Num. #3: Hyperbolic PDE equation : 1D conservation law

The aim of this session consists in solving conservation laws:

$$\partial_t u + \partial_x(f(u)) = 0, \tag{1a}$$

where the unknown is $u : [0,T] \times [0,L] \longrightarrow \mathbb{R}$. $f : \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be a C^2- continuous function and we define $\boldsymbol{a(u) = f'(u)}$. This equation is complemented with an initial datum :

$$u(0,x) = u_0(x),\ x \in [0,L] \tag{1b}$$

and, if needed, boundary conditions.

We will consider in the following a classical example, the Burgers equation, defined by

$$f(u) = \frac{u^2}{2}. \tag{2}$$

We briefly recall that the entropic solution for equation (2) with the initial datum $u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$ is equal to $u(x,t) = \begin{cases} 1, & \text{if } x < t/2 \\ 0, & \text{if } x > t/2 \end{cases}$ However, for the initial datum $u_0(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$, the exact solution is equal to $u(x,t) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{t}, & \text{if } 0 < x < t \\ 1, & \text{if } x > t \end{cases}$.

We discretize $[0,L]$ with a constant space step Δx and we consider the discrete points $(x_i)_{i\in\{0,\ldots,N_x\}}$, with:

$$x_i = i\Delta x,\ i \in \{0,\ldots,N_x\}.$$

We also define

$$x_{i+1/2} = \left(i + \frac{1}{2}\right)\Delta x,\ i \in \{0,\ldots,N_x\}$$

such that x_i is the middle of $]x_{i-1/2}, x_{i+1/2}[$. We discretize the time interval $[0,T]$ with a constant time step Δt and we consider the discrete times :

$$t^n = n\Delta t,\ n \in \{0,\ldots,N_t\}.$$

We denote by u_i^n the discrete approximation of the exact solution u at time t_n and at point x_i.

We consider the following numerical schemes in order to find an approximation of the solution to equation (1a) with initial condition (1b). Some of them are some generalizations of the schemes we saw on the previous session for the transport equation. They write as :

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n) \tag{3a}$$

with

$$F_{i+1/2}^n = \mathscr{F}(u_i^n, u_{i+1}^n), \tag{3b}$$

where $\mathscr{F}$ is the numerical flux. We consider the following fluxes satisfying the consistency condition $\mathscr{F}(u,u) = f(u)$:

- **Upwind conservative scheme**:

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \begin{cases} f(u_i^n) & \text{if } a\left(\dfrac{u_i^n + u_{i+1}^n}{2}\right) \geq 0, \\ f(u_{i+1}^n) & \text{if } a\left(\dfrac{u_i^n + u_{i+1}^n}{2}\right) < 0. \end{cases} \tag{4a}$$

- **Roe scheme**:

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \begin{cases} f(u_i^n) & \text{if } A(u_i^n, u_{i+1}^n) > 0 \\ f(u_{i+1}^n) & \text{if } A(u_i^n, u_{i+1}^n) < 0 \end{cases}, \text{ where } A(u,v) = \begin{cases} \dfrac{f(u) - f(v)}{u - v} & \text{if } u \neq v \\ f'(u) = a(u) & \text{if } u = v \end{cases}. \tag{4b}$$

- **Engquist-Osher scheme**:

$$\mathscr{F}(u_i^n, u_{i+1}^n) = f^+(u_i^n) + f^-(u_{i+1}^n) \text{ with } f^+(u) = \int_0^u \max(a(s), 0)\, ds \text{ and } f^-(u) = \int_0^u \min(a(s), 0)\, ds. \tag{4c}$$

We notice that $f^+(u) + f^-(u) = f(u)$.

