Numerics and Control of PDEs

Lecture 7

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Feedback stabilization of a 1D nonlinear model

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Plan of lecture 7

1. The nonlinear model
2. Eigenvalues of the linearized operator and of its adjoint
3. Stabilizability and detectability of the linearized model
4. Local feedback stabilization of the continuous model
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1. A Burgers equation with a Dirichlet boundary control

We consider the equation

$$\frac{\partial w}{\partial t} - \nu \frac{\partial^2 w}{\partial x^2} + w\frac{\partial w}{\partial x} = f_s \quad \text{in } (0,1)\times(0,\infty),$$

$$w(0,t) = u_s + u(t) \quad \text{and} \quad \nu \frac{\partial w}{\partial x}(1,t) = g_s \quad \text{for } t\in(0,\infty),$$

$$w(\cdot,0) = w_0 \quad \text{in } (0,1).$$

We assume that f_s, g_s and u_s are stationary data, and that w_s is the solution to the equation

$$-\nu\, w_{s,xx} + w_s\, w_{s,x} = f_s \quad \text{in } (0,1),$$

$$w_s(0) = u_s \quad \text{and} \quad \nu\, w_{s,x}(1) = g_s.$$

We set $z = w - w_s$. The equation satisfied by z is

$$\frac{\partial z}{\partial t} - \nu \frac{\partial^2 z}{\partial x^2} + w_s \frac{\partial z}{\partial x} + z \frac{\partial w_s}{\partial x} + z \frac{\partial z}{\partial x} = 0 \quad \text{in } (0,1) \times (0,\infty),$$

$$z(0,t) = u(t) \quad \text{and} \quad \nu \frac{\partial z}{\partial x}(1,t) = 0 \quad \text{for } t \in (0,T),$$

$$z(\cdot,0) = w_0 - w_s = z_0 \quad \text{in } (0,1).$$

The linearized model is

$$\frac{\partial z}{\partial t} - \nu \frac{\partial^2 z}{\partial x^2} + w_s \frac{\partial z}{\partial x} + z \frac{\partial w_s}{\partial x} = 0 \quad \text{in } (0,1) \times (0,\infty),$$

$$z(0,t) = u(t) \quad \text{and} \quad \nu \frac{\partial z}{\partial x}(1,t) = 0 \quad \text{for } t \in (0,T),$$

$$z(\cdot,0) = z_0 \quad \text{in } (0,1).$$

The linearized operator in $L^2(0,1)$ is defined by

$$D(A) = \{z \in H^2(0,1) \mid z(0) = 0,\ z_x(1) = 0\},$$

$$Az = \nu\, z_{xx} - w_s\, z_x - z\, w_{s,x}.$$

As in the case of the heat equation, it can be shown that the linearized controlled system may be written in the form

$$z' = Az + Bu, \quad z(0) = z_0,$$

with $B = (\lambda_0 - A)D$ for some $\lambda_0 > 0$, where the Dirichlet operator D is defined by $Du = w$ and w is the solution to the stationary equation

$$\lambda_0 w - \nu\, w_{xx} + w_s\, w_x + w\, w_{s,x} = 0 \quad \text{in } (0,1),$$
$$w(0) = u \quad \text{and} \quad \nu\, w_x(1) = 0.$$

The coefficient $\lambda_0 > 0$ is chosen so that the bilinear form

$$a(w,\phi) = \int_0^1 (\lambda_0\, w\, \phi + \nu w_x \phi_x + w_s\, w_x\, \phi + w\, w_{s,x}\, \phi)\, dx$$

is coercive in $H^1_{\{0\}}(0,1)$.

Measurements

We choose either a distributed measurement

$$H_d z(t) = \left(\frac{1}{|I_1|} \int_{I_1} z(t), \cdots, \frac{1}{|I_{N_o}|} \int_{I_{N_o}} z(t) \right) \in Y_o = \mathbb{R}^{N_o},$$

with $I_j \subset (0,1)$ for $1 \le j \le N_o$, or a boundary measurement

$$H_b z(t) = z(1,t).$$

The adjoint of $(A, D(A))$ is the operator $(A^*, D(A^*))$ defined by

$$D(A^*) = \{\phi \in H^2(0,1) \mid \phi(0) = 0,\ \phi_x(1) + w_s(1)\phi(1) = 0\},$$

$$A^*\phi = \nu\, \phi_{xx} + w_s\, \phi_x.$$

2. Eigenvalues of A and A^*

To study the stabilizability of the pair $(A + \omega I, B)$ is stabilizable and the detectability of the pairs $(A + \omega I, H_d)$ and $(A + \omega I, H_b)$, we have to characterize the spectrum of A and the eigenfunctions of A and A^*. Since A is not selfadjoint, it may admits complex eigenvalues. Let us first show that the eigenvalues of A are real.

