A REMARK ON CHARACTERIZING INNER PRODUCT SPACES VIA STRONG THREE-POINT HOMOGENEITY

SUJIT SAKHARAM DAMASE AND APOORVA KHARE

Abstract. We show that a normed linear space is isometrically isomorphic to an inner product space if and only if it is a strongly n-point homogeneous metric space for any (or every) $n \geqslant 3$. The counterpart for $n = 2$ is the Banach–Mazur problem.

Normed linear spaces and inner product spaces are central to much of mathematics, in particular analysis and probability. The goal of this short note is to provide a "metric" characterization that we were unable to find in the literature, of when the norm in a linear space arises from an inner product. This characterization is classical in spirit, is in terms of a "strong" 3-point homogeneity property that holds in all inner product spaces, and is adjacent to a well-known open question.

Two prevalent themes in the early 20th century involved exploring metric geometry – e.g. when a (finite or) separable metric space isometrically embeds into the Hilbert space of square-summable sequences $\ell^2 := \ell^2(\mathbb{N})$ – and exploring when a normed linear space $(\mathbb{B}, \|\cdot\|)$ is Hilbert, i.e. $\|\cdot\|^2$ arises from an inner product on $\mathbb{B}$. See e.g. [3], [9], [11]–[12], [14]–[18], [20], [25]; additional works on the latter theme can be found cited in [15]. There have also been books – see e.g. [5, 8, 10, 13] – as well as later works. This note, while squarely in the latter theme, is strongly inspired by the former theme – in which it is worth mentioning the Mathematics Kolloquium [23] of Karl Menger and others in Vienna, from 1928–36. This long-running lecture series saw contributions in metric geometry and related areas by Menger, Gödel, von Neumann, and others, and led to new developments in metric embeddings, metric convexity, fixed point theory, and more. The goal of our work is to provide such a distance-geometric characterization of an inner product.

We begin with some results from the latter theme. Jordan and von Neumann showed in [16] that the norm in a real or complex linear space $\mathbb{B}$ comes from an inner product if and only if the parallelogram law holds in $\mathbb{B}$; they also showed the (real and) complex polarization identity in *loc. cit.* We collect this and other equivalent conditions for the norm $\|\cdot\|$ in a real or complex linear space $\mathbb{B}$ to arise from an inner product:

Theorem 1. *Suppose $(\mathbb{B}, \|\cdot\|)$ is a real or complex inner product space. Its norm arises from an inner product if and only if any of the following equivalent conditions holds (for the last two, we work only over $\mathbb{R}$):*

- (IP1) *(Jordan and von Neumann, [16].) $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$ for all $f, g \in \mathbb{B}$. (The parallelogram law; it originally appears in the authors' work with Wigner, see [17, p. 32].)*
- (IP2) *(Ficken, [11].) If $f, g \in \mathbb{B}$ with $\|f\| = \|g\|$, then $\|\alpha f + \beta g\| = \|\beta f + \alpha g\|$ for all real α, β.*
- (IP3) *(Day, [9].) If $f, g \in \mathbb{B}$ with $\|f\| = \|g\| = 1$, then $\|f+g\|^2 + \|f-g\|^2 = 4$. (The parallelogram law, but only for rhombi.)*
- (IP4) *(James, [15], when $\dim \mathbb{B} \geqslant 3$.) For all $f, g \in \mathbb{B}$, $\|f + \alpha g\| \geqslant \|f\|$ for all scalars α, if and only if $\|g + \alpha f\| \geqslant \|g\|$ for all scalars α. (Symmetry of Birkhoff–James orthogonality.)*

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(IP5) *(Lorch, [20], over $\mathbb{R}$.) There exists a fixed constant $\gamma' \in \mathbb{R} \setminus \{0, \pm 1\}$*[1] *such that whenever $f', g' \in \mathbb{B}$ with $\|f'\| = \|g'\|$, we have $\|f' + \gamma' g'\| = \|g' + \gamma' f'\|$.*

(IP6) *(Lorch, [20], over $\mathbb{R}$.) There exists a fixed constant $\gamma \in \mathbb{R} \setminus \{0, \pm 1\}$ such that whenever $f, g \in \mathbb{B}$ with $\|f + g\| = \|f - g\|$, we have $\|f + \gamma g\| = \|f - \gamma g\|$.*

Indeed, that every inner product space satisfies these properties is immediate, while the implications (IP5) $\iff$ (IP6) $\implies$ (IP2) were shown by Lorch; and that (IP1), (IP2), (IP3), (IP4) imply that $\|\cdot\|^2$ arises from an inner product were shown by the respectively named authors above.