- **Lax-Friedrichs scheme**:

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \frac{1}{2}\big(f(u_i^n) + f(u_{i+1}^n)\big) - \frac{\Delta x}{2\Delta t}\big(u_{i+1}^n - u_i^n\big). \tag{4d}$$

- **Rusanov (or Local Lax-Friedrichs) scheme**:

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \frac{1}{2}\big(f(u_i^n) + f(u_{i+1}^n)\big) - \frac{a_{i+1/2}^n}{2}\big(u_{i+1}^n - u_i^n\big), \text{ where } a_{i+1/2}^n = \max_{u \in [u_i^n, u_{i+1}^n]} |a(u)|. \tag{4e}$$

- **Lax-Wendroff scheme**:

$$\mathscr{F}(u_i^n, u_{i+1}^n) = \frac{1}{2}\big(f(u_i^n) + f(u_{i+1}^n)\big) - \frac{\Delta t}{2\Delta x} a\big(\frac{u_{i+1}^n + u_i^n}{2}\big)\big(f(u_{i+1}^n) - f(u_i^n)\big). \tag{4f}$$

Fluxes (4a), (4b) and (4c) are some extensions of the upwind flux for the transport equation.

Equation (1) can also be developed as :

$$\partial_t u + f'(u)\partial_x(u) = 0,$$

i.e.

$$\partial_t u + a(u)\partial_x(u) = 0,$$

which is called the **non-conservative form.** We obtain therefore a non-conservative scheme, namely :

- **Upwind non-conservative scheme**:

$$\begin{aligned} \frac{u_i^{n+1}-u_i^n}{\Delta t} + a(u_i^n)\frac{u_i^n-u_{i-1}^n}{\Delta x} &= 0, \qquad \text{if} \quad a(u_i^n)\geq 0,\\ \frac{u_i^{n+1}-u_i^n}{\Delta t} + a(u_i^n)\frac{u_{i+1}^n-u_i^n}{\Delta x} &= 0, \qquad \text{if} \quad a(u_i^n)< 0, \end{aligned} \tag{4g}$$

which can be written as :

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + \left(a(u_i^n)+|a(u_i^n)|\right)\frac{u_i^n-u_{i-1}^n}{2\Delta x} + \left(a(u_i^n)-|a(u_i^n)|\right)\frac{u_{i+1}^n-u_i^n}{2\Delta x} = 0.$$

We will use the following initial data :

$$u_0(x) = \mathrm{e}^{-(x-2)^2/0.1} \tag{5a}$$

$$u_0(x) = \begin{cases} 1-|x-2| & \text{if } 1\leq x\leq 3\\ 0 & \text{otherwise} \end{cases} \tag{5b}$$

$$u_0(x) = \begin{cases} 1 & \text{if } 1\leq x\leq 2\\ 0 & \text{otherwise} \end{cases} \tag{5c}$$

$$u_0(x) = \begin{cases} -1 & \text{if } 0\leq x\leq 1\\ 1 & \text{if } 1\leq x\leq 2\\ -1 & \text{if } 2\leq x\leq 5 \end{cases} \tag{5d}$$

For all these schemes, the stability condition is equal to

$$\Delta t \leq \frac{\Delta x}{\max|f'(u)|}.$$

Exercise

1. Compute the functions f^+ and f^- of the Engquist-Osher flux in the case of equation (2).
2. Implement the resolution of equation (2) using the seven methods (4) presented above, using time step $\Delta t = 0.04$ until time $T = 1$. We consider the interval $[0,5]$ with a space step $\Delta x = 0.1$ and we will use function (5c) as an initial datum.We take periodic boundary conditions.
3. Compare the seven schemes in the case of the two other initial data (5a) and (5b). What is your conclusion ? Choose one of these schemes and plot the evolution of the solution with time.
4. Show the effect of the CFL condition on the stability of the various schemes.
5. Compare upwind conservative and upwind non-conservative scheme for equation (2) with initial datum (5c). What do you notice ?
6. What are the results of Roe scheme with equation (2) and initial datum (5d). What is your interpretation ? Do the other schemes have the same drawback ?