Num. #1: Ordinary differential equations (ODEs)

The aim of this session consists in solving ordinary differential equations (ODEs) of type :

$$\partial_t u = f(t,u) \tag{1a}$$

where the unknown is $u : [0,T] \longrightarrow \mathbb{R}^d$, $d \in \mathbb{N}$. $f : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ denotes a given function assumed to be continuous and Lipschitz continuous with respect to the state variable, in order to ensure the hypotheses of Cauchy-Lipschitz Theorem. We complement this equation with a given initial datum :

$$u(0) = a \in \mathbb{R}^d. \tag{1b}$$

We discretize the time interval $[0,T]$ with a constant time step Δt and we consider the discrete times :

$$t^n = n\Delta t,\ n \in \{0,\dots,N_t\}.$$

We denote by u^n the discrete approximation of the exact solution u at time t_n.

We study several methods to discretize equation (1a) with initial condition (1b) :

- **Forward Euler method** :
$$u^{n+1} = u^n + \Delta t f(t^n, u^n).$$

- **Midpoint (or Runge-Kutta 2) method** :
$$u^{n+1} = u^n + \Delta t f(t^n + \frac{\Delta t}{2}, u^n + \frac{\Delta t}{2} f(t^n, u^n)),$$
which can also be written as :
$$\begin{aligned} u^{n+1/2} &= u^n + \frac{\Delta t}{2} f(t^n, u^n) \\ u^{n+1} &= u^n + \Delta t f(t^n + \frac{\Delta t}{2}, u^{n+1/2}). \end{aligned}$$

- **Heun method** :
$$\begin{aligned} p_1^n &= f\left(t^n, u^n\right) \\ u^{n+1/2} &= u^n + \Delta t p_1^n \\ p_2^n &= f\left(t^n + \Delta t, u^{n+1/2}\right) \\ u^{n+1} &= u^n + \frac{\Delta t}{2}\left(p_1^n + p_2^n\right). \end{aligned}$$

- **Runge-Kutta 4 method** :

$$
\begin{aligned}
p_1^n &= f\left(t^n, u^n\right),\\
p_2^n &= f\left(t^n + \frac{\Delta t}{2}, u^n + \frac{\Delta t}{2} p_1^n\right)\\
p_3^n &= f\left(t^n + \frac{\Delta t}{2}, u^n + \frac{\Delta t}{2} p_2^n\right)\\
p_4^n &= f\left(t^n + \Delta t, u^n + \Delta t p_3^n\right)\\
u^{n+1} &= u^n + \frac{\Delta t}{6}\left(p_1^n + 2p_2^n + 2p_3^n + p_4^n\right).
\end{aligned}
$$

Exercise

1. Implement (with MATLAB) the resolution of the following equation :

$$\partial_t u = 1 - u^2 \text{ with } u(0) = 0 \tag{2}$$

   using the four methods presented above and a time step $\Delta t = 0.1$ until time $T = 1$.

2. Remark that the exact solution of equation (2) can be computed exactly and is equal to :

$$u(t) = tanh(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

   Enhance the convergence property of the Heun method by letting Δt go to 0.

3. We still consider equation (2). Compare the order of the four methods by plotting a graph in a log-log scale, which represents the evolution of L^2 error with respect to the time step Δt.

4. Same question using now the following equation

$$\partial_t u = e^{-u} - 1 + u \text{ with } u(0) = 1, \tag{3}$$

   for which no exact solution is known. The error will be defined as the L^2 norm of the difference between the solution computed with a time step Δt and the solution computed with a time step $\frac{\Delta t}{2}$.

5. Implement the resolution of the following system of equations, called the Lorenz system :

$$
\begin{cases}
\partial_t x = 10(y - x)\\
\partial_t y = x(28 - z) - y\\
\partial_t z = xy - \frac{8}{3} z
\end{cases}
\tag{4}
$$

   using the four methods presented above. The parameters will be : $(x_0, y_0, z_0) = (1, 1, 1)$, $T = 30$ and $\Delta t = 0.001$.