All of these characterizations of an inner product use the norm and the vector space structure (over $\mathbb{R}$ or $\mathbb{C}$) on $\mathbb{B}$. The goal of this note is to isolate the inner product using the metric in $\mathbb{B}$ but *avoiding* both the additive structure and the (real or complex) scalar multiplication. Thus, our result is in the spirit of both of the aforementioned classical themes: characterizing inner products in normed linear spaces, while using metric geometry alone.

1. The main result and its proof

To state our result, first recall from [27] or even [4, p. 470] that for an integer $n \geqslant 1$, a metric space (X, d) is *n-point homogeneous* if given two finite subsets $Y, Y' \subseteq X$ with $|Y| = |Y'| \leqslant n$, any isometry $T : Y \to Y'$ extends to an isometry $: X \to X$. We will require a somewhat more restrictive notion:

Definition 2. A metric space (X, d) will be termed *strongly n-point homogeneous* if given subsets $Y, Y' \subseteq X$ with $|Y| = |Y'| \leqslant n$, each isometry $T : Y \to Y'$ can be extended to an onto isometry $: X \twoheadrightarrow X$.

We now motivate our main result (and the above definition via onto isometries). It seems to be folklore that the Euclidean space $\mathbb{R}^k$ is n-point homogeneous for all n – and more strongly, satisfies that every isometry between finite subsets $Y, Y' \subseteq \mathbb{R}^k$, upon pre- and post- composing with suitable translations in order to send 0 to 0, extends to an orthogonal linear map $: \mathbb{R}^k \to \mathbb{R}^k$.

Remark 3. While not central to our main result, we will explain below why this property also holds for infinite subsets of $(\mathbb{R}^k, \|\cdot\|_2)$; a weakening of it was termed the *free mobility postulate* by Birkhoff [4, pp. 469–470]. However, this postulate is not satisfied by any infinite-dimensional Hilbert space for infinite subsets. This was already pointed out in 1944 by Birkhoff in *loc. cit.*; for a specific counterexample, see e.g. the proof of [28, Theorem 11.4] in ℓ^2, where the left-shift operator sending $Y := \{(x_n)_{n \geqslant 0} \in \ell^2 : x_1 = 0\}$ onto $Y' := \ell^2$ is an isometry that does not extend to $(1, 0, 0, \dots)$.

Returning to our motivation: in fact all inner product spaces satisfy this "orthogonal extension property" for all pairs of isometric finite subsets Y, Y', as we explain below. Here we are interested in the converse question, i.e.,

(a) if this "orthogonal extension property" (with Y, Y' finite) characterizes inner product spaces (among normed linear spaces); and
(b) if yes, then how much can this property be weakened without disturbing the characterization – and if it can in fact be weakened to use metric geometry alone. (In an arbitrary normed linear space, we necessarily cannot use orthogonality or inner products; but we also want to not use the vector space operations either.)

This note shows that indeed (a) holds. Moreover, (b) we can indeed weaken the orthogonal extension property to (i) replacing the orthogonal linear map by merely an onto isometry – not necessarily linear *a priori* – and (ii) working with 3-point subsets Y, Y'. More precisely:

[1] We correct a small typo in Lorch's (IP5) in [20]: he stated $\gamma' \neq 0, 1$ but omitted excluding -1; but clearly $\gamma' = -1$ "works" for every normed linear space $\mathbb{B}$ and all vectors $f', g' \in \mathbb{B}$, since $\|f' - g'\| = \|g' - f'\|$.

Theorem 4. *Suppose $(\mathbb{B}, \|\cdot\|)$ is a nonzero real or complex normed linear space. Then $\|\cdot\|^2$ arises from an inner product – real or complex, respectively – if and only if $\mathbb{B}$ is strongly n-point homogeneous for any (equivalently, every) $n \geqslant 3$.*

Remark 5. Below, we will provide additional equivalent – and *a priori* weaker – conditions to add to Theorem 4 in characterizing an inner product. See Theorems 12 and 15.

To the best of our ability – and that of a dozen experts – we were unable to find such a result proved in the literature. Before proceeding to its proof, we discuss the assertion for $n = 1, 2$. If $n = 1$ then Theorem 4 fails to hold, since every normed linear space $\mathbb{B}$ is strongly one-point homogeneous: given $x, y \in \mathbb{B}$, the translation $z \mapsto z + y - x$ is an onto isometry sending x to y.

