Numerical schemes for hyperbolic equations

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Motivation

Many problems in physics have the form

of **Conservation Laws** $\partial_t U + \nabla_x \cdot F(U) = 0$

where

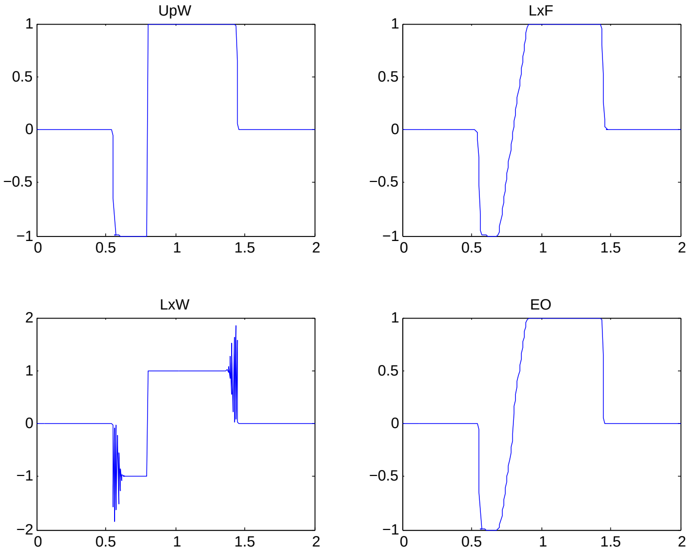
- U can be a scalar or a vector, thus $F(U)$ a vector or a matrix,
- x can be one- or multi-dimensional

or **Balance Laws** $\partial_t U + \nabla_x \cdot F(U) = S(U)$.

Goal: To design a "good and efficient" numerical method

- mathematical and physical criterion
- non linearities
- conservation of equilibria ($\nabla_x \cdot F(U_{\mathrm{eq}}) = S(U_{\mathrm{eq}})$).

Example ▸ Kr ▸ Adm



What does "conservation" mean?

Density of a passive tracer immersed in a fluid

- $\rho(t,x)$ = density of "particles":
  $\int_\Omega \rho(t,y)\,\mathrm{d}y$ = Mass contained in Ω at time t
- $u(t,x)$ = (God-given) velocity of the fluid
- **Mass balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(t,y)\,\mathrm{d}y = -\int_{\partial\Omega} \rho(t,y)u(t,y)\cdot\nu(y)\,\mathrm{d}\sigma(y).$$

- Integrating by parts yields the PDE $\partial_t\rho + \nabla_x\cdot(\rho u) = 0$.

Moto

To design numerical schemes by mimicking the physical derivation of the equation (Finite Volume Schemes)

The cornerstone is the notion of **flux**

- Given a physical quantity U, its evolution in a domain Ω is driven by gain/loss through the boundaries described by a **flux** Q so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} U(t,y)\,\mathrm{d}y = \int_{\partial\Omega} Q(t,y)\cdot\nu(y)\,\mathrm{d}\sigma(y).$$

- Then, **a physical law** prescribes how Q depends on U.
- Example: U =temperature, Fourier's law: $Q = -k\nabla_x U$. It leads to the **Heat Eq.** $\partial_t U = \nabla_x \cdot (k\nabla U)$. But this eq. does not belong to the framework of Hyperbolic problems... Main differences: **Infinite Speed of Propagation & Regularizing Effects**

Examples

Transport eq.

$\partial_t \rho + \nabla_x \cdot (\rho u) = 0$

Kinetic eq. (Statistical physics)

Description in phase space $\partial_t f + \nabla_x \cdot (\xi f) = \text{Interaction terms}$, with $f(t, x, \xi)$ depending on space **and** velocity.

Non linear models: traffic flows

Lighthill-Whitham-Richards' model: ρ=density of vehicles, the velocity is $u(t,x) = V_0(1-\rho)$ **depends on** ρ!

$$\partial_t \rho + \partial_x (V_0 \rho (1-\rho)) = 0, \qquad \text{the flux is } V_0 \rho (1-\rho)$$

Non linear models: Burgers eq.

