Numerics and Control of PDEs

Lecture 6

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Estimation of the 1D Heat equation

Stabilization in the case of partial information

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Plan of lecture 6

1. Detectability of the 1D Heat equation
2. Approximation of the estimator
3. Coupling between a feedback control law and an estimator

1. Detectability of the 1D heat equation with different types of measurements

We assume that we have a noisy model

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = \mu \quad \text{in } (0,1) \times (0,T),$$
$$z(0,t) = 0 \quad \text{and} \quad z(1,t) = u(t) \quad \text{for } t \in (0,T),$$
$$z(\cdot,0) = z_0 + \mu_0 \quad \text{in } (0,1).$$

In this setting, u and z_0 are known, but μ and μ_0 are not known. We would like to estimate $z(t)$ by using measurements.

Boundary measurements

We choose the measure operator H defined by

$$Hz(t) = z_x(0,t) \in Y_o = \mathbb{R}.$$

Thus $H \in \mathcal{L}(H^2(0,1), Y_o)$. We set $D(A) = H^2(0,1) \cap H_0^1(0,1)$, and $Az = \frac{d^2 z}{dx^2}$. The pair (A,H) is detectable. Indeed A is stable and $A + LH$ is stable with $L = 0$.

To increase the quality of the estimation we look for $L \in \mathcal{L}(\mathbb{R}; L^2(0,1))$ such that $A + \omega I + LH$ is stable for $\omega > 0$.

To study the detectability of the pair $(A + \omega I, H)$, we can use the Hautus criterion

$$\forall \lambda, \quad \text{Re}\,\lambda \geq 0, \quad \text{Ker}(\lambda I - A - \omega I) \cap \text{Ker}(H) = \{0\}.$$

Assume that

$$\cdots < \lambda_{N_\omega + 1} < -\omega < \lambda_{N_\omega} < \cdots < \lambda_1.$$

Thus we have to show that if

$$1 \leq k \leq N_\omega, \quad (A + \omega I)\xi = (\lambda_k + \omega)\xi \quad \textit{and} \quad H\xi = 0,$$

then $\xi = 0$. Such a function ξ is of the form $\xi = \alpha \xi_k$, and $H\xi = \alpha\, \xi_{k,x}(0) = \alpha\, \pi\, k$. Thus $H\xi = 0$ implies $\alpha = 0$ and $\xi = 0$.

Distributed measurements

We choose the measure operator H defined by

$$Hz = \left(\frac{1}{|I_1|} \int_{I_1} z, \cdots \frac{1}{|I_{N_o}|} \int_{I_{N_o}} z \right) \in Y_o = \mathbb{R}^{N_o}.$$

Thus $H \in \mathcal{L}(L^2(0,1), Y_o)$.

To prove that the pair $(A + \omega I, H)$ is detectable, we have to show that if

$$1 \le k \le N_\omega, \quad (A + \omega I)\xi = (\lambda_k + \omega)\xi \quad \textit{and} \quad H\xi = 0,$$

then $\xi = 0$. Such a function ξ is of the form $\xi = \alpha \xi_k$, and $H\xi = \alpha \left(\frac{1}{|I_1|} \int_{I_1} \xi_k, \cdots, \frac{1}{|I_{N_o}|} \int_{I_{N_o}} \xi_k \right)$. Thus, we have to choose the intervals I_1,...,I_{N_o} so that $\int_{I_{N_k}} \xi_k \neq 0$ at least for one interval N_k.

Detectability for distributed measurements with one interval

If we choose $N_o = 1$ and $I_1 = (0, 1)$. The eigenfunction ξ_1 is observable, while the eigenfunctions $\xi_2, \cdots, \xi_{2k}, \cdots$ are not observable. Thus the pair $(A + \omega I, H)$ is detectable if $0 < \omega < 4\pi^2$.

If we choose $N_o = 1$ and $I_1 = (0, 1/2)$. The eigenfunctions ξ_1, ξ_2, ξ_3 are observable, while the eigenfunctions $\xi_4,\ \xi_8 \cdots, \xi_{4k}, \cdots$ are not observable. Thus the pair $(A + \omega I, H)$ is detectable if $0 < \omega < 16\pi^2$.