If instead $n = 2$ then one is asking if there exists a (real) linear space $(\mathbb{B}, \|\cdot\|)$ with $\|\cdot\|^2$ not arising from an inner product, such that given any $f, f', g, g' \in \mathbb{B}$ with $\|f' - f\| = \|g' - g\|$, every isometry sending f, f' to g, g' respectively extends to an onto isometry of $\mathbb{B}$. By pre- and post-composing with translations, one can assume $f = g = 0$ and $\|f'\| = \|g'\|$; now one is asking if every real normed linear space with transitive group of onto-isometries fixing 0 (these are called *rotations*) is isometrically isomorphic to an inner product space. Thus we come to the well-known *Banach–Mazur problem* [2] – which was affirmatively answered by Mazur for finite-dimensional $\mathbb{B}$ [21], has counterexamples among non-separable $\mathbb{B}$ [24], and remains open for infinite-dimensional separable $\mathbb{B}$. (This is also called the *Mazur rotations problem*; see the recent survey [6].) This is when $n = 2$; and the $n \geqslant 3$ case is Theorem 4.

Remark 6. Given the preceding paragraph, one can assume in Theorem 4 that $\mathbb{B}$ is infinite-dimensional, since if $\dim \mathbb{B} < \infty$ then strong 3-point homogeneity implies strong 2-point homogeneity, which by Mazur's solution [21] to the Banach–Mazur problem implies $\mathbb{B}$ is Euclidean. That said, our proof of Theorem 4 works uniformly over all normed linear spaces, and we believe is simpler than using the Banach–Mazur problem (which moreover cannot be applied for all $\mathbb{B}$).

Proof of Theorem 4. We begin by proving the real case. We first explain the forward implication, starting with $\mathbb{B} \cong (\mathbb{R}^k, \|\cdot\|_2)$ for an integer $k \geqslant 1$. As is asserted (without proof) on [4, p. 470], $\mathbb{R}^k$ is n-point homogeneous for every $n \geqslant 1$; we now show that it is moreover strongly n-point homogeneous – in fact, that it satisfies the "orthogonal extension property" above:

Let $Y, Y' \subseteq \mathbb{R}^k$, and let $T : Y \to Y'$ be an isometry. By translating Y and Y', assume $0 \in Y \cap Y'$ and $T(0) = 0$. Then T extends to an orthogonal self-isometry $\widetilde{T}$ of $(\mathbb{R}^k, \|\cdot\|_2)$.

This claim can be proved from first principles and is likely folklore (e.g. see a skeleton argument in [5, Section 38]). However, for self-completeness, we present a detailed sketch via a "lurking isometry" argument, along with some supplementary remarks. For full details, see [19, Theorem 22.3] and its proof.

The first step is to note that vectors $y_0, \ldots, y_n$ in an inner product space are linearly dependent if and only if their Gram matrix $G := (\langle y_i, y_j \rangle)_{i,j=0}^n$ is singular, since

$$v := \sum_{j=0}^{n} c_j y_j = \mathbf{0} \quad \Longrightarrow \quad G\mathbf{c} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{c}^* G \mathbf{c} = 0 \quad \Longrightarrow \quad \|v\|^2 = 0. \tag{1.1}$$

This simple fact yields several noteworthy consequences; we mention two here, without proof. The first is an 1841 result by Cayley [7] (during his undergraduate days):

Lemma 7. *Let $(X = \{x_0, \ldots, x_n\}, d)$ be a finite metric space, and $\Psi : X \to \ell^2$ an isometry. Then the vectors $\Psi(X)$ are affine linearly-dependent (i.e. lie in an $(n-1)$-dimensional affine subspace)* *if and only if the Cayley–Menger matrix of X is singular:*

$$CM(X)_{(n+2)\times(n+2)} := \begin{pmatrix} 0 & d_{01}^2 & d_{02}^2 & \cdots & d_{0n}^2 & 1 \\ d_{10}^2 & 0 & d_{12}^2 & \cdots & d_{1n}^2 & 1 \\ d_{20}^2 & d_{21}^2 & 0 & \cdots & d_{2n}^2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{n0}^2 & d_{n1}^2 & d_{n2}^2 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \textit{where } d_{ij} = d_X(x_i, x_j). \tag{1.2}$$

The second consequence is used below, but also underlies the Global Positioning System (GPS) "trilateration" – i.e. that every point on say a Euclidean plane P is uniquely determined by its distances from three non-collinear points in P. More generally:

Proposition 8. *Fix $y_0 = 0, \dots, y_n \in \ell^2$. The following are equivalent for $y \in \ell^2$:*

1. *y is (uniquely) determined by its distances from $y_0, \dots, y_n$.*
2. *y is in the span of $y_1, \dots, y_n$.*

(See e.g. [19, Proposition 22.7].) Returning to the proof of the orthogonal extension property for $\mathbb{R}^k$: fix a maximal linearly independent subset $\{y_1, \dots, y_r\}$ in Y. Then we claim, so is $\{T(y_1), \dots, T(y_r)\}$. Indeed, by polarization we have

$$\begin{aligned} 2\langle y_i, y_j\rangle &= \|y_i - 0\|_2^2 + \|y_j - 0\|_2^2 - \|y_i - y_j\|_2^2 \\ &= \|T(y_i) - T(0)\|_2^2 + \|T(y_j) - T(0)\|_2^2 - \|T(y_i) - T(y_j)\|_2^2 = 2\langle T(y_i), T(y_j)\rangle \end{aligned} \tag{1.3}$$

for $1 \leqslant i, j \leqslant r$. Thus the Gram matrix of the $T(y_i)$ equals that of the y_i, and hence is invertible as well; so by (1.1) the $T(y_j)$ are linearly independent too. Moreover, for any other $y \in Y$, the Gram matrix of $y, y_1, \dots, y_r$ is singular by (1.1), hence so is the Gram matrix of $T(y), T(y_1), \dots, T(y_r)$ by (1.3).

Next, we claim that the linear extension $\widetilde{T}$ of T from $\{y_1 \dots, y_r\}$ to $\operatorname{span}_{\mathbb{R}}(Y)$ (hence mapping into $\operatorname{span}_{\mathbb{R}}(Y'))$ agrees with T on Y. Indeed, by maximality one writes $y \in \operatorname{span}_{\mathbb{R}}(Y)$ as $\sum_{j=1}^r c_j(y) y_j$ and $T(y) \in \operatorname{span}_{\mathbb{R}}(Y')$ as $\sum_{j=1}^r c'_j(y) T(y_j)$; then (1.3) and Proposition 8 show that $c_j \equiv c'_j$ on Y (for all j). Hence $\widetilde{T} \equiv T$ on Y.

It also follows from (1.3) that $\widetilde{T}$ preserves lengths, hence is injective. Finally, choose orthonormal bases of the orthocomplements in $\mathbb{R}^k$ of $\operatorname{span}_{\mathbb{R}}(Y)$ and of $\widetilde{T}(\operatorname{span}_{\mathbb{R}}(Y))$, and map the first of these bases (within $\operatorname{span}_{\mathbb{R}}(Y)^\perp$) bijectively onto the second; then extend $\widetilde{T}$ to all of $\mathbb{R}^k$ by linearity. Direct-summing these two orthogonal linear maps yields the desired extension of T to a linear self-isometry $\widetilde{T}$ of $(\mathbb{R}^k, \|\cdot\|_2)$.

This linear orthogonal map $\widetilde{T}$ is necessarily injective on $\mathbb{R}^k$, hence surjective as well. This shows the orthogonal extension property for all (possibly infinite) isometric subsets $Y, Y' \subseteq \mathbb{R}^k$, and hence the forward implication in the main result for $\mathbb{B} \cong \mathbb{R}^k$.

If instead $\mathbb{B}$ is an infinite-dimensional inner product space, and T an isometry between $Y, Y' \subset \mathbb{B}$ of common size at most n, first pre- and post- compose by translations to assume $0 \in Y \cap Y'$ and $T(0) = 0$. Now let $\mathbb{B}_0 \cong (\mathbb{R}^k, \|\cdot\|_2)$ be the span of $Y \cup Y'$. By the above analysis, $T : Y \to Y'$ extends to an orthogonal operator on $\mathbb{B}_0$, which we still denote by T; as $\mathbb{B}_0$ is a complete subspace of $\mathbb{B}$, by the "projection theorem" (or from first principles) we get $\mathbb{B} = \mathbb{B}_0 \oplus \mathbb{B}_0^\perp$. Now the bijective orthogonal map $T|_{\mathbb{B}_0} \oplus \operatorname{Id}|_{\mathbb{B}_0^\perp}$ completes the proof of the forward implication – for any $n \geqslant 1$.