A toy model for gas dynamics

$$\partial_t \rho + \partial_x (\rho^2/2) = 0$$

Examples Contn'd

Waves eq. (linear system)

$\partial_t u + c\partial_x v = 0, \qquad \partial_t v + c\partial_x u = 0$

- leads to $\partial_{tt}^2 u - c^2 \partial_{xx}^2 u = 0$.
- Set $W_\pm = u \pm v$, then $\partial_t W_\pm \pm c\partial_x W_\pm = 0$ that is a system of transport eq. (or a kinetic model with 2 velocities).

Euler system

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho u^2/2 + p)u \end{pmatrix} = 0$$

with $E = u^2/2 + e$, $p = p(\rho, e)$. (For instance $p = 2\rho e$.)

Moto

A numerical scheme for a complex system should first work on simple equations!

The **NON-CONSERVATIVE** transport equation

$$\partial_t \rho + u \partial_x \rho = 0$$

Define the Characteristics

Assume that $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies $|u(t,x)| \leq C(1+|x|)$. Then we can apply the **Cauchy-Lipschitz theorem** and define the **Characteristic Curves**

$$\frac{\mathrm{d}}{\mathrm{d}s} X(s;t,x) = u(s, X(s;t,x)), \qquad X(t;t,x) = x.$$

$X(s;t,x)$ is the position occupied at time s by a particle which starts from position x at time t.

Go back to the PDE

- Chain Rule: $\frac{\mathrm{d}}{\mathrm{d}s}\Big[\rho(s, X(s;t,x))\Big] = (\partial_t \rho + u \cdot \nabla_x \rho)(s, X(s;t,x)) = 0$
- Integrate between $s = 0$ and $s = t$: $\rho(t,x) = \rho_{\mathrm{Init}}(X(0;t,x))$.

The **CONSERVATIVE** transport equation

$$\partial_t \rho + \partial_x(\rho u) = \partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0$$

becomes $\dfrac{\mathrm{d}}{\mathrm{d}s}\Big[\rho(s, X(s; t, x))\Big] = -\rho \partial_x u(s, X(s; t, x))$.
Therefore $\rho(t, x) = \rho_{\mathrm{Init}}(X(0; t, x))\ J(0; t, x)$ with

$$J(s; t, x) = \exp\left(- \int_s^t \partial_x u(\sigma, X(\sigma; t, x))\, \mathrm{d}\sigma \right).$$

Interpretation: J is the jacobian of $y = X(s; t, x)$

$$\begin{cases} \partial_s\big(\partial_x X(s; t, x)\big) = (\partial_x u)(s, X(s; t, x))\ \partial_x X(s; t, x), \\ \partial_x X(t; t, x) = 1. \end{cases}$$

We deduce that

$$\partial_x X(s; t, x) = \exp\left(+ \int_t^s \partial_x u(\sigma, X(\sigma; t, x)\, \mathrm{d}\sigma \right) = J(s; t, x)$$

and $\mathrm{d}y = J(s; t, x)\, \mathrm{d}x$.

Fundamental observations

- Maximum principle: for the non-conservative case if $0 \leq \rho_{\rm Init}(x) \leq M$, then $0 \leq \rho(t,x) \leq M$; for the conservative case if $\rho_{\rm Init}(x) \geq 0$ then $\rho(t,x) \geq 0$.
- Mass conservation: for the conservative case $\int_{\mathbb{R}} \rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} \rho_{\rm Init}(y)\,\mathrm{d}y$.
- For $\rho_{\rm Init} \in C^1$, we get solutions in C^1 (no gain of regularity)
- The discussion extends to the multi-dimensional framework.
- The formulae generalize to data in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$: it provides a (unique) solution in $C^0([0,T], L^p(\mathbb{R}))$ for $1 \leq p < \infty$, in $C^0([0,T], L^\infty(\mathbb{R}) - \mathrm{weak} - \star)$ for $p = \infty$.

Hints for proving uniqueness (Conservative case)

Weak solution

For any trial function $\phi \in C_c^1([0,\infty) \times \mathbb{R})$,

$$-\int_0^\infty \int_{\mathbb{R}} \rho(t,x)\big(\partial_t\phi(t,x)+u(t,x)\partial_x\phi(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}} \rho_{\mathrm{Init}}(x)\phi(0,x)\,\mathrm{d}x = 0.$$

