Num #6: Kinetic schemes

This work aims at implementing a kinetic scheme, first for Burgers equation : and then for the 1D monoatomic compressible Euler system. In both cases, the equations are considered as the limit of a kinetic equation and to solve it numerically, we will use a splitting scheme.

1 Burgers equation

We consider Burgers equation

$$\partial_t u + \partial_x \mathscr{F}(u) = 0, \text{ with } \mathscr{F}(u) = \frac{u^2}{2}, \tag{1}$$

as the limit of the following kinetic equation when $\varepsilon \to 0$:

$$\partial_t f + a(v)\partial_x f = \frac{1}{\varepsilon}\left(\chi_u(v) - f\right), \tag{2a}$$

with:

$$a(v) = \mathscr{F}'(v) = v. \tag{2b}$$

The function $\chi_u(v)$ is defined by:

$$\chi_u(v) = \begin{cases} 1 & \text{if } 0 < v < u, \\ -1 & \text{if } u < v < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the unknows of Equation (1) and System (2) are linked by the relationship:

$$u(t,x) = \int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v.$$

We use a spliting scheme to approach the solution of System (2):

- **Step 1: Linear transport step**

  $$\partial_t f + a(v)\partial_x f = 0.$$

  We consider here an upwind scheme:

  $$f_i^{n+\frac{1}{2}}(v) = f_i^n(v) - \frac{\Delta t}{\Delta x}\left[a^+(v)\left(f_i^n(v) - f_{i-1}^n(v)\right) - a^-(v)\left(f_{i+1}^n(v) - f_i^n(v)\right)\right] \tag{3}$$

  where $a^+(v) = \max(a(v), 0)$ and $a^-(v) = \max(-a(v), 0)$.

- **Step 2: Collision step**

  $$\partial_t f = \frac{1}{\varepsilon}\left(\chi_u - f\right).$$

  When $\varepsilon \to 0$, this step reduces to $f = \chi_u$ and thus, the second step of the scheme consists in solving:

  $$f_i^{n+1}(v) = \chi_{u_i^{n+\frac{1}{2}}}(v),$$

  where $u_i^{n+\frac{1}{2}} = \int_{\mathbb{R}} f_i^{n+\frac{1}{2}}(v)\,\mathrm{d}v$.

Noting that $u_i^{n+1} = \int_{\mathbb{R}} f_i^{n+1}(v)\,\mathrm{d}v = \int_{\mathbb{R}} \chi_{u_i^{n+\frac{1}{2}}}(v)\,\mathrm{d}v = u_i^{n+\frac{1}{2}}$, we can now return to variables of interest, that is u_i^n. Integrating Eq. (3) with respect to $v \in \mathbb{R}$ yields an equation of the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

where the flux $F_{i+1/2}^n$ is defined by:

$$F_{i+1/2}^n = \int_{\mathbb{R}} \Big[a^+(v)\chi_{u_i^n}(v) - a^-(v)\chi_{u_{i+1}^n}(v)\Big]\,\mathrm{d}v.$$

Exercise

1. Give the expression of fluxes $F_{i+\frac{1}{2}}$. Implement the scheme (with periodic boundary conditions).
2. Perform a test case with the following initial datum on the interval $[0,5]$:

$$u_0(x) = \mathrm{e}^{-(x-2)^2/0.1}.$$

3. Highlight the CFL condition.

2 Compressible Euler system

We now consider the compressible Euler system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t E + \partial_x(Eu + pu) = 0. \end{cases}$$

and we rewrite it in the (ρ, j, E)- variables as :

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \partial_t j + \partial_x(2E) = 0, \\ \partial_t E + \partial_x(3\frac{Ej}{\rho} - \frac{j^3}{\rho^2}) = 0, \end{cases} \tag{4}$$