For a noisy observation

$$y_{obs}(t) = Hz(t) + \eta,$$

The estimator will be of the form

$$z_e' = Az_e + Bu + L(Hz_e - y_{obs}), \quad z_e(0) = z_0,$$

with $L = -P_e H^* R_\eta$, P_e is the solution to

$$P_e = P_e^* \geq 0, \quad P_e(A^* + \omega I) + P_e(A + \omega I) - P_e H^* R_\eta^{-1} H P_e + Q_\mu = 0,$$

$Q_\mu \in \mathcal{L}(L^2(0,1))$, $R_\eta \in \mathcal{L}(Y_o)$, $Q_\mu = Q_\mu^* \geq 0$ and $R_\eta = R_\eta^* > 0$. The operators Q_μ and R_η are chosen according to the 'a priori' knowledge we have on the model noise and the measurement noise.

This is an infinite dimensional Riccati equation which has to be approximated.

2. Approximation of the estimator

We observe the system

$$z' = Az + Bu + \mu, \quad z(0) = z_0 + \mu_0.$$

The solution z to the continuous model is approximated by

$$z(t) = \sum_{i=1}^{N-1} z_i(t)\phi_i + u(t)\phi_N, \quad \text{with } \mathbf{z}(t) = (z_1, \cdots, z_{N-1})^T.$$

The approximate state equation is

$$\mathbf{E}\mathbf{z}' = \mathbf{A}\mathbf{z} + \mathbf{B}u + \boldsymbol{\mu}, \quad \mathbf{z}(0) = \mathbf{z}_0 + \boldsymbol{\mu}_0.$$

We would like to deduce from $y_{obs}(t) = Hz(t) + \eta$ an observation for **z**. The measure of the approximate state $z(t)$ is

$$Hz(t) = \sum_{i=1}^{N-1} z_i(t)H\phi_i + u(t)H\phi_N.$$

Let us consider the case of a boundary observation or a distributed observation for which $H\phi_N = 0$. In that case we can set

$$\mathbf{Hz}(t) = \sum_{i=1}^{N-1} z_i(t) H\phi_i \quad \text{and} \quad \mathbf{y}_{obs}(t) = \mathbf{Hz}(t) + \boldsymbol{\eta}(t),$$

where $\boldsymbol{\eta}$ is a discrete approximation of η.
If we have a numerical approximation $H_h z(t)$ of $Hz(t)$ for which

$$H_h z(t) \neq \sum_{i=1}^{N-1} z_i(t) H\phi_i,$$

and which gives a better accuracy of the measure of the exact state, we can use it in the definition of $\mathbf{Hz}(t)$.
We assume now that $\mathbf{Hz}(t) = H_h \left(\sum_{i=1}^{N} z_i(t)\phi_i \right) = H_h z(t)$ and $\mathbf{y}_{obs}(t) = \mathbf{Hz}(t) + \boldsymbol{\eta}$ are given.
To find the filtering gain, we have to solve

$$\begin{aligned} & \mathbf{P}_e \in \mathcal{L}(\mathbb{R}^{N-1}), \quad \mathbf{P}_e = \mathbf{P}_e^T \geq 0, \\ (GMRE)_e \qquad & \mathbf{E}\mathbf{P}_e\mathbf{A}_\omega^T + \mathbf{A}_\omega^T\mathbf{P}_e\mathbf{E} - \mathbf{E}\mathbf{P}_e\mathbf{H}^T R_\eta^{-1}\mathbf{H}\mathbf{P}_e\mathbf{E} + Q_\mu = 0, \\ & (\mathbf{E}, \mathbf{A}_\omega - \mathbf{E}\mathbf{P}_e\mathbf{H}^T R_\eta^{-1}\mathbf{H}) \quad \text{is stable.} \end{aligned}$$

The estimator is

$$\mathbf{E}\mathbf{z}_e' = \mathbf{A}\mathbf{z}_e + \mathbf{B}u - \mathbf{E}\mathbf{P}_e\mathbf{H}^T R_\eta^{-1}(\mathbf{H}\mathbf{z}_e - \mathbf{y}_{obs}), \quad \mathbf{z}_e(0) = \mathbf{z}_0,$$
$$\mathbf{E}\mathbf{z}' = \mathbf{A}\mathbf{z} + \mathbf{B}u + \boldsymbol{\mu}, \quad \mathbf{z}(0) = \mathbf{z}_0 + \boldsymbol{\mu}_0,$$
$$\mathbf{y}_{obs}(t) = \mathbf{H}\mathbf{z}(t) + \boldsymbol{\eta}.$$

We solve the state equation to generate the measures $\mathbf{y}_{obs}(t)$. Next we calculate $\mathbf{z}_e$.
The operator $\mathbf{P}_e$ belongs to $\mathbb{R}^{N_z \times N_z}$. If N_z is too large, we have to look at an estimator of smaller dimeension.

