Num. #2: Hyperbolic PDE equation : transport equation

The aim of this session consists in solving partial differential equations (PDEs) of transport type:

$$\partial_t u + \partial_x(a(t,x)u) = 0, \tag{1a}$$

where the unknown is $u : [0,T] \times [0,L] \longrightarrow \mathbb{R}$. $a : [0,T] \times [0,L] \longrightarrow \mathbb{R}$ denotes a given vector field assumed to be continuous. This equation is complemented with an initial datum :

$$u(0,x) = u_0(x),\ x \in [0,L] \tag{1b}$$

and, if needed, boundary conditions. This problem is restricted to the periodic boundary condition case in the present work.

In the case when $a(x) = a$ is a constant, the exact solution is given by $u(x,t) = u_0(x - at)$. Otherwise, we compute the characteristic curves $X(s;t,x)$, which are solutions to the following ODE :

$$\frac{d}{ds}X(s) = a(s,X(s)), \qquad X(t) = x,$$

and the solution of system (1) is given by :

$$u(t,x) = u_0(X(0))\ \exp\left(-\int_0^t (\partial_x a)(s,X(s))\,ds\right),$$

We discretize $[0,L]$ with a constant space step Δx and we consider the discrete points $(x_i)_{i\in\{0,\dots,N_x\}}$, with:

$$x_i = i\Delta x,\ i \in \{0,\dots,N_x\}.$$

We define cells of which x_i is the middle and we note $x_{i+1/2} = \left(i+\frac{1}{2}\right)\Delta x$ such that x_i is the middle of $]x_{i-1/2}, x_{i+1/2}[$. We now discretize the time interval $[0,T]$ with a constant time step Δt and we consider the discrete times :

$$t^n = n\Delta t,\ n \in \{0,\dots,N_t\}.$$

We denote by u_i^n the discrete approximation of the exact solution u at time t_n and at point x_i (the unknowns are located in the center of the cells).

We consider the following numerical schemes in order to find an approximation of the solution to equation (1a) with initial condition (1b). They are all defined under the same common form :

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n) \tag{2a}$$

with

$$F_{i+1/2}^n = \mathscr{F}(u_i^n, u_{i+1}^n), \tag{2b}$$

where $\mathscr{F}$ is the numerical flux.

We consider the following fluxes satisfying the consistency condition $\mathscr{F}(u,u) = a(t,x)u$:

- **Upwind scheme** :

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \begin{cases} a(t^n, x_{i+1/2})u_i^n & \text{if } a(t^n, x_{i+1/2}) \geq 0, \\ a(t^n, x_{i+1/2})u_{i+1}^n & \text{if } a(t^n, x_{i+1/2}) < 0 \end{cases}, \text{ where } x_{i+1/2} = \frac{x_i + x_{i+1}}{2},$$

  which can be written as :

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \frac{a(t^n, x_{i+1/2})}{2}(u_i^n + u_{i+1}^n) - \frac{|a(t^n, x_{i+1/2})|}{2}(u_{i+1}^n - u_i^n). \tag{3a}$$

- **Lax-Friedrichs scheme** :

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \frac{1}{2}\big(a(t^n, x_i)u_i^n + a(t^n, x_{i+1})u_{i+1}^n\big) - \frac{\Delta x}{2\Delta t}\big(u_{i+1}^n - u_i^n\big). \tag{3b}$$

- **Lax-Wendroff scheme** :

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \frac{1}{2}\big(a(t^n, x_i)u_i^n + a(t^n, x_{i+1})u_{i+1}^n\big) - \frac{\Delta t}{2\Delta x}a(t^n, x_{i+1/2})\big(a(t^n, x_{i+1})u_{i+1}^n - a(t^n, x_i)u_i^n\big). \tag{3c}$$

For all these schemes, the stability condition is equal to

$$\Delta t \leq \frac{\Delta x}{\max|a|}.$$

Exercise

1. To begin with, define a vector with the discrete points (x_i) which discretize the interval $[0,5]$ with a space step $\Delta x = 0.1$. Define a discretization of the three following initial data and plot them :

$$u_0(x) = \mathrm{e}^{-(x-2)^2/0.1} \tag{4a}$$

$$u_0(x) = \begin{cases} 1 - |x-2| & \text{if } 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases} \tag{4b}$$

$$u_0(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{4c}$$

2. Implement the resolution of the following equations :

$$\partial_t u + \partial_x u = 0, \tag{5a}$$

$$\partial_t u + \partial_x (\sin(\frac{2\pi}{L}x)u) = 0, \tag{5b}$$

   using the three methods presented above and a time step $\Delta t = 0.04$ until time $T = 1$. We still consider the interval $[0,5]$ with a space step $\Delta x = 0.1$ and we will use function (4a) as an initial datum. We take some periodic boundary conditions $u(t,0) = u(t,L)$.

3. Use the previous program for equation (5a) with initial datum (4a) with the following time steps : $\Delta t = 0.2, 0.1, 0.09, 0.05$. What do you notice ?

4. We still consider equation (5a) with initial datum (4a) and the following parameters $T = 1$, $L = 5$, $\Delta t = 0.95\Delta x$. Compare the order of the three methods by plotting a graph in a log-log scale, which represents the evolution of L^2 error with respect to the space step Δx.

5. Compare the three schemes for equation (5a) in the case of the two other initial data (4b) and (4c), with $T = 1$, $L = 5$, $\Delta x = 0.01$, $\Delta t = 0.95\Delta x$. . What do you notice ?