Numerics and Control of PDEs

Lecture 5

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Boundary stabilization of the 1D Heat equation in the case of full information

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1. Boundary stabilization of the 1D heat equation

We start with

$$(NHBC)\qquad \begin{aligned} &\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1)\times(0,T),\\ &z(0,t) = 0 \quad \text{and} \quad z(1,t) = u(t) \quad \text{for } t\in(0,T),\\ &z(x,0) = z_0(x) \quad \text{in } (0,1). \end{aligned}$$

We already know that if $u = 0$, then

$$\|z(t)\|_{L^2(0,1)} \leq e^{\lambda_1 t}\|z_0\|_{L^2(0,1)} = e^{-\pi^2 t}\|z_0\|_{L^2(0,1)}.$$

Thus the solution is already stable, and we have nothing to do. If we look for a faster exponential decay $e^{-\omega t}$ with $\omega > 0$, we can introduce

$$\widehat{z} = e^{\omega t} z \quad \text{and} \quad \widehat{u} = e^{\omega t} u,$$

the PDE satisfied by $\widehat{z}$ is

$$(HE)\qquad \begin{aligned} &\frac{\partial \widehat{z}}{\partial t} - \frac{\partial^2 \widehat{z}}{\partial x^2} - \omega\widehat{z} = 0 \quad \text{in } (0,1)\times(0,T),\\ &\widehat{z}(0,t) = 0 \quad \text{and} \quad \widehat{z}(1,t) = \widehat{u}(t) \quad \text{for } t\in(0,T),\\ &\widehat{z}(x,0) = z_0(x) \quad \text{in } (0,1). \end{aligned}$$

Now

$$\|\widehat{z}(t)\|_{L^2(0,1)} \geq e^{(\lambda_1+\omega)t}|\zeta_{0,1}| = e^{(\omega-\pi^2)t}|\zeta_{0,1}| \longrightarrow +\infty \quad as \quad t \longrightarrow +\infty,$$

if $\zeta_{0,1} \neq 0$ and $\omega > \pi^2$.

If we choose a control $\widehat{u}$ stabilizing $\widehat{z}$, the corresponding control u will stabilize z with the exponential decay rate $e^{-\omega t}$. The heat equation (HE) may be written as an infinite dimensional system satisfied by the Fourier coefficients of $\widehat{z}$

$$(IDS) \qquad \begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \end{bmatrix}' = \begin{bmatrix} \omega+\lambda_1 & 0 & \dots \\ 0 & \omega+\lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \end{bmatrix} + \widehat{u}(t) \begin{bmatrix} -\xi_{1,x}(1) \\ -\xi_{2,x}(1) \\ \vdots \end{bmatrix},$$

where $\xi_{i,x}(1)$ denotes $\frac{d\xi_i}{dx}(1)$.

The stabilizability of (HE) is equivalent to the stabilizability of (IDS).
We set $D(A) = H^2(0,1) \cap H_0^1(0,1)$, and $Az = \frac{d^2 z}{dx^2}$. We notice that $A = A^*$. We can always choose $\omega > 0$ so that $-\omega \notin \sigma(A)$. We have

$$\dots < \lambda_{N_\omega+1} < -\omega < \lambda_{N_\omega} < \dots < \lambda_1.$$

where $\lambda_k = -k^2\,\pi^2$ are the eigenvalue of A.

The stabilizability of (IDS) is equivalent to the stabilizability of the projected unstable system

$$\begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \\ \widehat{\zeta}_{N_\omega} \end{bmatrix}' = \begin{bmatrix} \omega+\lambda_1 & 0 & \dots & 0 \\ 0 & \omega+\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \omega+\lambda_{N_\omega} \end{bmatrix} \begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \\ \widehat{\zeta}_{N_\omega} \end{bmatrix} + \widehat{u}(t) \begin{bmatrix} -\xi_{1,x}(1) \\ -\xi_{2,x}(1) \\ \vdots \\ -\xi_{N_\omega,x}(1) \end{bmatrix}.$$

We can use the Hautus criterion for studying the stabilizability

$$\forall j \in \{1,\cdots,N_\omega\}, \quad \mathrm{Ker}(\lambda_j I - A^* - \omega I) \cap \mathrm{Ker}(B^*) = \{0\}.$$

or

$$\forall j \in \{1,\cdots,N_\omega\}, \quad \mathrm{Ker}(\lambda_j I - \Lambda_{\omega,u}) \cap \mathrm{Ker}(\mathbb{B}^*_{\omega,u}) = \{0\}.$$

The eigenvectors belonging to $\mathrm{Ker}(\lambda_j I - A^* - \omega I)$ are $\alpha\,\xi_j$. And $B^*(\alpha\,\xi_j) = -\alpha\,\xi_{j,x}(1)$. Thus, if $B^*(\alpha\,\xi_j) = 0$ then $\alpha = 0$. This means that the Hautus criterion is satisfied and the system is stabilizable.