Estimator of small dimension

As for the calculation of the feedback gain, rather than estimating the solution to the equation

$$\mathbf{E}\mathbf{z}' = \mathbf{A}\mathbf{z} + \mathbf{B}u + \boldsymbol{\mu}, \quad \mathbf{z}(0) = \mathbf{z}_0 + \boldsymbol{\mu}_0,$$

we can work with the system

$$\zeta' = \boldsymbol{\Lambda}\zeta + \mathbb{B}\, u + \Sigma^{-1}\boldsymbol{\mu}, \quad \zeta(0) = \zeta_0 + \Sigma^{-1}\boldsymbol{\mu}_0,$$

or with the projected system

$$\zeta'_{\omega,u} = \boldsymbol{\Lambda}_{\omega,u}\zeta_{\omega,u} + \mathbb{B}_{\omega,u}\, u + \pi_{\omega,u}\Sigma^{-1}\boldsymbol{\mu}, \quad \zeta_{\omega,u}(0) = \pi_{\omega,u}\zeta_0 + \pi_{\omega,u}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0,$$

where $\zeta_{\omega,u} = \pi_{\omega,u}\zeta$ and $\pi_{\omega,u} \in \mathcal{L}(\mathbb{R}^{N-1}, \mathbb{R}^{N_\omega})$. The notation are the ones of the previous lecture. Since we make the change of variable

$$\mathbf{z} = \Sigma\zeta,$$

we introduce $\mathbb{H}$ defined by

$$\mathbb{H}\zeta = \mathbf{H}\mathbf{z}.$$

When we work with $\zeta_{\omega,u} = \pi_{\omega,u}\zeta$, it is convenient to set

$$\mathbb{H}_{\omega,u} = \mathbb{H}\pi_{\omega,u}.$$

It is convenient to estimate $\zeta_{\omega,u}$ rather than ζ since N_ω is much smaller than $N-1$.
To find a filtering gain, we can solve the Riccati equation

$$\begin{aligned}
&\mathbb{P}_e \in \mathcal{L}(\mathbb{R}^{N_\omega}), \quad \mathbb{P}_e = \mathbb{P}_e^T \geq 0, \\
&\mathbb{P}_e \Lambda_{\omega,u}^T + \Lambda_{\omega,u}\mathbb{P}_e - \mathbb{P}_e \mathbb{H}_{\omega,u}^T R_\eta^{-1} \mathbb{H}_{\omega,u}\mathbb{P}_e + \mathbb{Q}_\mu = 0, \\
&\Lambda_{\omega,u} - \mathbb{P}_e \mathbb{H}_{\omega,u}^T R_\eta^{-1} \mathbb{H}_{\omega,u} \quad \text{is stable,}
\end{aligned}$$

where $\mathbb{Q}_\mu$ stands for the covariance operator of $\Sigma^{-1}\boldsymbol{\mu}$.
We find the corresponding $\mathbf{P}_e \in \mathcal{L}(\mathbb{R}^{N-1})$ such that $(\mathbf{E}, \mathbf{A}_\omega - \mathbf{E}\mathbf{P}_e\mathbf{H}^T R_\eta^{-1}\mathbf{H})$ is stable by setting

$$[\mathbf{P}_e]_{i,j} = \sum_{k=1}^{N_\omega} \sum_{\ell=1}^{N_\omega} \boldsymbol{\xi}_{i,k}\, [\mathbb{P}_e]_{k,\ell}\, \boldsymbol{\xi}_{j,\ell}.$$

The estimation equation will be the same one as above.
Since we have solved the Riccati equation for the pair $(\Lambda_{\omega,u}, \mathbb{H}_{\omega,u})$ rather than $(\Lambda_\omega, \mathbb{H})$, this means that we have neglected the measure

$$\mathbb{H} \sum_{k=N_\omega+1}^{N-1} \zeta_k \boldsymbol{\xi}_k.$$

Since, for $N_\omega + 1 \leq k \leq N - 1$, ζ_k is rapidly decreasing the error that we introduce is small and the accuracy is good.