The eigenvalue problem for the linearized model is

$$
\begin{aligned}
&-\nu\, z_{xx} + w_s\, z_x + z\, w_{s,x} = \lambda\, z \quad \text{in } (0,1),\\
&z(0) = 0 \quad \text{and} \quad \nu\, z_x(1) = 0.
\end{aligned}
$$

Since the operator is not selfadjoint, we make the following change of unknowns

$$z(x) = e^{\beta(x)} v.$$

We have

$$
\begin{aligned}
z_x &= \beta_x\, e^{\beta(x)} v + e^{\beta(x)} v_x\\
z_{xx} &= \beta_{xx}\, e^{\beta(x)} v + \beta_x^2\, e^{\beta(x)} v + 2\beta_x\, e^{\beta(x)} v_x + e^{\beta(x)} v_{xx}.
\end{aligned}
$$

Thus, the equation for v is

$$-\nu\, v_{xx} + (-2\nu\, \beta_x + w_s)\, v_x + \left(-\nu\, \beta_{xx} - \nu\, \beta_x^2 + w_s\, \beta_x + w_{s,x}\right) v = \lambda\, v \quad \text{in } (0,1)$$

$$v(0) = 0 \quad \text{and} \quad v_x(1) + \beta_x\, v(1) = 0.$$

We choose

$$\beta_x = \frac{w_s}{2\nu} \quad \text{and} \quad \beta_{xx} = \frac{w_{s,x}}{2\nu}.$$

We have

$$-\nu\, v_{xx} + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v = \lambda\, v \quad \text{in } (0,1),$$

$$v(0) = 0 \quad \text{and} \quad v_x(1) + \frac{w_s(1)}{2\nu}\, v(1) = 0.$$

Now, we have a selfadjoint operator. Therefore we have shown that the eigenvalues of A are real. In addition, if $\lambda_0 > 0$ is chosen so that the bilinear form

$$a(w,\phi) = \int_0^1 \left(\lambda_0\, w\, \phi + \nu w_x \phi_x + w_s\, w_x\, \phi + w\, w_{s,x}\, \phi\right)\, dx$$

is coercive in $H^1_{\{0\}}(0,1)$, we can show that $\lambda_0 - A$ is an isomorphism from $D(A)$ into $L^2(0,1)$, and its inverse $(\lambda_0 - A)^{-1}$ is a compact operator in $L^2(0,1)$. In that case, it can be shown that the eigenvalues of A are isolated and of finite multiplicity.

3. Stabilizability of (A, B) and detectability of (A, H)

Theorem. For any $\omega > 0$, the pair $(A + \omega I, B)$ is stabilizable and the pair $(A + \omega I, H_b)$ is detectable. The family $(I_j)_{1 \leq j \leq N_o}$ may be chosen so that the pair $(A + \omega I, H_d)$ is detectable.

To check the stabilizability of the pair $(A + \omega I, B)$, we have to verify the Hautus criterion. The adjoint of B is defined on $D(A^*)$ by

$$B^* \phi = \nu \phi_x(0), \quad \forall \phi \in D(A^*).$$

If $\phi \in H^1_{\{0\}}(0, 1)$ satisfies

$$-\nu \, \phi_{xx} - w_s \, \phi_x = \lambda \, \phi \quad \text{in } (0, 1),$$

$$\phi(0) = 0, \quad \text{and} \quad \nu \, \phi_x(1) + w_s(1) \phi(1) = 0, \quad \text{and} \quad \phi_x(0) = 0,$$

then necessarily $\phi = 0$, because ϕ is the solution to a second order homogeneous differential equation with a zero initial condition.

Thus the pair $(A + \omega I, B)$ is stabilizable.

Let us check the detectability of $(A+\omega I, H_b)$. If $\xi \in H^1_{\{0\}}(0,1)$ satisfies

$$-\nu\,\xi_{xx} + w_s\,\xi_x + \xi\,w_{s,x} = \lambda\,\xi \quad \text{in } (0,1),$$
$$\xi(0)=0, \quad \nu\,\xi_x(1)=0 \quad \text{and} \quad \xi(1)=0,$$

then necessarily $\xi = 0$, because $\xi_x(1)=0 \quad \text{and} \quad \xi(1)=0$. Thus the pair $(A+\omega I, H_b)$ is detectable.