Hölmgren's method

Let $\psi \in C_c^\infty((0,+\infty) \times \mathbb{R})$. $\psi(t,\cdot) = 0$ for $t \geq T$. Solve $\partial_t\phi + u\partial_x\phi = \psi$ with final data $\phi\big|_{t=T} = 0$. Precisely, we have

$$\phi(t,x) = \int_T^t \psi(\sigma, X(\sigma;t,x))\,\mathrm{d}\sigma \in C_c^1([0,+\infty) \times \mathbb{R}).$$

thus $$\int_0^\infty \int_{\mathbb{R}} \rho(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Fundamental example

Linear Transport with Constant Speed

Let $c \in \mathbb{R}$. Consider the PDE

$$\partial_t \rho + c\partial_x \rho = \partial_t \rho + \partial_x(c\rho) = 0.$$

Exact solution is known: $\rho(t,x) = \rho_{\mathrm{Init}}(x - ct)$.

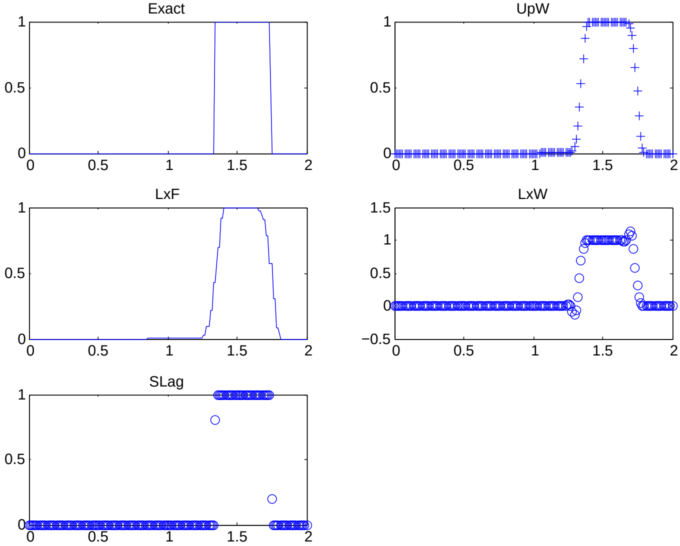
To be compared with the solution of the heat equation $\partial_t \rho = k\partial^2_{xx}\rho$ which is given by

$$\rho(t,x) = \frac{1}{\sqrt{4\pi t/k}} \int_{\mathbb{R}} e^{-k|x-y|^2/(4t)} \rho_{\mathrm{Init}}(y)\,\mathrm{d}y.$$

(Infinite speed of propagation and regularization of the data.)

Exercise: Find the solution of $\partial_t \rho_\epsilon + \partial_x(c\rho_\epsilon) = \epsilon\partial^2_{xx}\rho_\epsilon$
and its limit as $\epsilon \to 0$.

Behavior of different schemes (initial data=step, speed>0)



Nonlinear problems

Linear transport: For C^k data, we get a C^k solution.
Let us try to reproduce the reasoning for a non linear problem:

$$\text{Burgers eq.} \quad \partial_t \rho + \partial_x \rho^2/2 = 0 = (\partial_t + \rho \partial_x)\rho.$$

We still get $\rho(t,x) = \rho_{\text{Init}}(X(0;t,x))$. BUT now the characteristics depend on the solution itself

$$\frac{\mathrm{d}}{\mathrm{d}s} X(s;t,x) = \rho(s, X(s;t,x)), \qquad X(t;t,x) = x.$$

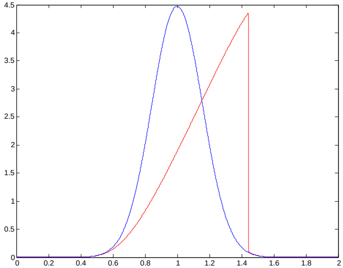
Singularities might appear in finite time

Let $v(t,x) = \partial_x \rho(t,x)$: $(\partial_t + \rho \partial_x) v = -v^2$. Along characteristics we recognize the **Ricatti eq.**

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[v(s, X(s;t,x))\right] = -v^2(s, X(s;t,x))$$

We get $v(t,x) = \left(t + \dfrac{1}{\partial_x \rho_{\text{Init}}(X(0;t,x))}\right)^{-1}$. Blow up when $\partial_x \rho_{\text{Init}} \leq 0$.