2. Infinite time horizon optimal control problem

We recall the equation satisfied by $(\widehat{z}, \widehat{u})$ that, for simplicity, we denote by (z, u)

$$(HE_0^\infty) \qquad \begin{aligned} &\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} - \omega z = 0 \quad \text{in } (0,1)\times(0,\infty),\\ &z(0,t) = 0 \quad \text{and} \quad z(1,t) = u(t) \quad \text{for } t \in (0,\infty),\\ &z(x,0) = z_0(x) \quad \text{in } (0,1). \end{aligned}$$

We rewrite (HE) in the form

$$z' = (A + \omega I)z + Bu = A_\omega z + Bu, \quad z(0) = z_0.$$

We introduce the functional

$$J_0^\infty(z,u) = \frac{1}{2}\int_0^\infty |Cz(t)|_Y^2 + \frac{R}{2}\int_0^\infty |u(t)|^2,$$

where $C \in \mathcal{L}(Z, Y)$ and Y is another Hilbert space. We look for u solution to the optimal control problem

$$(\mathcal{P}_{0,z_0}^\infty) \qquad \inf\{J_0^\infty(z,u) \mid (z,u) \text{ obeys } (HE_0^\infty)\}.$$

Theorem. For all $z_0 \in Z$, problem $(\mathcal{P}_{0,z_0}^{\infty})$ admits a unique solution $u_{z_0}^{\infty}$.

Proof. Since the pair (A, B) is stabilizable, there exists $u \in L^2(0, \infty)$ such that

$$J_0^{\infty}(z_{z_0,u}, u) < \infty.$$

The existence of solutions can be proved by using a minimizing sequence and by passing to the limit.

To determine this optimal control, We approximate the problem $(\mathcal{P}_{0,z_0}^{\infty})$ by a sequence of finite time horizon control problem $(\mathcal{P}_{0,z_0}^{k})$ defined over the time interval $(0, k)$. We determine the solution $u_{z_0}^k$ of $(\mathcal{P}_{0,z_0}^{k})$. We denote by $\tilde{u}_{z_0}^k$ the extension of $u_{z_0}^k$ by zero to (k, ∞). We show that

$$\tilde{u}_{z_0}^k \longrightarrow u_{z_0} \quad \text{in } L^2(0, \infty) \quad \text{as } k \to \infty.$$

3. Finite time horizon optimal control problem

We approximate the problem $(\mathcal{P}^{\infty}_{0,z_0})$ by a sequence of finite time horizon control problem. We introduce the functional

$$J_0^k(z,u) = \frac{1}{2}\int_0^k |Cz(t)|_Y^2 + \frac{R}{2}\int_0^k |u(t)|^2,$$

and the equation

$$(HE_0^k) \qquad z' = A_\omega z + Bu \quad \text{in } (0,k), \quad z(0) = z_0.$$

We look for u solution to the optimal control problem

$$(\mathcal{P}^k_{0,z_0}) \qquad \inf\{J_0^k(z,u) \mid (z,u) \text{ obeys } (HE_0^k)\}.$$

Theorem. For all $z_0 \in Z$, problem $(\mathcal{P}^k_{0,z_0})$ admits a unique solution $u^k_{z_0}$. If $z^k_{z_0}$ is the solution of (HE) corresponding to $u^k_{z_0}$, then $u^k_{z_0} = -R^{-1}B^*\phi$, where ϕ is the solution to the adjoint equation

$$-\phi' = A_\omega\phi + C^*Cz^k_{z_0}, \quad \phi(k) = 0.$$

Conversely, the system

$$z' = A_\omega z - BR^{-1}B^*\phi, \quad z(0) = z_0,$$
$$-\phi' = A_\omega\phi + C^*Cz, \quad \phi(k) = 0,$$

admits a unique solution $(z^k_{z_0}, \phi^k_{z_0})$ and the optimal control is defined by $u^k_{z_0} = -R^{-1}B^*\phi^k_{z_0}$.

The infimum value is

$$\inf(\mathcal{P}^k_{0,z_0}) = \frac{1}{2}\left(z_0, \phi^k_{z_0}(0)\right)_{L^2(0,1)}.$$

The function $\phi^k_{z_0} \in C([0,k];Z)$ and the mapping

$$P(k) \ : \ z_0 \longmapsto \phi^k_{z_0}(0),$$

is linear and continuous in Z. Moreover $P(k) = P(k)^* \geq 0$.

4. Closed loop stabilization - Feedback control - Riccati equation

Passage to the limit when $k \to \infty$.