4. Local feedback stabilization of the continuous model

We choose a feedback gain $K \in \mathcal{L}(Z, U)$ such that $(A + \omega I, B)$ is stable and a filtering gain $L \in \mathcal{L}(Y_o, Z)$ such that $(A + \omega I, H)$ is stable. Here H stands either for H_d or H_b. It η denotes a measurement error, μ is a model error and μ_0 is an error in the initial condition, we have to solve the system

$$
\begin{aligned}
&\frac{\partial z}{\partial t} - \nu \frac{\partial^2 z}{\partial x^2} + w_s \frac{\partial z}{\partial x} + z \frac{\partial w_s}{\partial x} + z \frac{\partial z}{\partial x} = \mu \quad \text{in } (0,1) \times (0,\infty),\\
&z(0,t) = K z_e(t) \quad \text{and} \quad \nu \frac{\partial z}{\partial x}(1,t) = 0, \quad \text{for } t \in (0,\infty),\\
&z(\cdot,0) = z_0 + \mu_0 \quad \text{in } (0,1),\\
&\frac{\partial z_e}{\partial t} - \nu \frac{\partial^2 z_e}{\partial x^2} + w_s \frac{\partial z_e}{\partial x} + z_e \frac{\partial w_s}{\partial x} = L(H z_e - H z + \eta) \quad \text{in } (0,1) \times (0,\infty),\\
&z_e(0,t) = K z_e(t) \quad \text{and} \quad \nu \frac{\partial z_e}{\partial x}(1,t) = 0 \quad \text{for } t \in (0,\infty),\\
&z_e(\cdot,0) = z_0 \quad \text{in } (0,1).
\end{aligned}
$$

Theorem – Local stabilization of the nonlinear system. Let us choose $s \in (0, 1/2)$. There exist $C_0 > 0$ and $c_1 > 0$, such that if $0 < C \leq C_0$ and

$$\|z_0\|_{H^s(0,1)} + \|\mu_0\|_{H^s(0,1)} + \|e^{\cdot\,\omega}\mu\|_{L^2(0,\infty;H^{-1+s}(0,1))} + \|e^{\cdot\,\omega}\eta\|_{L^2(0,\infty;Y_o)} \leq c_1\, C,$$

then the nonlinear system coupling z and z_e admits a unique solution in the space

$$\{(z, z_e) \in H^{1+s,1/2+s/2}((0,1)\times(0,\infty)) \mid \|(z, z_e)\|_{H^{1+s,1/2+s/2}((0,1)\times(0,\infty))} \leq C\}.$$

5. Approximation of the closed loop nonlinear system

For the finite dimensional approximate system, we have to solve

$$\mathbf{E}\mathbf{z}_e' = \mathbf{A}\mathbf{z}_e + \mathbf{B}\mathbf{K}\mathbf{z}_e + \mathbf{L}(\mathbf{H}\mathbf{z}_e - \mathbf{H}\mathbf{z} - \eta), \quad \mathbf{z}_e(0) = \mathbf{z}_0,$$
$$\mathbf{E}\mathbf{z}' = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{K}\mathbf{z}_e + F(\mathbf{z}) + \boldsymbol{\mu}, \quad \mathbf{z}(0) = \mathbf{z}_0 + \boldsymbol{\mu}_0.$$

6. Explicit determination of unstable steady states

We are going to show that there exist steady states of the Burgers equation for which the linearized operators about these steady states are unstable in the case of mixed boubdary conditions. We shall also consider the case of Dirichlet conditions without being able to give a positive answer.

Case of mixed boundary conditions. We denote by w_s a solution to

$$\begin{aligned} &-\nu\, w_{s,xx} + w_s\, w_{s,x} = f_s \quad \text{in } (0,1), \\ &w_s(0) = u_s \quad \text{and} \quad \nu\, w_{s,x}(1) = g_s. \end{aligned}$$

The eigenvalue problem for the linearized model is

$$\begin{aligned} &-\nu\, z_{xx} + w_s\, z_x + z\, w_{s,x} = \lambda\, z \quad \text{in } (0,1), \\ &z(0) = 0 \quad \text{and} \quad \nu\, z_x(1) = 0. \end{aligned}$$

