Suppose \( n \geq 0 \) are non-negative integers. The Schur polynomial \( s_\lambda(n_1, \ldots, n_k) \) is the symmetric sum of the monomials \( x_1^{n_1}x_2^{n_2} \cdots x_k^{n_k} \), where \( \lambda \) is a partition with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \).

Schur polynomials are homogeneous and symmetric. (Character of irreducible polynomial representations of \( GL_n(C) \).)

**Definition 3 (Cauchy), Weyl Character Formula:** \( s_\lambda(u) = \frac{\det(u_\lambda^{(m)}(u))}{\det(u_\lambda^{(n)}(u))} \)

**Ratios of Schur polynomials:** Monotonicity, via Schur positivity

Suppose \( u \in [0, \infty)^n \). How do the ratios of these Schur polynomials behave on the positive orthant?

\[
  f(u) = \frac{s_{\lambda|u}^{(u)}}{s_{\mu|u}^{(u)}} \quad u \in [0, \infty)^N.
\]

**Example 5.** Suppose \( m = (2, 2) \) as above, and \( n = (3, 2) \). Then:

\[
  f(w, x) = \frac{(x_1 + x_2 + x_3)(x_1 + 2x_2 + x_3)}{(x_1 + x_2 + x_3)(x_1 + 2x_2 + x_3)}.
\]

Note: both numerator and denominator are non-negative, hence no decreasing in each coordinate.

**Claim:** \( f \) is also non-decreasing in each coordinate.

(Why?) Applying the quotient rule of differentiation to \( f \),

\[
  f(x, y, z) = \frac{a_1(x) a_2(y) a_3(z)}{a_1(y) a_2(z) a_3(x)},
\]

and this is monomial-positive. In fact, more holds: if we write this as a product of \( a_j(x) \), then each \( a_j(x) \) is Schur-positive, i.e., a sum of Schur polynomials.

\[
  f(x, y, z) = f_1(x) f_2(y) f_3(z).
\]

This happens in general.

### Main result

**Theorem 10 [2016].**

Fix \( N \geq 1 \) and \( n_1 < \cdots < n_{N-1} \leq M \), and let

\[
  f(x) = \left( \sum_{n=1}^{M} x_n e_{n+1}^{-1} \right)^N.
\]

where \( e_0 = 1 \). Let \( 0 < \rho < \infty \). Then the following are equivalent.

1. \( f(x) \) preserves positivity on \( P_N(\rho) \).
2. \( f(x) \) preserves positivity on \( P_M(\rho) \).
3. \( \sum_{j=1}^{N} \frac{s_j(\rho) e_j(\rho) a_j(\rho)}{s_j(\rho) a_j(\rho)} \geq 0 \)

where \( a_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_M) \).

Some consequences of Theorem 10:

1. Examples of polynomial preservers with negative coefficients.
2. More strongly, we can use this to characterize the sign patterns of power series preservers on \( P_N(\rho) \).
3. Provides the first construction of polynomials that preserve positivity on \( P_N(\rho) \), but not on \( P_M(\rho) \). Thus Horn’s result is sharp.

**From determinants to Schur polynomials**

Sketch of proof of Theorem 10: (1) \( \implies \) (2): Immediate.

(3) \( \implies \) (2): We will only show how Schur polynomials and the threshold \( \rho \) arise out of Theorem 102, assuming that \( a_j > 0 \).

First note that for any vector \( u \in \mathbb{R}_r^N \),

\[
  f(u) = \sum_{j=1}^{N} \frac{s_j(u) e_j(u) a_j(u)}{s_j(u) a_j(u)}.
\]

Let \( V(u) = \frac{1}{\sum_{j=1}^{N} s_j(u)} \) be the “Vandermonde determinant for \( u \).” Then by the Cauchy–Binet formula and Definition 2,

\[
  \det(f(u)) = V(u)^{-1} \sum_{j=1}^{N} s_j(u) e_j(u) a_j(u) \geq 0 \quad (13)
\]

Now suppose the coordinates of \( u \) are distinct and \( 0 < \sqrt{\rho} \). Solving for \( \sqrt{\rho} \) yields

\[
  \rho \geq \frac{\sqrt{\rho}}{\sqrt{\rho} \sqrt{\rho}}.
\]

Finally, if Theorem 103 holds, then \( 1/\rho \) must exceed the right-hand side for all \( u \in [0, \infty)^N \) with distinct coordinates, hence for all \( u \in [0, \infty)^N \).

**Cutter-Greene-Skandera conjecture, and weak majorization**

**Definition 14.** Given integers \( 0 \leq m_1 \leq \cdots \leq m_N \leq \cdots \leq m_{N+1} \), the vector \( m = (m_1, \ldots, m_{N+1}) \) weakly majorizes \( n = (n_1, \ldots, n_{N+1}) \) if

\[
  \sum_{k=1}^{N} k m_k \geq \sum_{k=1}^{N} k n_k \quad \forall k.
\]

If the final inequality is an equality, we say \( m \) majorizes \( n \).

A conjecture of Cutter-Greene-Skandera [JST J. Comb. 2011] says that \( m \) is majorizes if and only if the following inequality holds:

\[
  \sum_{k=1}^{N} k m_k \geq \sum_{k=1}^{N} k n_k.
\]

The conjecture was recently proved by Sia [JST J. Comb. 2016], and Ai-Haddad and Mazza [Found. Comput. Math. 2018].

Compare this with Theorem 6, which says that if \( m \geq n \) coordinate-wise,

\[
  \sum_{k=1}^{N} k m_k \geq \sum_{k=1}^{N} k n_k.
\]

**Question 15.** How can these two inequalities be reconciled?

The above three papers all assume \( m_i = \sum_{j=1}^{i} n_j \). Replace by an inequality -- novel characterization of weak majorization.

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