We next come to the reverse implication; now $n \geqslant 3$. From the definitions, it suffices to work with $n = 3$. Moreover, the cases of $\dim \mathbb{B} = 0, 1$ are trivial since $\mathbb{B}$ is then an inner product space, so the reader may also assume $\dim \mathbb{B} \in [2, \infty]$ in the sequel, if required.

We work via contradiction: let $(\mathbb{B}, \|\cdot\|)$ be a nonzero 3-point homogeneous normed linear space, with $\|\cdot\|^2$ not induced by an inner product. Then Lorch's condition (IP5) fails to hold for any

$\gamma' \in \mathbb{R} \setminus \{0, \pm 1\}$. Fix such a scalar γ'; then there exist vectors $f', g' \in \mathbb{B}$ with $\|f'\| = \|g'\|$ but $\|f' + \gamma' g'\| \neq \|g' + \gamma' f'\|$. In particular, $0, f', g'$ are distinct. Now let $Y = Y' = \{0, f', g'\}$, and consider the isometry $T : Y \to Y'$ which fixes 0 and interchanges f', g'. By hypothesis, T extends to an onto isometry : $\mathbb{B} \to \mathbb{B}$, which we also denote by T. But then T is affine-linear by the Mazur–Ulam theorem [22], hence is a linear isometry as $T(0) = 0$. This yields

$$\|f' + \gamma' g'\| = \|f' - (-\gamma')g'\| = \|T(f') - T(-\gamma' g')\| = \|g' + \gamma' f'\|,$$

which provides the desired contradiction, and proves the reverse implication.

Remark 9. An alternate argument to the one provided above (for the reverse implication) goes as follows. *By the parallelogram law (IP1), it suffices to show every plane $P \subseteq \mathbb{B}$ is Euclidean. Suppose $0 \neq f, g \in P$ with $\|f\| = \|g\|$; then the isometry of $\{0, f, g\}$ that fixes 0 and exchanges f, g extends to an onto isometry of $\mathbb{B}$. By Mazur–Ulam* [22]*, T is affine-linear (hence linear as $T(0) = 0$), and thus sends P onto itself. But then $\|\alpha f + \beta g\| = \|T(\alpha f + \beta g)\| = \|\beta f + \alpha g\|$ for all $\alpha, \beta \in \mathbb{R}$. By Ficken's result (IP2) for P, P is Euclidean.* We note that this argument is somewhat more involved than the one above, as it makes use of (i) Ficken's characterization (IP2), which as Lorch wrote [20] is *a priori* more involved than Lorch's (IP5); and (ii) the parallelogram law (IP1), which our argument does not use (nor did Lorch).[2]

The above analysis proves the real case; now suppose $(\mathbb{B}, \|\cdot\|)$ is a nonzero complex normed linear space. The forward implication follows from the real case, since $(\mathbb{B}, \|\cdot\|)$ is also a real normed space – which we denote by $\mathbb{B}|_{\mathbb{R}}$. For the same reason, in the reverse direction we obtain that $\mathbb{B}|_{\mathbb{R}}$ is isometrically isomorphic to a real inner product space: $\|f\| = \sqrt{\langle f, f\rangle_{\mathbb{R}}}$ for a real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ on $\mathbb{B}|_{\mathbb{R}}$ and all $f \in \mathbb{B}|_{\mathbb{R}}$. Now the complex polarization trick of Jordan–von Neumann [16] gives that

$$\langle f, g\rangle := \langle f, g\rangle_{\mathbb{R}} - i\langle if, g\rangle_{\mathbb{R}} \tag{1.4}$$

is indeed a complex inner product on $\mathbb{B}$ satisfying: $\|f\| = \sqrt{\langle f, f\rangle}$ for all f. □

2. A second proof; Lorch's characterizations for complex linear spaces

We next provide a second proof of the "reverse implication" of Theorem 4 for complex linear spaces $\mathbb{B} = \mathbb{B}|_{\mathbb{C}}$, which requires the *complex version* of Lorch's condition (IP5) above. Note that the above argument over $\mathbb{R}$ cannot immediately proceed verbatim over $\mathbb{C}$ for two reasons:

(a) Lorch's condition (IP5) needs to be verified as characterizing a complex inner product.
(b) The Mazur–Ulam theorem does not go through over $\mathbb{C}$ – e.g., the isometry $\eta : (z, w) \mapsto (\overline{z}, w)$ of the Hilbert space $(\mathbb{C}^2, \|\cdot\|_2)$ sends $(0, 0)$ to itself, but is neither $\mathbb{C}$-linear nor $\mathbb{C}$-antilinear.