As above, we make the following change of unknowns

$$z(x) = e^{\beta(x)} v.$$

The equation for v is

$$-\nu\, v_{xx} + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v = \lambda\, v \quad \text{in } (0,1),$$

$$v(0) = 0 \quad \text{and} \quad \nu\, v_x(1) + \frac{w_s(1)}{2}\, v(1) = 0.$$

if we choose

$$\beta_x = \frac{w_s}{2\nu} \quad \text{and} \quad \beta_{xx} = \frac{w_{s,x}}{2\nu}.$$

We set

$$\mathcal{D}(\mathcal{A}) = \{v \in H^2(0,1) \mid v(0) = 0,\; v_x(1) + \frac{w_s(1)}{2\nu}\, v(1) = 0\},$$

and

$$\mathcal{A}v = -\nu\, v_{xx} + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v.$$

Integrating by part, we obtain

$$\int_0^1 \left(-\nu\, v_{xx} + \left(\frac{w_s}{2} + \frac{w_s^2}{4\nu}\right) v\right) v\, dx$$

$$= \int_0^1 \left(\nu\, v_x^2 + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v^2\right)\, dx - \nu\, (v\, v_x)\,|_{x=1}$$

$$= \int_0^1 \left(\nu\, v_x^2 + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v^2\right)\, dx + \frac{w_s(1)}{2}\, v^2(1).$$

We have

$$\int_0^1 v^2\, dx \leq \frac{16}{\pi^2} \int_0^1 v_x^2\, dx \quad \text{for all } v \in H^1_{\{0\}}(0,1),$$

and the equality holds for

$$v(x) = \sin\left(\frac{\pi x}{4}\right).$$

Thus for such a function v, it yields

$$\int_0^1 \left(\nu\, v_x^2 + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v^2\right)\, dx + \frac{w_s(1)}{2}\, v^2(1)$$

$$= \int_0^1 \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu} + \nu\frac{\pi^2}{16}\right) v^2\, dx + \frac{w_s(1)}{2}\, v^2(1).$$

We look for w_s such that $w_s(1) < 0$ and

$$\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu} + \nu\frac{\pi^2}{16} < 0.$$

We look for the solution to the equation

$$2\nu\, w' + w^2 + \nu^2\pi^2/4 = -\nu^2\pi^2(2\varepsilon + \varepsilon^2)/4,$$

where $0 < \varepsilon < 1$ has to be chosen. We have

$$\frac{2\nu\, w'}{\nu^2\pi^2(1+\varepsilon)^2/4} + \frac{w^2}{\nu^2\pi^2(1+\varepsilon)^2/4} + 1 = 0.$$

Thus

$$\frac{\frac{2}{\pi(1+\varepsilon)/2}\,\frac{w'}{\nu\pi(1+\varepsilon)/2}}{\left(\frac{w}{\nu\pi(1+\varepsilon)/2}\right)^2 + 1} = -1.$$

Integrating from 0 to x, we have

$$\frac{4}{\pi(1+\varepsilon)}\arctan\left(\frac{w}{\nu\pi(1+\varepsilon)/2}\right) - \frac{4}{\pi(1+\varepsilon)}\arctan\left(\frac{w_0}{\nu\pi(1+\varepsilon)/2}\right) = -x,$$

and

$$w(x) = -\frac{\nu\pi(1+\varepsilon)}{2}\tan\left(\frac{\pi(1+\varepsilon)x}{4} + C_0\right).$$

For $x = 1$

$$w(1) = -\frac{\nu\pi(1+\varepsilon)}{2}\tan\left(\frac{\pi(1+\varepsilon)}{4} + C_0\right).$$

Thus, we can choose $0 < \varepsilon < 1$ so that $w(1)$ takes any prescribed value in the interval $(-\infty, 0)$.

In particular we would like to have

$$v_x(1) + \frac{w_s(1)}{2\nu}\,v(1) = 0,$$

with $v(x) = \sin\left(\frac{\pi x}{4}\right)$.

$$v_x(1) = \frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{8}, \quad v(1) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

and

$$w_s(1) = -\frac{\pi\nu}{2}.$$

We have to solve

$$\frac{\nu\pi(1+\varepsilon)}{2}\tan\left(\frac{\pi(1+\varepsilon)}{4} + C_0\right) = \frac{\pi\nu}{2},$$

that is

$$\tan\left(\frac{\pi(1+\varepsilon)}{4} + C_0\right) = \frac{\pi}{4}\left(\frac{4}{\pi(1+\varepsilon)}\right).$$