Loss of regularity for nonlinear problems



- ρ remains bounded but $\partial_x \rho$ becomes singular
- Characteristics not well-defined: Cauchy-Lipschitz th. does not apply

Another way to think of the loss of regularity

- Sol constant along characteristics
- $\frac{\mathrm{d}}{\mathrm{d}t} X(t;0,x) = f'\big((\rho(t;X(t;0,x))\big) = f'\big(\rho_{\mathrm{Init}}(x)\big)$ hence $X(t;0,x) = x + tf'\big(\rho_{\mathrm{Init}}(x)\big) = \phi_t(x)$
- To find $\rho(t,x)$ by means of $\rho_{\mathrm{Init}}(x)$, one needs to invert $x \mapsto \phi_t(x)$. But $\phi_t'(x) = 1 + tf''\big(\rho_{\mathrm{Init}}(x)\big)\rho_{\mathrm{Init}}'(x)$ might change sign.

Inria

(We need) Weak solution for Scalar Conservation Laws

Definition

For any trial function $\phi \in C_c^1([0,\infty) \times \mathbb{R})$,

$$-\int_0^\infty \int_{\mathbb{R}} \big(\rho \partial_t \phi + f(\rho) \partial_x \phi\big)(t,x) \,\mathrm{d}x \,\mathrm{d}t - \int_{\mathbb{R}} \rho_{\mathrm{Init}}(x) \phi(0,x) \,\mathrm{d}x = 0.$$

Rankine-Hugoniot conditions

Discontinuities satisfy $[\![f(\rho)]\!] = \dot{s} [\![\rho]\!]$.

Non uniqueness

Burgers eq. with $\rho_{\mathrm{Init}} = 0$: $\rho_1(t,x) = 0$ and $\rho_2(t,x) = \mathbf{1}_{0<x<t/2} - \mathbf{1}_{-t/2<x<0}$ are both weak solutions!

How to select among weak sol.: entropy criterion

Observe that for **smooth** solutions of $\partial_t \rho + \partial_x f(\rho) = 0$, we have

$$\partial_t \eta(\rho) + \partial_x q(\rho) = 0, \qquad q'(z) = \eta'(z) f'(z).$$

But, discontinuous solutions DO NOT verify this relation. ▸ Det

Definition

A weak solution ρ is said to be entropic, if, *for any convex function* η, we have

$$-\int_0^\infty \int_{\mathbb{R}} \big(\eta(\rho)\partial_t \phi + q(\rho)\partial_x \phi\big)(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} \eta(\rho_{\mathrm{Init}})(x)\phi(0,x)\,\mathrm{d}x \leq 0$$

for any non negative trial function $\phi \geq 0$. ("$\partial_t \eta(\rho) + \partial_x q(\rho) \leq 0$")

Kruzkov's Theorem

The SCL admits a unique weak-entropic solution with $\rho \in C^0([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$. ▸ ExBu

Admissible discontinuities

Go back to the Rankine-Hugoniot condition: $[\![q(u)]\!] \leq \dot{s}[\![\eta(u)]\!]$.
Use Kruzkhov's entropies

$$\eta_k(u) = |u - k|, \qquad q_k(u) = (f(u) - f(k))\operatorname{sgn}(u - k).$$

- with $k < \min(u_\ell, u_r)$, and $k > \max(u_\ell, u_r)$: back to RH.
- with $k = \theta u_\ell + (1 - \theta) u_r$ it leads to

$$\operatorname{sgn}(u_r - u_\ell)\Big(\theta f(u_\ell) + (1 - \theta) f(u_r) - f(\theta u_\ell + (1 - \theta) u_r)\Big) \leq 0$$

Letting $\theta \to 0$, $\theta \to 1$, it yields the Lax criterion

$$f'(u_r) \leq \dot{s} \leq f'(u_\ell).$$

- In particular, when the flux f is **convex**, admissible discontinuities satisfy $u_r \leq u_\ell$. ExBu

Entropy and vanishing viscosity approach

- Owing to regularizing effects, one can prove the existence of solutions for the regularized problem

$$\partial_t \rho_\epsilon + \partial_x f(\rho_\epsilon) = \epsilon \partial^2_{xx} \rho_\epsilon$$

