Num. #5: 1D hyperbolic PDEs system : 2nd order schemes

The aim of this session consists in finding 2nd order methods for

$$\partial_t u + \partial_x (f(u)) = 0, \tag{1}$$

where the unknown is $u : [0, T] \times [0, L] \longrightarrow \mathbb{R}$.

The schemes we presented in the previous sessions are mostly of order one, except Lax-Wendroff scheme which is very oscillatory. Let us explain how to construct TVD 2nd-order scheme, first for the transport equation and then for a conservation law.

1 Transport equation

We first consider the transport equation :

$$\partial_t u + a\partial_x u = 0, \tag{2}$$

where the unknown is $u : [0, T] \times [0, L] \longrightarrow \mathbb{R}$ and $a \in \mathbb{R}$ is a constant. The initial data will be :

$$u_0(x) = \mathrm{e}^{-(x-2)^2/0.1}. \tag{3a}$$

$$u_0(x) = \begin{cases} 1 & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{3b}$$

We first recall the upwind and the Lax-Wendroff schemes, which write as :

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

with the following fluxes :

- **Upwind scheme** :
$$F_{i+1/2}^n = \frac{a}{2}(u_i^n + u_{i+1}^n) - \frac{|a|}{2}(u_{i+1}^n - u_i^n),$$

- **Lax-Wendroff scheme** :
$$F_{i+1/2}^n = \frac{a}{2}\big(u_i^n + u_{i+1}^n\big) - \frac{\Delta t}{2\Delta x}a^2\big(u_{i+1}^n - u_i^n\big).$$

We can therefore rewrite Lax-Wendroff flux as a perturbation of upwind flux, as :

$$F_{i+1/2}^n = \frac{a}{2}\big(u_i^n + u_{i+1}^n\big) - \frac{|a|}{2}(u_{i+1}^n - u_i^n) + \frac{|a|}{2}\left(1 - |a|\frac{\Delta t}{\Delta x}\right)(u_{i+1}^n - u_i^n)$$

and the flux for the **flux-limiters methods in the case** $a > 0$ is defined as :

$$F_{i+1/2}^n = \frac{a}{2}\big(u_i^n + u_{i+1}^n\big) - \frac{|a|}{2}(u_{i+1}^n - u_i^n) + \frac{|a|}{2}\left(1 - |a|\frac{\Delta t}{\Delta x}\right)(u_{i+1}^n - u_i^n)\boldsymbol{\Phi}(\boldsymbol{\theta}_{i+1/2}^{+,n}), \tag{4}$$

where

$$\theta^{+,n}_{i+1/2} = \frac{u_i^n - u_{i-1}^n}{u_{i+1}^n - u_i^n}.$$

In the case where $u_{i+1}^n = u_i^n$, the limiter part will be put to 0.

In the case when $a < 0$, we define

$$\theta^{-,n}_{i+1/2} = \frac{u_{i+2}^n - u_{i+1}^n}{u_{i+1}^n - u_i^n}.$$

We consider here three different flux limiters functions, namely :

- **Minmod limiter** :

$$\Phi(r) = \max(0, \min(r, 1)) \tag{5a}$$

- **Roe's superbee limiter**:

$$\Phi(r) = \max\left(0, \min\left(2r, 1\right), \min\left(r, 2\right)\right) \tag{5b}$$

- **Van Leer's limiter**:

$$\Phi(r) = \frac{r + |r|}{1 + |r|} = \begin{cases} \dfrac{2r}{1+r} & \text{if } r \geq 0 \\ 0 & \text{if } r \leq 0 \end{cases}. \tag{5c}$$

Exercise

1. Implement the flux limiters methods presented here in the case $a = 1$, with initial datum (3a). Use periodic boundary conditions, $\Delta x = 0.01$ and $\Delta t = 0.95\Delta x$.
2. Choose one of the previous limiter functions and highlight the order of the scheme. The error will be defined as the difference between the exact solution and the solution computed.
3. What happens with initial datum (3b) ?

2 Conservation law

This idea can be generalized to a scalar conservation law of the form :

$$\partial_t u + \partial_x (f(u)) = 0, \tag{6}$$

where the unknown is $u : [0, T] \times [0, L] \longrightarrow \mathbb{R}$. $f : \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be a C^2- continuous function and we define $\boldsymbol{a(u) = f'(u)}$.

We still use initial data (3a) and (3b).

In that case, the velocity a remains positive and the numeral flux of the flux limiters method is defined as :

$$F^n_{i+1/2} = f(u_i^n) + \frac{1}{2}\hat{a}_i^n \left(1 - \frac{\Delta t}{\Delta x}\hat{a}_i^n\right)(u_{i+1}^n - u_i^n)\boldsymbol{\Phi(\theta^{+,n}_{i+1/2})}, \tag{7}$$

where

$$\hat{a}_i^n = \begin{cases} \dfrac{f(u_{i+1}^n) - f(u_i^n)}{u_{i+1}^n - u_i^n} & \text{if } u_{i+1}^n \neq u_i^n \\ a(u_i^n) & \text{if } u_{i+1}^n = u_i^n \end{cases}$$

and

$$\theta^{+,n}_{i+1/2} = \frac{1 - \dfrac{\Delta t}{\Delta x}\hat{a}^n_{i-1}}{1 - \dfrac{\Delta t}{\Delta x}\hat{a}^n_i} \times \frac{f(u^n_i) - f(u^n_{i-1})}{f(u^n_{i+1}) - f(u^n_i)}.$$

The more general case, where a can change sign, can be found in Godlewski-Raviart's book, p. 187.

We still consider flux limiters functions (5).

Exercise

1. Implement the flux limiters methods presented here in the case of the Burgers equation $f(u) = \dfrac{u^2}{2}$ with initial data (3a) and (3b). Use periodic boundary conditions, $\Delta x = 0.01$ and $\Delta t = 0.95\Delta x$.