For $C_0 = 0$, the equation

$$\tan\left(\frac{\pi(1+\varepsilon)}{4}\right) = \frac{\pi}{4}\left(\frac{4}{\pi(1+\varepsilon)}\right)$$

admits the solution

$$\frac{\pi(1+\varepsilon)}{4} = \frac{\pi}{4}, \quad \varepsilon = 0.$$

For $-\pi/4 < C_0 < 0$, we have a solution ε such that

$$\frac{\pi(1+\varepsilon)}{4} > \frac{\pi}{4}.$$

Thus we can find $\varepsilon > 0$ for which $w_s(1) = -\frac{\pi\nu}{2}$. For such a function w_s, we have

$$\int_0^1 \left(\nu\, v_x^2 + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v^2\right) dx + \frac{w_s(1)}{2}\, v^2(1)$$
$$= -\frac{\nu^2\pi^2(2\varepsilon+\varepsilon^2)}{4}\int_0^1 v^2\, dx - \frac{\pi\nu}{4}\, v^2(1).$$

This means that the linearized Burgers operator has negative eigenvalues. Indeed, $v \in \mathcal{D}(\mathcal{A})$, and

$$\int_0^1 (\mathcal{A}v\, v)\, dx = -\frac{\nu^2\pi^2(2\varepsilon + \varepsilon^2)}{4} \int_0^1 v^2\, dx - \frac{\pi\nu}{4}\, v^2(1) < 0.$$

Since $\mathcal{A}$ is self-adjoint, this inequality ensures the existence of negative eigenvalues.

Case of Dirichlet boundary conditions.

In the case of Dirichlet boundary conditions, using the same change of unknowns as above, the eigenvalue prtoblem for v is

$$-\nu\, v_{xx} + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v = \lambda\, v \quad \text{in } (0,1),$$
$$v(0) = 0 \quad \text{and} \quad v(1) = 0.$$

Integrating by part, we obtain

$$\int_0^1 \left(-\nu\, v_{xx} + \left(\frac{w_s}{2} + \frac{w_s^2}{4\nu}\right) v\right) v\, dx$$
$$= \int_0^1 \left(\nu\, v_x^2 + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v^2\right)\, dx.$$

We have

$$\int_0^1 v^2\, dx \leq \frac{1}{\pi^2} \int_0^1 v_x^2\, dx \quad \text{for all } v \in H_0^1(0,1),$$

and the equality holds for

$$v(x) = \sin\left(\pi x\right).$$

We choose $v(x) = \sin\left(\pi x\right)$, which is the function giving the best constant in the above inequality. We have

$$\int_0^1 \left(\nu\, v_x^2 + \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu}\right) v^2\right)\, dx = \int_0^1 \left(\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu} + \nu\pi^2\right) v^2\, dx.$$

If

$$\frac{w_{s,x}}{2} + \frac{w_s^2}{4\nu} + \nu\pi^2 < 0,$$

we shall have negative eigenvalues. We look for the solution to the equation

$$2\nu\, w' + w^2 + 4\nu^2\pi^2 = -4\nu^2\pi^2(2\varepsilon + \varepsilon^2),$$

where $0 < \varepsilon < 1$ has to be chosen. We have

$$\frac{2\nu\, w'}{4\nu^2\pi^2(1+\varepsilon)^2} + \frac{w^2}{4\nu^2\pi^2(1+\varepsilon)^2} + 1 = 0.$$

Thus

$$\frac{\frac{1}{\pi(1+\varepsilon)}\frac{w'}{2\nu\pi(1+\varepsilon)}}{\left(\frac{w}{2\nu\pi(1+\varepsilon)}\right)^2 + 1} = -1.$$

Integrating from 0 to x, we have

$$\frac{1}{\pi(1+\varepsilon)}\arctan\left(\frac{w}{2\nu\pi(1+\varepsilon)}\right) - \frac{1}{\pi(1+\varepsilon)}\arctan\left(\frac{w_0}{2\nu\pi(1+\varepsilon)}\right) = -x,$$

and

$$w(x) = -2\nu\pi(1+\varepsilon)\tan\left(\pi(1+\varepsilon)x + C_0\right).$$

We do not obtain a continuous function over $[0,1]$.

Explicit expression of w_s

We choose $C_0 = -\pi/8$. We have to solve the equation

$$\tan\left(\frac{\pi}{4}(1+\varepsilon) - \frac{\pi}{8}\right) = \frac{1}{1+\varepsilon}.$$

e find the approximate solution

$$\varepsilon = 0.323745909529.$$

The corresponding w_s is defined by

$$w(x) = -\frac{\nu\pi(1+\varepsilon)}{2}\tan\left(\frac{\pi(1+\varepsilon)x}{4} - \frac{\pi}{8}\right).$$