Convergence of the sequence of controls. We denote by $(z_{z_0}^k, u_{z_0}^k)$ the solution to $(\mathcal{P}_{0,z_0}^k)$, by $(\tilde{z}_{z_0}^k, \tilde{u}_{z_0}^k)$ the extension of $(z_{z_0}^k, u_{z_0}^k)$ by 0 to (k, ∞). We have

$$J_0^\infty(\tilde{z}_{z_0}^k, \tilde{u}_{z_0}^k) \le J_0^k(z_{z_0}^k, u_{z_0}^k) \le J_0^k(z_{z_0}^\infty, u_{z_0}^\infty) \le J_0^\infty(z_{z_0}^\infty, u_{z_0}^\infty) < \infty.$$

Thus the sequence $(\tilde{u}_{z_0}^k)_k$ is bounded in $L^2(0,\infty)$. From any subsequnce, we can extract a subsequence converging weakly in $L^2(0,\infty)$. We show that the weak limit is $u_{z_0}^\infty$, and next we show that the convergence is strong in $L^2(0,\infty)$.

Convergence of the sequence $P(k)$. The mapping $k \mapsto (P(k)z_0, z_0)_{L^2(0,1)}$ is increasing and bounded. We can show that for all $z_0 \in Z$, the sequence $(P(k)z_0)_k$ converges to Pz_0 and that

$$\inf(\mathcal{P}_{0,z_0}^\infty) = \frac{1}{2}\left(Pz_0, z_0\right)_{L^2(0,1)}.$$

Moreover the operator P is the unique solution of the following Riccati equation

$$(ARE) \qquad \begin{aligned} & P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \\ & PA_\omega + A_\omega^* - PBR^{-1}B^*P + C^*C = 0, \\ & A_\omega - BR^{-1}B^*P \quad \text{is stable.} \end{aligned}$$

To stabilize the heat equation with the exponential decay rate $e^{-\omega t}$, in the case of full information, we solve the system

$$(CLS) \qquad \begin{aligned} & \frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1) \times (0,\infty), \\ & z(0,t) = 0, \quad z(1,t) = -B^*Pz = \frac{\partial Pz}{\partial x}\Big|_{x=1}, \quad \text{for } t \in (0,\infty), \\ & z(x,0) = z_0(x) \quad \text{in } (0,1), \end{aligned}$$

where P is the solution to the (ARE).

5. Finite dimensional approximation of the feedback control

Using the P_1 FEM we obtain the system

$$\mathbf{E}\mathbf{z}'(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{B}u(t), \quad \mathbf{E}\mathbf{z}(0) = ((z_0, \phi_i)_{L^2})_{1\le i\le N-1},$$

where $\mathbf{z} = (z_1, \cdots, z_{N-1})^T$ and the solution to the approximate heat equation is

$$z = \sum_{i=1}^{N-1} z_i\, \phi_i + u\, \phi_N.$$

We set

$$\mathbf{A}_\omega = \mathbf{A} + \omega\mathbf{E},$$

and to calculate the feedback gain, we can solve the matrix Riccati equation

$$\begin{aligned} & \mathbf{P} \in \mathcal{L}(\mathbb{R}^{N-1}), \quad \mathbf{P} = \mathbf{P}^T \ge 0, \\ (MRE) \qquad & \mathbf{P}\mathbf{E}^{-1}\mathbf{A}_\omega + \mathbf{A}_\omega^T\mathbf{E}^{-1}\mathbf{P} - \mathbf{P}\mathbf{E}^{-1}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{E}^{-1}\mathbf{P} + \mathbf{C}^T\mathbf{C} = 0, \\ & \mathbf{E}^{-1}\mathbf{A}_\omega - \mathbf{E}^{-1}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{E}^{-1}\mathbf{P} \quad \text{is stable,} \end{aligned}$$

where $\mathbf{C}\mathbf{z}$ is a discrete approximation of Cz. For example if $C = I_z$ we choose $\mathbf{C} = \mathbf{E}^{1/2}$.

Let us notice that $\mathbf{P}$ is the solution to the (MRE) if and only if $\mathbf{\Pi} = \mathbf{E}^{-1}\mathbf{P}\mathbf{E}^{-1}$ is the solution to the Generalized Matrix Riccati Equation ($GMRE$)

$$(GMRE) \quad \begin{aligned} & \mathbf{\Pi} \in \mathcal{L}(\mathbb{R}^{N-1}), \quad \mathbf{\Pi} = \mathbf{\Pi}^T \geq 0, \\ & \mathbf{\Pi}\mathbf{A}_\omega\mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{A}_\omega^T\mathbf{\Pi} - \mathbf{\Pi}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{\Pi} + \mathbf{E}^{-1}\mathbf{C}^T\mathbf{C}\mathbf{E}^{-1} = 0, \\ & \mathbf{E}\mathbf{z}' = (\mathbf{A}_\omega - \mathbf{B}R^{-1}\mathbf{B}^T\mathbf{\Pi}\mathbf{E})\mathbf{z} \quad \text{is stable,} \end{aligned}$$

