Num. #4: 1D hyperbolic PDEs system

The aim of this session consists in solving systems of hyperbolic type:

$$\partial_t U + \partial_x (f(U)) = 0, \tag{1a}$$

where the unknown is $U : [0,T] \times [0,L] \longrightarrow \mathbb{R}^d$, $d \in \mathbb{N}$. $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is assumed to be a C^2- continuous function. This equation is complemented with an initial datum :

$$U(0,x) = U_0(x),\ x \in [0,L]. \tag{1b}$$

For this session, we will consider Neumann boundary conditions.

We will consider in the following two classical examples. The first one is the shallow-water system :

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x (hu^2 + \frac{g}{2} h^2) = 0, \end{cases} \tag{2}$$

where h denotes the height of water and u its velocity. $g > 0$ is the gravitational constant and is almost equal to $9.81 m/s^2$.

We consider in this case initial data defined on $[0,5]$ of the form :

$$h_0(x) = \mathrm{e}^{-(x-2)^2/0.1} \text{ and } u_0(x) = 0, \tag{3a}$$

and

$$h_0(x) = \begin{cases} 2 & \text{if } x < 2.5 \\ 1 & \text{if } x \geq 2.5 \end{cases}, \text{ and } u_0(x) = 0, \tag{3b}$$

which corresponds to the "dam-break problem".

The second one is the compressible Euler system:

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p) = 0, \\ \partial_t E + \partial_x (Eu + pu) = 0. \end{cases}$$

where ρ denotes the density of the gas, u its velocity and E its total energy. We introduce the internal energy, linked to the total energy by the following relation :

$$E = \frac{1}{2}\rho u^2 + \rho e.$$

The previous system is closed by the equation of state which links the pressure to the internal energy:

$$p = 2\rho e.$$

We denote by $j = \rho u$ the momentum in order to rewrite previous system in the (ρ, j, E)- variables as :

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \partial_t j + \partial_x (2E) = 0, \\ \partial_t E + \partial_x (3\frac{Ej}{\rho} - \frac{j^3}{\rho^2}) = 0, \end{cases} \tag{4}$$

We set this system on $[0,1]$ and we complement this system with the following initial data :

$$\rho(0,x) = \begin{cases} 1 & \text{if} \quad x \leqslant 0.5, \\ 0.125 & \text{if} \quad x > 0.5, \end{cases} \qquad p(0,x) = \begin{cases} 1 & \text{if} \quad x \leqslant 0.5, \\ 0.1 & \text{if} \quad x > 0.5, \end{cases} \qquad \text{and} \qquad u(0,x) = 0. \tag{5}$$

To implement these systems, we will use an extension of the **Rusanov scheme** (also called Local Lax-Friedrichs scheme) for conservation law, which is defined as follows :

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n) \tag{6a}$$

and the numerical flux is equal to :

$$F_{i+1/2}^n = \frac{1}{2}\big(f(U_i^n) + f(U_{i+1}^n)\big) - \frac{A_{i+1/2}^n}{2}\big(U_{i+1}^n - U_i^n\big), \tag{6b}$$

where

$$A_{i+1/2}^n = \max(\rho(f'(U_i^n)), \rho(f'(U_{i+1}^n))).$$

Here $\rho(M)$ denotes the spectral radius of the matrix M.

The stability condition is equal to

$$\Delta t \leq \frac{\Delta x}{\max \rho(f'(u))}$$

and the Neumann boundary condition will be discretized as follows :

$$h_{-1} = h_0, \quad h_{N_x} = h_{N_x-1} \text{ and } u_{-1} = -u_0, \quad u_{N_x} = -u_{N_x-1}$$

in the case of the shallow-water system (2) and

$$\rho_{-1} = \rho_0, \quad \rho_{N_x} = \rho_{N_x-1} \text{ and } j_{-1} = -j_0, \quad j_{N_x} = -j_{N_x-1} \text{ and } E_{-1} = E_0, \quad E_{N_x} = E_{N_x-1}$$

in the case of the compressible Euler system (4).

Exercise

1. Compute the eigenvalues of the jacobian of systems (2) and (4).

2. Implement the Rusanov (or Local Lax-Friedrichs) scheme.

3. Test it for the shallow-water system (2) with initial data (3a) and (3b) on the domain $[0,5]$ with $\Delta x =$

0.01. The time step will be adapted at every step according to the stability condition. We will use Neumann boundary conditions.

4. Test it for the compressible Euler system (4) with initial data (5) on the domain $[0,1]$ with $\Delta x = 0.01$. The time step will be adapted at every step according to the stability condition. We will use Neumann boundary conditions.