- Compare Euler and Navier-Stokes: ϵ plays the role of "viscosity". Besides, "good" numerical schemes induce such kind of regularization.
- Entropy estimates

$$\begin{aligned} \partial_t \eta(\rho_\epsilon) + \partial_x q(\rho_\epsilon) &= \epsilon \eta'(\rho_\epsilon) \partial^2_{xx} \rho_\epsilon \\ &= \epsilon \partial_x \big(\eta'(\rho_\epsilon) \partial_x \rho_\epsilon\big) - \epsilon \eta''(\rho_\epsilon) |\partial_x \rho_\epsilon|^2 \end{aligned}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \eta(\rho_\epsilon)\,\mathrm{d}x + \epsilon \int \eta''(\rho_\epsilon) |\partial_x \rho_\epsilon|^2 \,\mathrm{d}x = 0.$$

- In particular, with $\eta(\rho) = \rho^2/2$, we deduce that

ρ_ϵ is bounded in $L^\infty(0, T; L^2(\mathbb{R}))$,

$\sqrt{\epsilon}\partial_x \rho_\epsilon$ is bounded in $L^2((0, T \times \mathbb{R})$.

Entropy and vanishing viscosity approach, Contn'd

- We know that

$$\rho_\epsilon \text{ is bounded in } L^\infty(0,T;L^2(\mathbb{R}))$$

$$\sqrt{\epsilon}\partial_x\rho_\epsilon \text{ is bounded in } L^2((0,T)\times\mathbb{R})$$

- Similarly we can obtain L^∞ estimates (use for instance $\eta(\rho) = \left[\rho - \|\rho_{\text{Init}}\|_\infty\right]_-^2$)
- Therefore

$$\partial_t\rho_\epsilon + \partial_x f(\rho_\epsilon) = \sqrt{\epsilon}\partial_x\left(\sqrt{\epsilon}\partial_x\rho_\epsilon\right) \xrightarrow[\epsilon\to 0]{} 0$$

and, on the same token,

$$\partial_t\eta(\rho_\epsilon) + \partial_x q(\rho_\epsilon) = \underbrace{\sqrt{\epsilon}\partial_x\left(\sqrt{\epsilon}\eta'(\rho_\epsilon)\partial_x\rho_\epsilon\right)}_{\xrightarrow[\epsilon\to 0]{} 0} \underbrace{-\epsilon\eta''(\rho_\epsilon)|\partial_x\rho_\epsilon|^2}_{\leq 0}$$

Discontinuous solutins and entropies ▸ Kr

For discontinuous solutions, we make the following quantity appear (by reproducing the computations that lead to RH relations)

$$
\begin{aligned}
&[\![\eta(\rho)]\!]\dot{s} - [\![q(\rho)]\!] = (\eta(\rho_r) - \eta(\rho_\ell))\dot{s} - (q(\rho_r) - q(\rho_\ell)) \\
&= \int_{\rho_\ell}^{\rho_r} \eta'(z)\dot{s}\,\mathrm{d}z - \int_{\rho_\ell}^{\rho_r} q'(z)\dot{s}\,\mathrm{d}z \\
&= \int_{\rho_\ell}^{\rho_r} \eta'(z)\dot{s}\,\mathrm{d}z - \int_{\rho_\ell}^{\rho_r} \eta' f'(z)\,\mathrm{d}z \\
&= -\int_{\rho_\ell}^{\rho_r} \eta''(z)\Big(\frac{f(\rho_r) - f(\rho_\ell)}{\rho_r - \rho_\ell}(z - \rho_\ell) - \big(f(z) - f(\rho_\ell)\big)\Big)\,\mathrm{d}z \\
&= -\int_{\rho_\ell}^{\rho_r} \eta''(z)(z - \rho_\ell)\,\left(\frac{f(\rho_r) - f(\rho_\ell)}{\rho_r - \rho_\ell} - \frac{f(z) - f(\rho_\ell)}{z - \rho_\ell}\right)\mathrm{d}z.
\end{aligned}
$$

Since η is convex, $z \mapsto \eta''(z)(z - \rho_\ell)$ has a constant sign on the interval I defined by ρ_r and ρ_ℓ. Assuming that f is convex or concave on I, the integrand has a constant sign and $[\![\eta(\rho)]\!]\dot{s} - [\![q(\rho)]\!]$ vanishes iff

$f(z) - f(\rho_\ell) = \dfrac{f(\rho_r) - f(\rho_\ell)}{\rho_r - \rho_\ell}(z - \rho_\ell)$ on I. It would mean that f is an affine function on I, a case that we exclude by assumption

(the flux is assumed “genuinely non linear”).