or similarly to the equivalent equation

$$(GMRE) \qquad \begin{aligned} & \mathbf{\Pi} \in \mathcal{L}(\mathbb{R}^{N-1}), \quad \mathbf{\Pi} = \mathbf{\Pi}^T \geq 0, \\ & \mathbf{E}\mathbf{\Pi}\mathbf{A}_\omega + \mathbf{A}_\omega^T\mathbf{\Pi}\mathbf{E} - \mathbf{E}\mathbf{\Pi}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{\Pi}\mathbf{E} + \mathbf{C}^T\mathbf{C} = 0, \\ & \mathbf{E}\mathbf{z}' = (\mathbf{A}_\omega - \mathbf{B}R^{-1}\mathbf{B}^T\mathbf{\Pi}\mathbf{E})\mathbf{z} \quad \text{is stable.} \end{aligned}$$

The stability condition

$$\mathbf{E}\mathbf{z}' = (\mathbf{A}_\omega - \mathbf{B}R^{-1}\mathbf{B}^T\mathbf{\Pi}\mathbf{E})\mathbf{z} \quad \text{is stable,}$$

will be written as

$$(\mathbf{E}, \mathbf{A}_\omega - \mathbf{B}R^{-1}\mathbf{B}^T\mathbf{\Pi}\mathbf{E}) \quad \text{is stable.}$$

The closed loop system (*CLS*) is approximated by

$$\mathbf{E}\mathbf{z}'(t) = (\mathbf{A} - \mathbf{B}R^{-1}\mathbf{B}^T\mathbf{\Pi}\mathbf{E})\mathbf{z}(t), \quad \mathbf{E}\mathbf{z}(0) = ((z_0, \phi_i)_{L^2})_{1\le i\le N-1},$$

and the optimal control is

$$u(t) = -R^{-1}\mathbf{B}^T\mathbf{\Pi}\mathbf{E}\mathbf{z}(t).$$

We are going to compare particular choices for C. When $C = 0$, the *ARE* is called the Bernoulli equation. It gives the control of minimal norm.

6.a. Bernoulli equation. C $= 0$

We know consider the problem

$$(\mathcal{P}^\infty_{0,z_0}) \qquad \inf\{J_0^\infty(z,u) \mid (z,u) \text{ obeys } (HE) \text{ and } \lim_{t\to\infty} \|z(t)\|_Z = 0\}.$$

with

$$(HE) \qquad \begin{aligned} &\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} - \omega z = 0 \quad \text{in } (0,1)\times(0,\infty), \\ &z(0,t) = 0 \quad \text{and} \quad z(1,t) = u(t) \quad \text{for } t \in (0,T), \\ &z(x,0) = z_0(x) \quad \text{in } (0,1), \end{aligned}$$

with $\omega = 10$ and

$$J_0^\infty(z,u) = \frac{1}{2}\int_0^\infty |u(t)|^2.$$

We look at the solution to the following Bernoulli equation

$$(ABE) \qquad \begin{aligned} &P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \\ &PA_\omega + A_\omega^* - PBR^{-1}B^*P = 0, \\ &A_\omega - BR^{-1}B^*P \quad \text{is stable.} \end{aligned}$$

Representing the system by using the Hilbertian basis $(\xi_i)_{i\in\mathbb{N}^*}$, we have to deal with the infinite dimensional system

$$(IDS) \quad \begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \end{bmatrix}' = \begin{bmatrix} 10+\lambda_1 & 0 & \dots \\ 0 & 10+\lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \end{bmatrix} + \widehat{u}(t) \begin{bmatrix} -\xi_{1,x}(1) \\ -\xi_{2,x}(1) \\ \vdots \end{bmatrix}.$$

We introduce the operators

$$\Lambda_\omega = \begin{bmatrix} 10+\lambda_1 & 0 & \dots \\ 0 & 10+\lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad \mathbb{B} = \begin{bmatrix} -\xi_{1,x}(1) \\ -\xi_{2,x}(1) \\ \vdots \end{bmatrix}.$$

Since $10+\lambda_1$ is the only unstable eigenvalue of Λ_ω, we use a decomposition by blocks of the form

$$\Lambda_\omega = \begin{bmatrix} 10+\lambda_1 & 0 \\ 0 & \Lambda_{\infty,\infty} \end{bmatrix} \quad \text{and} \quad \mathbb{B}\mathbb{B}^* = \begin{bmatrix} S_{1,1} & S_{1,\infty} \\ S_{\infty,1} & S_{\infty,\infty} \end{bmatrix},$$

with $S_{1,1} = (\xi_{1,x}(1))^2$.