However, since η is $\mathbb{R}$-linear on $\mathbb{C}^2$, this reveals how to potentially fix (b) for a second proof of the reverse implication: it suffices to use Lorch's condition (IP5) for γ' still real – and avoiding $0, \pm 1$ as above – now for all vectors $f', g' \in \mathbb{B}|_{\mathbb{C}}$. If this still characterizes a complex inner product, then one could continue the above proof verbatim, using the Mazur–Ulam theorem for *real* normed linear spaces and replacing the word "linear" twice by "$\mathbb{R}$-linear" in the proof.

Thus we need to verify if Lorch's condition (IP5) characterizes a complex inner product. More broadly, one can ask which of Lorch's conditions (I_1)–(I_6) and $(I_1') =$ (IP5) in [20] – which characterized an inner product in a *real* normed linear space – now do the same over $\mathbb{C}$:

Theorem 10 (Lorch, [20])**.** *Suppose $\mathbb{B}$ is a real normed linear space. Then the norm is induced by an inner product if and only if any of the following equivalent conditions hold:*

[2] One can also see [8, (2.8)], where Dan mentions the flip map of an isosceles triangle. However, Dan requires the isometry T to be *linear*, which is strictly stronger than our characterization-hypothesis of strong homogeneity. Moreover, this linearity assumption is indeed required by Dan, since he mentions in the very next line that linearity can be dropped if $\mathbb{B}$ is assumed to be strictly convex (see Theorem 13). In contrast, Theorem 4 makes no assumption either about the isometry T (other than surjectivity), nor about the normed linear space $\mathbb{B}$.

(I_1) (Stated above as (IP6).) *There exists a fixed constant* $\gamma \in \mathbb{R} \setminus \{0, \pm 1\}$ *such that whenever* $f, g \in \mathbb{B}$ *with* $\|f+g\| = \|f-g\|$, *we have* $\|f+\gamma g\| = \|f - \gamma g\|$.

(I_1') (Stated above as (IP5).) *There exists a fixed constant* $\gamma' \in \mathbb{R} \setminus \{0, \pm 1\}$ *such that whenever* $f', g' \in \mathbb{B}$ *with* $\|f'\| = \|g'\|$, *we have* $\|f' + \gamma' g'\| = \|g' + \gamma' f'\|$.

(I_2) *A triangle is isosceles if and only if two medians are equal:*

$$f + g + h = 0, \ \|f\| = \|g\| \quad \Longrightarrow \quad \|f-h\| = \|g-h\|, \qquad \forall f,g,h \in \mathbb{B}.$$

(I_3) *For all* $f,g,h,k \in \mathbb{B}$,

$$f+g+h+k = 0, \ \|f\| = \|g\|, \ \|h\| = \|k\| \quad \Longrightarrow \quad \|f-h\| = \|g-k\| \text{ and } \|g-h\| = \|f-k\|.$$

(I_4) *For all* $f_1, f_2 \in \mathbb{B}$, *the expression*

$$\phi(f_1, f_2; g) = \|f_1 + f_2 + g\|^2 + \|f_1 + f_2 - g\|^2 - \|f_1 - f_2 - g\|^2 - \|f_1 - f_2 + g\|^2$$

is independent of $g \in \mathbb{B}$.

(I_5) *If* $f, g \in \mathbb{B}$ *and* $\|f\| = \|g\|$, *then* $\|\alpha f + \alpha^{-1} g\| \geqslant \|f+g\|$ *for all* $0 \neq \alpha \in \mathbb{R}$.

(I_6) *For a fixed integer* $n \geqslant 3$, *and vectors* $f_1, \dots, f_n \in \mathbb{B}$, *we have*

$$f_1 + \cdots + f_n = 0 \quad \Longrightarrow \quad \sum_{i<j} \|f_i - f_j\|^2 = 2n \sum_{i=1}^n \|f_i\|^2.$$

In fact, these characterizations have since been cited and applied in many papers that work over real linear spaces. The need to work over complex normed linear spaces – as well as the fact that several characterizations in [9, 11, 16] before Lorch held uniformly over both $\mathbb{R}$ and $\mathbb{C}$ – provide natural additional reasons to ask if Lorch's conditions also characterize complex inner products.