3. Coupling between a feedback control law and an estimator

We assume that we have a noisy model

$$
\begin{aligned}
&\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = \mu \quad \text{in } (0,1) \times (0,T), \\
&z(0,t) = 0 \quad \text{and} \quad z(1,t) = u(t) \quad \text{for } t \in (0,T), \\
&z(\cdot,0) = z_0 + \mu_0 \quad \text{in } (0,1),
\end{aligned}
$$

and a noisy observation

$$y_{obs}(t) = Hz(t) + \eta.$$

We choose some $\omega > 0$, such that

$$\cdots < \lambda_{N_\omega+1} < -\omega < \lambda_{N_\omega} < \cdots < \lambda_1.$$

where $\lambda_k = -k^2\pi^2$ are the eigenvalue of A. We introduce the projected system

$$\begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \\ \widehat{\zeta}_{N_\omega} \end{bmatrix}' = \begin{bmatrix} \omega+\lambda_1 & 0 & \dots & 0 \\ 0 & \omega+\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \omega+\lambda_{N_\omega} \end{bmatrix} \begin{bmatrix} \widehat{\zeta}_1 \\ \widehat{\zeta}_2 \\ \vdots \\ \widehat{\zeta}_{N_\omega} \end{bmatrix} + \widehat{u}(t) \begin{bmatrix} -\xi_{1,x}(1) \\ -\xi_{2,x}(1) \\ \vdots \\ -\xi_{N_\omega,x}(1) \end{bmatrix}.$$

We set

$$\Lambda_{\omega,u} = \begin{bmatrix} \omega+\lambda_1 & 0 & \dots & 0 \\ 0 & \omega+\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \omega+\lambda_{N_\omega} \end{bmatrix} \quad \text{and} \quad \mathbb{B}_{\omega,u} = \begin{bmatrix} -\xi_{1,x}(1) \\ -\xi_{2,x}(1) \\ \vdots \\ -\xi_{N_\omega,x}(1) \end{bmatrix}.$$

We solve the matrix Riccati equation

$$\mathbb{P}_{\omega,u} \in \mathbb{R}^{N_\omega \times N_\omega},\ \mathbb{P}_{\omega,u} = \mathbb{P}_{\omega,u}^T,$$

$$\mathbb{P}_{\omega,u}\Lambda_{\omega,u} + \Lambda_{\omega,u}\mathbb{P}_{\omega,u} - \mathbb{P}_{\omega,u}\mathbb{B}_{\omega,u}\mathbb{B}_{\omega,u}^*\mathbb{P}_{\omega,u} + \delta I_{\mathbb{R}^{N_\omega}} = 0, \quad \text{with } \delta \geq 0,$$

$$\Lambda_{\omega,u} - \mathbb{B}_{\omega,u}\mathbb{B}_{\omega,u}^*\mathbb{P}_{\omega,u} \quad \text{is stable.}$$

The system coupling the feedback gain and the estimator is

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = \mu \quad \text{in } (0,1) \times (0,T),$$

$$z(0,t) = 0 \quad \text{and} \quad z(1,t) = -\mathbb{B}^*_{\omega,u}\mathbb{P}_{\omega,u} \begin{bmatrix} (z_e(t),\xi_1) \\ (z_e(t),\xi_2) \\ \vdots \\ (z_e(t),\xi_{N_\omega}) \end{bmatrix} \quad \text{for } t \in (0,T),$$

$$z(\cdot,0) = z_0 + \mu_0 \quad \text{in } (0,1),$$

$$\frac{\partial z_e}{\partial t} - \frac{\partial^2 z_e}{\partial x^2} = L(Hz_e - y_{obs}(t)) \quad \text{in } (0,1) \times (0,T),$$

$$z_e(0,t) = 0 \quad \text{and} \quad z_e(1,t) = -\mathbb{B}^*_{\omega,u}\mathbb{P}_{\omega,u} \begin{bmatrix} (z_e(t),\xi_1) \\ (z_e(t),\xi_2) \\ \vdots \\ (z_e(t),\xi_{N_\omega}) \end{bmatrix} \quad \text{for } t \in (0,T),$$

$$z_e(\cdot,0) = z_0 \quad \text{in } (0,1),$$

$$y_{obs}(t) = Hz(t) + \eta,$$

where $L = -P_e H^* R_\eta$ is the filtering gain.