We choose $R = 1$. We look for the solution to the (ABE) in the form

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_{1,1} & \mathbb{P}_{1,\infty} \\ \mathbb{P}_{\infty,1} & \mathbb{P}_{\infty,\infty} \end{bmatrix}.$$

We know that the solution is unique. We look for a solution such that

$$\mathbb{P}_{1,\infty} = 0, \quad \mathbb{P}_{\infty,1} = 0, \quad \mathbb{P}_{\infty,\infty} = 0.$$

If such a solution exists $\mathbb{P}_{1,1} \in \mathbb{R}$ is the solution to the following Bernoulli equation in $\mathbb{R}$

$$\mathbb{P}_{1,1} > 0, \quad \mathbb{P}_{1,1}(10 + \lambda_1) + (10 + \lambda_1)\mathbb{P}_{1,1} - \mathbb{P}_{1,1}(\xi_{1,x}(1))^2\mathbb{P}_{1,1} = 0.$$

with $\xi_{1,x}(1) = -\pi\sqrt{2}$. Notice that this equation is nothing else than

$$\mathbb{P}_{\omega,u} > 0, \quad \mathbb{P}_{\omega,u}\Lambda_{\omega,u} + \Lambda_{\omega,u}\mathbb{P}_{\omega,u} - \mathbb{P}_{\omega,u}\mathbb{B}_{\omega,u}\mathbb{B}^*_{\omega,u}\mathbb{P}_{\omega,u} = 0.$$

We obtain $\mathbb{P}_{1,1} = \frac{2(10-\pi^2)}{2\pi^2}$. To verify that

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_{1,1} & 0 \\ 0 & 0 \end{bmatrix},$$

solves the Bernoulli equation, we have to prove that the operator $\Lambda_\omega - \mathbb{B}\mathbb{B}^*\mathbb{P}$ is stable.

Before proving that $\Lambda_\omega - \mathbb{B}\mathbb{B}^*\mathbb{P}$ is stable, we can observe that looking at the solution $\mathbb{P}$ of the form

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_{1,1} & 0 \\ 0 & 0 \end{bmatrix},$$

is equivalent to looking at the feedback stabilization of the first component.
Indeed the equation satisfied by ζ_1 is

$$\widehat{\zeta}_1' = (10 - \pi^2)\widehat{\zeta}_1 - \widehat{u}(t)\xi_{1,x}(1) = (10 - \pi^2)\widehat{\zeta}_1 + \widehat{u}(t)\,\pi\sqrt{2},$$
$$\widehat{\zeta}_1(0) = (z_0, \xi_1)_{L^2(0,1)}.$$

The Bernoulli equation for this system is

$$p > 0, \quad 2(10 - \pi^2)p - (\pi\sqrt{2})^2 p^2 = 0.$$

Thus

$$p = \frac{2(10 - \pi^2)}{2\pi^2}.$$

We notice that $p = \mathbb{P}_{1,1}$ and the closed loop linear system satisfied by $\widehat{\zeta}_1$ is

$$\widehat{\zeta}_1' = -(10 - \pi^2)\widehat{\zeta}_1, \quad \widehat{\zeta}_1(0) = (z_0, \xi_1)_{L^2(0,1)}.$$

The closed loop system for $\widehat{z}$ is

$$\frac{\partial \widehat{z}}{\partial t} - \frac{\partial^2 \widehat{z}}{\partial x^2} - 10\widehat{z} = 0 \quad \text{in } (0,1)\times(0,\infty),$$

$$\widehat{z}(0,t) = 0 \quad \text{and} \quad \widehat{z}(1,t) = -\frac{\sqrt{2}(10-\pi^2)}{\pi}\,(\widehat{z}(t),\xi_1)_{L^2(0,1)}, \quad \text{for } t\in(0,\infty),$$

$$\widehat{z}(x,0) = z_0(x) \quad \text{in } (0,1).$$

Let us prove that this system is stable. We know that

$$\widehat{\zeta}_1(t) = e^{-(10-\pi^2)t}\zeta_{0,1} \quad \text{and} \quad \widehat{u}(t) = -\frac{\sqrt{2}(10-\pi^2)}{\pi}\,\widehat{\zeta}_1(t).$$

For the other components, we have

$$\widehat{\zeta}_k' = (10-k^2\pi^2)\widehat{\zeta}_k + \widehat{u}(t)\xi_{k,x}(1), \quad \widehat{\zeta}_k(0) = \zeta_{0,k}.$$

Thus

$$\widehat{\zeta}_k(t) = e^{(10-k^2\pi^2)t}\zeta_{0,k} + \int_0^t e^{(10-k^2\pi^2)(t-s)}\widehat{u}(s)\xi_{k,x}(1)ds.$$