We quickly explain why this does hold. Indeed, Lorch's characterizations themselves involve real scalars, so even if one starts with a complex normed linear space $\mathbb{B}$, one still obtains a real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ on $\mathbb{B} = \mathbb{B}|_{\mathbb{R}}$. Now the discussion around (1.4) recovers the complex inner product from $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. This yields

Theorem 11 ("Complex Lorch")**.** *A complex nonzero normed linear space* $(\mathbb{B}, \|\cdot\|)$ *is isometrically isomorphic to a complex inner product space if and only if any of the following conditions of Lorch* [20] *holds:* (I_1) = *(IP6),* (I_1') = *(IP5), or* (I_2), …, (I_6) *– now stated verbatim in* $\mathbb{B}$, *with the constants* γ, γ', α *still being real.*

In particular, this provides a second proof of one implication in Theorem 4 over $\mathbb{C}$.

3. Isosceles triangle characterization; weaker notions of homogeneity

Note that the arguments in Section 1 in fact show a strengthening of Theorem 4. Namely: continuing the discussion preceding Theorem 4, the "orthogonal extension property" can be weakened to strong 3-point homogeneity, and even weaker – wherein one only works with $Y = Y'$ the vertices of an isosceles triangle in $\mathbb{B}$ (with the two equal sides of specified length).

Theorem 12. *The following are equivalent for a nonzero real/complex normed linear space* $(\mathbb{B}, \|\cdot\|)$.

1. $\|\cdot\|^2$ *arises from an (real or complex) inner product on* $\mathbb{B}$.
2. *If* $Y, Y' \subset \mathbb{B}$ *are finite subsets, with* $|Y| = |Y'|$, *then any isometry* $: Y \to Y'$ *– up to pre- and post- composing by translations – extends to an* $\mathbb{R}$*-linear onto isometry* $: \mathbb{B} \to \mathbb{B}$.
3. $\mathbb{B}$ *is strongly* n*-point homogeneous for any (equivalently, every)* $n \geqslant 3$.
4. *Given an isosceles triangle in* $\mathbb{B}$ *with vertex set* $Y = Y' = \{0, f', g'\}$ *such that* $\|f'\| = \|g'\| = 1$, *the isometry* $: Y \to Y'$ *that fixes* 0 *and flips* f', g' *extends to an onto isometry of* $\mathbb{B}$.

Indeed, we showed above that (4) $\implies$ (1) $\implies$ (2) (as the f', g' in Lorch's condition (IP5) can be simultaneously rescaled), while (2) $\implies$ (3) $\implies$ (4) is trivial. To see why one cannot assert $\mathbb{C}$-linearity in the second statement, let

$$\mathbb{B} = (\mathbb{C}^2, \|\cdot\|_2), \qquad Y = \{(0,0),(1,0),(i,0)\}, \qquad Y' = \{(0,0),(1,0),(0,1)\},$$

and let $T : Y \to Y'$ fix $(0,0)$ and $(1,0)$, and send $(i,0)$ to $(0,1)$. Then T necessarily cannot extend to a $\mathbb{C}$-linear map. Also note that akin to Theorem 4, here too the final three assertions each characterize inner products using (isosceles) metric geometry alone, and without appealing to the vector space structure in $\mathbb{B}$.

We conclude by exploring two weakenings of the notion of "strong" n-point homogeneity that was used to characterize when $\mathbb{B}$ is an inner product space. The first question is to ask if one can work with "usual" n-point homogeneity, i.e. if one can remove the "onto" part of that definition. Note that in the literature, the requirement of extending subset-isometries to onto-isometries of the whole (metric) space is perhaps more natural than merely into-isometries, given their appearance in well-known results and open problems in the Banach space literature – the Mazur–Ulam theorem [22], the Banach–Mazur problem [2, 21], and Tingley's problem [26] among others – and also given the folklore fact that this holds for all inner product spaces *and* in our results above. Now having studied the "onto" picture in detail, the first question (above) is natural.

To answer it, note that the proof of Theorem 4 required the "onto" hypothesis solely to invoke the Mazur–Ulam theorem. This hypothesis can thus be bypassed, at a cost:

Theorem 13. *Suppose* $(\mathbb{B}, \|\cdot\|)$ *is a nonzero real or complex normed linear space that is moreover strictly convex:*

$$\|a+b\| = \|a\| + \|b\| \qquad \implies \qquad a, b \in \mathbb{B} \text{ are linearly dependent.}$$

Then $\|\cdot\|^2$ *arises from a (real or complex, respectively) inner product if and only if* $\mathbb{B}$ *is* n*-point homogeneous for any (every)* $n \geqslant 3$ *– equivalently, if the flip map on the vertices of each isosceles triangle extends to a self-isometry of* $\mathbb{B}$*.*

In particular, this reveals yet another equivalent condition to the inner product property: $\mathbb{B}$ is strictly convex and ("usual") 3-point homogeneous.

