -P_eH^*R_\eta^{-1}$.

When we compare the Riccati equation for P_e and the Riccati equation used to define the control law

$$P = P^* \geq 0, \quad A^*P + PA - PBR^{-1}B^*P + Q = 0,$$

we notice that the roles of A and A^* are interchanged.

Next we use the filtering equation to determine the control by solving

$$z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_0.$$

After that we prove that the original system with the feedback coming from the estimator

$$z' = Az + BKz_e + \mu, \quad z(0) = z_0 + \mu_0,$$

is stable. Indeed the system satisfied by $(z, z_e)^T$ is

$$\begin{pmatrix} z \\ z_e \end{pmatrix}' = \begin{pmatrix} A & BK \\ -LH & A + BK + LH \end{pmatrix} \begin{pmatrix} z \\ z_e \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}.$$

Theorem. If $(e^{t(A+BK)})_{t\geq 0}$ is exponentially stable and if $(e^{t(A+LH)})_{t\geq 0}$ is exponentially stable, then the semigroup generated by

$$\mathcal{A} = \begin{bmatrix} A & BK \\ -LH & A + BK + LH \end{bmatrix}$$

is also exponentially stable on $Z \times Z$.

Proof. If $e = z - z_e$, we have

$$\begin{pmatrix} z \\ e \end{pmatrix}' = \begin{pmatrix} A + BK & -BK \\ 0 & A + LH \end{pmatrix} \begin{pmatrix} z \\ e \end{pmatrix} + \begin{pmatrix} \mu \\ L\eta \end{pmatrix}.$$

5. Local stabilization of nonlinear systems

Let us recall that we want to stabilize the nonlinear system

$$z' = Az + Bu + N(z) + R(z)u + \mu, \quad z(0) = z_0 + \mu_0,$$

where $z = (x, x', \theta, \theta')^T$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{M\ell} \end{pmatrix},$$

$$z_0 = (x_0, x_1, \theta_0, \theta_1)^T$$

with the measure

$$y_{obs}(t) = Hz(t) + \eta(t), \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The two nonlinear terms N and R obey $N(0) = 0$ and $R(0) = 0$.

We use the feedback and the filtering gains K and L determined for the linearized model, and we solve

$$\begin{aligned} z' &= Az + Bu + N(z) + R(z)u + \mu, \quad z(0) = z_0 + \mu_0, \\ z_e' &= Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_0, \\ y_{obs}(t) &= Hz(t) + \eta(t). \end{aligned}$$

We show, and we verify numerically, that this system is locally stable.