Replacing $\widehat{u}$ by its expression in terms of $\widehat{\zeta}_1$, we have

$$\begin{aligned}
&\int_0^t e^{(10-k^2\pi^2)(t-s)}\,\widehat{u}(s)\,\xi_{k,x}(1)ds \\
&= \frac{\sqrt{2}(10-\pi^2)}{\pi}\,\xi_{k,x}(1)\int_0^t e^{(10-k^2\pi^2)(t-s)}e^{-(10-\pi^2)s}\zeta_{0,1}ds \\
&= \frac{\sqrt{2}(10-\pi^2)}{\pi}\,\xi_{k,x}(1)\,\zeta_{0,1}\,e^{(10-k^2\pi^2)t}\int_0^t e^{-(20-k^2\pi^2-k^2\pi^2)s} \\
&= \frac{\sqrt{2}(10-\pi^2)}{\pi((k^2+1)\pi^2-20)}\,\xi_{k,x}(1)\,\zeta_{0,1}\,\left(e^{-(10-\pi^2)t}-e^{(10-k^2\pi^2)t}\right). \\
&= \frac{\sqrt{2}(10-\pi^2)}{(k^2+1)\pi^2-20}\,k(-1)^k\,\zeta_{0,1}\,\left(e^{-(10-\pi^2)t}-e^{(10-k^2\pi^2)t}\right).
\end{aligned}$$

Notice that

$$\left|e^{-(10-\pi^2)t}-e^{(10-k^2\pi^2)t}\right|\leq 2e^{-(10-\pi^2)t}$$

and

$$2\frac{\sqrt{2}(10-\pi^2)}{(k^2+1)\pi^2-20}\,k\leq\frac{1}{k}.$$

Thus

$$|\widehat{\zeta}_k(t)|^2\leq e^{2(10-k^2\pi^2)t}|\zeta_{0,k}|^2+\frac{2}{k^2}e^{-2(10-\pi^2)t}|\zeta_{0,1}|^2,$$

and

$$\|\widehat{z}(t)\|^2_{L^2(0,1)} \leq C\, e^{-2(10-\pi^2)t}\|z_0\|^2_{L^2(0,1)}.$$

Since this system is stable, it means that the solution z to the equation

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1)\times(0,\infty),$$

$$z(0,t) = 0 \quad \text{and} \quad z(1,t) = -\frac{\sqrt{2}(10-\pi^2)}{\pi}\,(z(t),\xi_1)_{L^2(0,1)}, \quad \text{for } t \in (0,\infty),$$

$$z(x,0) = z_0(x) \quad \text{in } (0,1),$$

obeys

$$\|z(t)\|_{L^2(0,1)} \leq Ce^{-10t}\|z_0\|_{L^2(0,1)}.$$

Therefore we have shown that $\Lambda_\omega - \mathbb{B}\mathbb{B}^*\mathbb{P}$ is stable.

Exponential decay with $\omega = 40$.

In that case the system satisfied by $\widehat{z} = e^{\omega t} z$ and $\widehat{u} = e^{\omega t} u$ is

$$\frac{\partial \widehat{z}}{\partial t} - \frac{\partial^2 \widehat{z}}{\partial x^2} - 40\widehat{z} = 0 \quad \text{in } (0,1) \times (0,T),$$
$$\widehat{z}(0,t) = 0 \quad \text{and} \quad \widehat{z}(1,t) = \widehat{u}(t) \quad \text{for } t \in (0,T),$$
$$\widehat{z}(x,0) = z_0(x) \quad \text{in } (0,1).$$

We have two unstable eigenvalues $40 - \pi^2$ and $40 - 2\pi^2$. The next one $40 - 9\pi^2$ is negative. As above, to solve the Bernoulli equation for this system, we can look for a feedback control stabilizing the projected unstable system.

Thus, we have to look for a feedback control stabilizing the following system in $\mathbb{R}^2$

$$\begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix}' = \begin{bmatrix}40+\lambda_1 & 0\\ 0 & 40+\lambda_2\end{bmatrix}\begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix} + \widehat{u}(t)\begin{bmatrix}-\xi_{1,x}(1)\\ -\xi_{2,x}(1)\end{bmatrix}.$$