Proof. The reverse implication over $\mathbb{R}$ follows from the corresponding part of Theorem 4, using Baker's result [1] that the Mazur–Ulam theorem does not require the surjectivity hypothesis if $\mathbb{B}$ is strictly convex. The reverse implication over $\mathbb{C}$ is now shown as in Theorem 4; and the forward implication follows from Theorem 4. ☐

We formulate the natural question corresponding to the remaining case.

Question 14. Suppose $\mathbb{B}$ is a (real) normed linear space that is not strictly convex (in particular, not strongly 3-point homogeneous or equivalently an inner product space). Can $\mathbb{B}$ be 3-point homogeneous? (To start, one can assume $\mathbb{B}$ separable and complete.)

We end by answering a natural question that arises from Theorem 12(4). Note that (strong) n-point homogeneity concerns subsets Y, Y' of size at most n. However, Theorem 12(4) shows that one can work with subsets of size exactly $n = 3$. Given Theorem 4, one can ask if subsets of size exactly 4, or any fixed integer > 4, will suffice to isolate an inner product. Our final result provides an affirmative answer, thereby adding to Theorem 12:

Theorem 15. *The conditions in Theorem 12 are further equivalent to:*

(5) *For any fixed* $3 \leqslant n < \infty$*, every isometry* $T : Y \to Y'$ *of* n*-point subsets* $Y, Y' \subseteq X$ *extends to an onto isometry of* $\mathbb{B}$*.*

Proof. Clearly (2) $\implies$ (5). Conversely, it suffices to assume (5) and show Theorem 12(1). We follow the proof of Theorem 4: if $\|\cdot\|$ is not induced by an inner product, there exist $\gamma' \in \mathbb{R} \setminus \{0, \pm 1\}$ and $f', g' \in \mathbb{B}$ with $\|f'\| = \|g'\|$, but $\|f' + \gamma' g'\| \neq \|g' + \gamma' f'\|$. Thus, $0, f', g'$ are distinct.

There are now two cases. First if $g' = -f'$ then the flip map extends to the onto isometry $T \equiv -\mathrm{id}_{\mathbb{B}}$. Else f', g' are $\mathbb{R}$-linearly independent, so their $\mathbb{R}$-span is a real normed plane $P \subseteq \mathbb{B}$, say. As all norms on $\mathbb{R}^2$ are equivalent, $(P, \|\cdot\|)$ is linearly bi-Lipschitz – hence homeomorphic – to $(\mathbb{R}^2, \|\cdot\|_2)$.

We now claim that the "equidistant locus in P" – i.e., the locus $Z_{f',g'} := \{h \in P : \|h - f'\| = \|h - g'\|\}$ is uncountable. Indeed, if this is not true then $Z_{f',g'}$ is at most countable, so $P \setminus Z_{f',g'}$ is homeomorphic to a countably-punctured Euclidean plane, hence is path-connected. As the function

$$\varphi : P \to \mathbb{R}; \quad h \mapsto \|h - f'\| - \|h - g'\|$$

is continuous on $P \setminus Z_{f',g'}$, and switches signs from f' to g', it must vanish at an "intermediate" point in $P \setminus Z_{f',g'}$. This contradiction shows the claim.

Finally, as $n \geqslant 3$, choose distinct points $h_1, \dots, h_{n-3} \in Z_{f',g'} \setminus \{0\}$, and define Y, Y' via:

$$Y = Y' := \{0, f', g',\ h_1, \dots, h_{n-3}\}.$$

Let $T : Y \to Y'$ interchange f', g' and fix all other points in Y. By hypothesis, T extends to an onto isometry of $\mathbb{B}$. Now repeating the proof of Theorem 4, it follows that $\mathbb{B}$ is a real inner product space. Finally, if $\mathbb{B}$ was a complex linear space then the discussion around (1.4) reveals the complex inner product structure. □

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(S.S. Damase) Department of Mathematics, Indian Institute of Science, Bangalore 560012, India; PhD student

Email address: sujits@iisc.ac.in

(A. Khare) Department of Mathematics, Indian Institute of Science, Bangalore 560012, India and Analysis & Probability Research Group, Bangalore 560012, India

Email address: khare@iisc.ac.in