To prove the exponential decay of z, we write the system satisfied by $\hat{z} = e^{\omega t} z$ and $\hat{e} = \hat{z} - \hat{z}_e$:

$$\frac{\partial \hat{z}}{\partial t} - \frac{\partial^2 \hat{z}}{\partial x^2} - \omega \hat{z} = \hat{\mu} \quad \text{in } (0,1) \times (0,\infty), \quad \hat{z}(0,t) = 0$$

$$\hat{z}(1,t) = -\mathbb{B}^*_{\omega,u} \mathbb{P}_{\omega,u} \begin{bmatrix} (\hat{z}(t), \xi_1) \\ (\hat{z}(t), \xi_2) \\ \vdots \\ (\hat{z}(t), \xi_{N_\omega}) \end{bmatrix} + \mathbb{B}^*_{\omega,u} \mathbb{P}_{\omega,u} \begin{bmatrix} (\hat{e}(t), \xi_1) \\ (\hat{e}(t), \xi_2) \\ \vdots \\ (\hat{e}(t), \xi_{N_\omega}) \end{bmatrix}$$

for $t \in (0,\infty)$,

$$\hat{z}(\cdot,0) = z_0 + \mu_0 \quad \text{in } (0,1),$$

$$\frac{\partial \hat{e}}{\partial t} - \frac{\partial^2 \hat{e}}{\partial x^2} - \omega \hat{e} = LH\hat{e} - L\hat{\eta} \quad \text{in } (0,1) \times (0,\infty),$$

$$\hat{e}(0,t) = 0 \quad \text{and} \quad \hat{e}(1,t) = 0 \quad \text{for } t \in (0,\infty),$$

$$\hat{e}(\cdot,0) = \mu_0 \quad \text{in } (0,1).$$

We know that $A + LH$ is exponentially stable, that is $\|e^{t(A+LH)}\| \leq Ce^{-\delta_o t}$ for some $\delta_o > 0$. Thus if $e^{-\delta_\eta}\hat{\eta} \in L^2(0, \infty; L^2(Y_o))$ for some $0 < \delta_\eta < \delta_o$, we can prove that

$$\|\hat{e}(t)\|_Z \leq Ce^{-\delta_e t} \quad \text{for some } 0 < \delta_e < \delta_\eta.$$

Next, we use $\|e^{t(A+BK)}\| \leq Ce^{-\delta_c t}$ for some $\delta_c > 0$. If $e^{-\delta_\mu}\hat{\mu} \in L^2(0, \infty; L^2(Z))$ for some $\delta_\mu > 0$, using the estimate of $\hat{e}$ in the equation satisfied by $\hat{z}$ we prove that

$$\|\hat{z}(t)\|_Z \leq Ce^{-\delta_z t} \quad \text{with } 0 < \delta_z < \min(\delta_c, \delta_\mu, \delta_e).$$

For the discrete model, the system coupling the estimator and the feedback gain is

$$\mathbf{E}\mathbf{z}_e' = \mathbf{A}\mathbf{z}_e - \mathbf{B}\mathbf{B}^T\mathbf{P}\mathbf{E}\mathbf{z}_e - \mathbf{E}\mathbf{P}_e\mathbf{H}^T R_\eta^{-1}(\mathbf{H}\mathbf{z}_e - \mathbf{y}_{obs}), \quad \mathbf{z}_e(0) = \mathbf{z}_0,$$
$$\mathbf{E}\mathbf{z}' = \mathbf{A}\mathbf{z} - \mathbf{B}\mathbf{B}^T\mathbf{P}\mathbf{E}\mathbf{z}_e + \boldsymbol{\mu}, \quad \mathbf{z}(0) = \mathbf{z}_0 + \boldsymbol{\mu}_0,$$
$$\mathbf{y}_{obs}(t) = \mathbf{H}\mathbf{z}(t) + \boldsymbol{\eta}.$$

This is a coupled system for $(\mathbf{z}, \mathbf{z}_e)^T$. We solve it with the Backward Differentiation Formula of order 2.

Let us summarize the different steps for finding the matrices $\mathbf{P} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $\mathbf{P}_e \in \mathbb{R}^{(N-1)\times(N-1)}$ of the previous system.

- We first determine the pairs $(\boldsymbol{\lambda}_k, \xi_k)_{1\leq N_\omega}$ of eigenvalues and eigenvectors of $(\mathbf{E}, \mathbf{A})$.
- Next, we build the matrices $\mathbb{A}_{\omega,u}$, $\mathbb{B}_{\omega,u}$ and $\mathbb{H}_{\omega,u}$.
- We determine $\mathbb{P}_{\omega,u}$ by solving the Riccati equation associated with the pair $(\mathbb{A}_{\omega,u}, \mathbb{B}_{\omega,u})$.
- We determine $\mathbb{P}_e$ by solving the Riccati equation associated with the pair $(\mathbb{A}_{\omega,u}, \mathbb{H}_{\omega,u})$.
- We calculate $\mathbf{P}$ from $\mathbb{P}_{\omega,u}$ and $\mathbf{P}_e$ from $\mathbb{P}_e$.