We set

$$\Lambda_{\omega,u} = \begin{bmatrix}40+\lambda_1 & 0\\ 0 & 40+\lambda_2\end{bmatrix}, \quad \mathbb{B}_{\omega,u} = \begin{bmatrix}-\xi_{1,x}(1)\\ -\xi_{2,x}(1)\end{bmatrix} = \begin{bmatrix}\pi\\ -2\pi\end{bmatrix},$$

and we can find a feedback for this 2×2 system by solving the MRE

$$\mathbb{P}_{\omega,u}\in\mathbb{R}^{2\times 2}, \quad \mathbb{P}_{\omega,u} = \mathbb{P}^*_{\omega,u} > 0,$$

$$\mathbb{P}_{\omega,u}\Lambda_{\omega,u} + \Lambda_{\omega,u}\mathbb{P}_{\omega,u} - \mathbb{P}_{\omega,u}\mathbb{B}_{\omega,u}\mathbb{B}^*_{\omega,u}\mathbb{P}_{\omega,u} = 0,$$

$$\Lambda_{\omega,u} - \mathbb{B}_{\omega,u}\mathbb{B}^*_{\omega,u}\mathbb{P}_{\omega,u} \quad \text{is stable.}$$

The corresponding control is obtained by solving

$$\begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix}' = (\Lambda_{\omega,u} - \mathbb{B}_{\omega,u}\mathbb{B}^*_{\omega,u}\mathbb{P})\begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix}, \quad \begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix}(0) = \begin{bmatrix}\zeta_{0,1}\\ \zeta_{0,2}\end{bmatrix},$$

and by setting

$$\widehat{u}(t) = -\mathbb{B}^*_{\omega,u}\mathbb{P}\begin{bmatrix}\widehat{\zeta}_1(t)\\ \widehat{\zeta}_2(t)\end{bmatrix}.$$

The closed loop system is

$$\begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix}' = \begin{bmatrix}-\lambda_1 - 40 & 0\\ 0 & -\lambda_2 - 40\end{bmatrix}\begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix}, \quad \begin{bmatrix}\widehat{\zeta}_1\\ \widehat{\zeta}_2\end{bmatrix}(0) = \begin{bmatrix}\zeta_{0,1}\\ \zeta_{0,2}\end{bmatrix}.$$

Thus

$$\begin{bmatrix}\widehat{\zeta}_1(t)\\ \widehat{\zeta}_2(t)\end{bmatrix} = \begin{bmatrix}e^{(\pi^2-40)t}\zeta_{0,1}\\ e^{(4\pi^2-40)t}\zeta_{0,2}\end{bmatrix}$$

and

$$\widehat{u}(t) = -\mathbb{B}^*_{\omega,u}\mathbb{P}\begin{bmatrix}\widehat{\zeta}_1(t)\\ \widehat{\zeta}_2(t)\end{bmatrix}.$$

As previously, we can prove that the closed loop system obtained with this control law is stable:

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1)\times(0,\infty),$$

$$z(0,t) = 0 \quad \text{and} \quad z(1,t) = -\mathbb{B}^*_{\omega,u}\mathbb{P}\begin{bmatrix}(z(t),\xi_1)\\ (z(t),\xi_2)\end{bmatrix}, \quad \text{for } t\in(0,\infty),$$

$$z(x,0) = z_0(x) \quad \text{in } (0,1),$$

obeys

$$\|z(t)\|_{L^2(0,1)} \le Ce^{-40t}\|z_0\|_{L^2(0,1)}.$$

6.b. Numerical approximation of the Bernoulli equation

Step 1. We have to find the unstable eigenvalues and the corresponding eigenvectors. The discrete eigenvalue problem is

$$\text{Determine } \boldsymbol{\lambda} \in \mathbb{R} \text{ and } \boldsymbol{\xi} \in \mathbb{R}^{N-1} \text{ such that}$$
$$\mathbf{A}\xi = \lambda \mathbf{E}\xi.$$

Since the matrices **A** and **E** are symmetric, we only look for real eigenvalues. We have

$$\boldsymbol{\lambda}_{N-1} \cdots \leq \boldsymbol{\lambda}_{N_\omega+1} < -\omega < \boldsymbol{\lambda}_{N_\omega} < \cdots < \boldsymbol{\lambda}_1.$$

We assume that the space discretization is fine enough so that $\boldsymbol{\lambda}_k$ is a very good approximation of λ_k for $1 \leq k \leq N_\omega + 1$. We have $N >> N_\omega + 1$. In particular, we assume that, for $1 \leq k \leq N_\omega + 1$, the eigenvalues $\boldsymbol{\lambda}_k$ are simple. For simplicity if one of the eigenvalue less than $\boldsymbol{\lambda}_{N_\omega+1}$ is not simple, we repeat it with its order of multiplicity. We denote by $\boldsymbol{\xi}_k$ a normalized eigenvector associated with the eigenvalue $\boldsymbol{\lambda}_k$.