We set this system on $[0,1]$ and for later purpose, we also introduce:

$$T = 2e = 2\frac{E}{\rho} - u^2 = 2\frac{E}{\rho} - \frac{j^2}{\rho^2}.$$

This system is viewed as the limite $\varepsilon \to 0$ of the BGK equation:

$$\partial_t f + v\partial_x f = \frac{1}{\varepsilon}\Big(M[f] - f\Big).$$

It holds in the sense that f is expected to converge towards $M[f]$, with macroscopic quantities solutions of system (4).
We follow the same procedure as in the previous section. The scheme is therefore written in two steps:

1. **Step 1: Linear transport step**:
$$\partial_t f + v\partial_x f = 0.$$
We adopt an upwind scheme to solve this problem. This yields:
$$f_i^{n+\frac{1}{2}} = f_i^n - \frac{\Delta t}{\Delta x}\Big[\frac{v+|v|}{2}\big(f_i^n - f_{i-1}^n\big) + \frac{v-|v|}{2}\big(f_{i+1}^n - f_i^n\big)\Big]$$

2. **Step 2: Collision step** :
$$\partial_t f = \frac{1}{\varepsilon}\Big(M[f] - f\Big).$$
The first remark is to note that during this step, macroscopic quantities associated to f do not change. This is due to the fact that macroscopic quantities of f and $M[f]$ are the same and consequently their partial derivatives vanish during this step.

   When ε goes to zero, this step reduces to:
$$f_i^{n+1} = M_i^{n+\frac{1}{2}}\Big(= M_i^{n+1}\Big),$$
where M_i^n stands for $M[f_i^n]$ and the last equality holds owing to previous remark and the fact that $M[f]$ only depends on macroscopic quantities.

In pratice, we do not resolve the previous scheme: we go back to macroscopic variable by integrating previous equations. Indeed, macroscopic quantities associated to f are defined as follows:

$$\begin{aligned}
\rho(t,x) &= \int_{\mathbb{R}} f(t,x,v)\,dv,\\
\rho u(t,x) &= \int_{\mathbb{R}} v f(t,x,v)\,dv,\\
E(t,x) &= \int_{\mathbb{R}} \frac{v^2}{2} f(t,x,v)\,dv.
\end{aligned}$$

Hence, intregrating previous scheme with respect to v yields:

$$U_i^{n+1} = U_i^{n+\frac{1}{2}} = U_i^n - \frac{\Delta t}{\Delta x}\int_R \Big(1, v, \frac{v^2}{2}\Big)\Big[\frac{v+|v|}{2}\big(M_i^n - M_{i-1}^n\big) + \frac{v-|v|}{2}\big(M_{i+1}^n - M_i^n\big)\Big]dv,$$

where U_i^n stands for the vector $\big(\rho_i^n,\ j_i^n,\ E_i^n\big)$.

In conclusion, the scheme reads as follows:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\Big[F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\Big]$$

where the flux $F_{i+\frac{1}{2}}$ is given by:

$$F_{i+\frac{1}{2}} = \int_{v\leqslant 0} v\Big(1, v, \frac{v^2}{2}\Big) M^n_{i+1}\, dv + \int_{v\geqslant 0} v\Big(1, v, \frac{v^2}{2}\Big) M^n_i\, dv.$$

In the case where the "equilibrium function" $M[f]$ is defined by:

$$M[f](v) = \frac{\rho}{2\sqrt{3T}} \mathbb{1}_{|v-u|\leqslant\sqrt{3T}},$$

(ρ, u and T are macroscopic quantities associated to f), it is possible to derive a simple explicit expression of fluxes $F_{i+\frac{1}{2}}$.

Exercise

1. Give the expression of fluxes $F_{i+\frac{1}{2}}$. Implement the scheme.
2. Perform a test case with the following initial data on the domain $[0,1]$ with Neumann boundary conditions, as in session #4:

$$\rho(0,x) = \begin{cases} 1 & \text{if} \quad x \leqslant 0.5, \\ 0.125 & \text{if} \quad x > 0.5, \end{cases} \qquad p(0,x) = \begin{cases} 1 & \text{if} \quad x \leqslant 0.5, \\ 0.1 & \text{if} \quad x > 0.5, \end{cases} \qquad \text{and} \qquad u(0,x) = 0.$$