We notice that $(\boldsymbol{\xi}_k)_{1 \leq k \leq N-1}$ is a basis of $\mathbb{R}^{N-1}$ constituted of eigenvectors of the pair $(\mathbf{E}, \mathbf{A})$. This basis obeys

$$\boldsymbol{\xi}_j^T \mathbf{E} \boldsymbol{\xi}_i = \delta_{i,j} \quad \text{for } 1 \leq i, j \leq N-1.$$

In the case of a multiple eigenvalue we can always choose the corresponding eigenvectors so that this orthoganility condition is still satisfied. We can express the vector $\mathbf{z}(t) = (z_1(t), \cdots, z_{N-1}(t))^T \in \mathbb{R}^{N-1}$ in this basis. If we denote by $\Sigma \in \mathbb{R}^{(N-1)\times(N-1)}$ the matrix whose columns are the vectors $\boldsymbol{\xi}_k$, we have

$$\mathbf{z} = \Sigma\,\zeta.$$

By replacing $\mathbf{z}$ by $\Sigma\,\zeta$ in the equation

$$\mathbf{E}\mathbf{z}' = \mathbf{A}_\omega \mathbf{z} + \mathbf{B}u,$$

we obtain

$$\mathbf{E}\Sigma\,\zeta' = \mathbf{A}_\omega \Sigma\,\zeta + \mathbf{B}u.$$

With the orthogonality condition $\boldsymbol{\xi}_j^T \mathbf{E} \boldsymbol{\xi}_i = \delta_{i,j}$ and the identity $\boldsymbol{\xi}_j^T \mathbf{A} \boldsymbol{\xi}_i = \delta_{i,j} \boldsymbol{\lambda}_i$, we have

$$\zeta_k' = (\boldsymbol{\lambda}_k + \omega)\, \zeta_k + \boldsymbol{\xi}_k^T \mathbf{B} u \quad \text{for } 1 \leq k \leq N-1.$$

We set

$$\Lambda_{\omega,u} = \begin{bmatrix} \omega + \boldsymbol{\lambda}_1 & 0 & \dots & 0 \\ 0 & \omega + \boldsymbol{\lambda}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \omega + \boldsymbol{\lambda}_{N_\omega} \end{bmatrix},\ [\mathbb{B}_{\omega,u}]_i = \boldsymbol{\xi}_i^T \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/h \end{bmatrix},\ 1 \leq i \leq N_\omega,$$

and

$$\Lambda_{\omega} = \begin{bmatrix} \omega + \boldsymbol{\lambda}_1 & 0 & \dots & 0 \\ 0 & \omega + \boldsymbol{\lambda}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \omega + \boldsymbol{\lambda}_{N-1} \end{bmatrix},\ [\mathbb{B}]_i = \boldsymbol{\xi}_i^T \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/h \end{bmatrix},\ 1 \leq i \leq N-1.$$

We have to stabilize the system

$$\zeta' = \Lambda\zeta + \mathbb{B}\, u, \quad \zeta(0) = \zeta_0,$$

or the projected system

$$\zeta'_{\omega,u} = \Lambda_{\omega,u}\zeta_{\omega,u} + \mathbb{B}_{\omega,u}\, u, \quad \zeta_{\omega,u}(0) = \pi_{\omega,u}\zeta_0,$$

where $\zeta_{\omega,u} = \pi_{\omega,u}\zeta$ and $\pi_{\omega,u} \in \mathcal{L}(\mathbb{R}^{N-1}, \mathbb{R}^{N_\omega})$ consists in taking the first N_ω components.
To find a feedback gain, we can solve the Bernoulli equation

$$\begin{aligned}
&\mathbb{P}_{\omega,u} \in \mathcal{L}(\mathbb{R}^{N_\omega}), \quad \mathbb{P}_{\omega,u} = \mathbb{P}_{\omega,u}^T \geq 0,\\
&\mathbb{P}_{\omega,u}\Lambda_{\omega,u} + \Lambda_{\omega,u}^T\mathbb{P}_{\omega,u} - \mathbb{P}_{\omega,u}\mathbb{B}_{\omega,u}\mathbb{B}_{\omega,u}^T\mathbb{P}_{\omega,u} = 0,\\
&\Lambda_{\omega,u} - \mathbb{B}_{\omega,u}\mathbb{B}_{\omega,u}^T\mathbb{P}_{\omega,u} \quad \text{is stable.}
\end{aligned}$$

We find the corresponding $\mathbf{P} \in \mathcal{L}(\mathbb{R}^{N-1})$ such that $(\mathbf{E}, \mathbf{A}_\omega - \mathbf{B}\mathbf{B}^T\mathbf{P}\mathbf{E})$ is stable by setting

$$[\mathbf{P}]_{i,j} = \sum_{k=1}^{N_\omega}\sum_{\ell=1}^{N_\omega} \boldsymbol{\xi}_{i,k}\, [\mathbb{P}_{\omega,u}]_{k,\ell}\, \boldsymbol{\xi}_{j,\ell}.$$