

A WEIGHT-FORMULA FOR ALL HIGHEST WEIGHT MODULES, AND A HIGHER ORDER PARABOLIC CATEGORY \mathcal{O}

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ABSTRACT. Let \mathfrak{g} be a complex Kac–Moody algebra, with Cartan subalgebra \mathfrak{h} . Also fix a weight $\lambda \in \mathfrak{h}^*$. For $M(\lambda) \twoheadrightarrow V$ an arbitrary highest weight \mathfrak{g} -module, we provide a cancellation-free, non-recursive formula for the weights of V . This is novel even in finite type, and is obtained from λ and a collection $\mathcal{H} = \mathcal{H}_V$ of independent sets in the Dynkin diagram of \mathfrak{g} that are associated to V .

Our proofs use and reveal a finite family (for each λ) of “higher order Verma modules” $\mathbb{M}(\lambda, \mathcal{H})$ – these are all of the universal modules for weight-considerations. They (i) generalize and subsume parabolic Verma modules $M(\lambda, J)$, and (ii) have pairwise distinct weight-sets, which exhaust the weight-sets of all modules $M(\lambda) \twoheadrightarrow V$. As an application, we explain the sense in which the modules $M(\lambda)$ of Verma and $M(\lambda, J_V)$ of Lepowsky are respectively the zeroth and first order upper-approximations of every V , and continue to higher order upper-approximations $\mathbb{M}_k(\lambda, \mathcal{H}_V)$ (and to lower-approximations). We also determine the k th order integrability of V , for all $k \geq 0$.

We then introduce the category $\mathcal{O}^{\mathcal{H}} \subseteq \mathcal{O}$, which is a higher order parabolic analogue that contains the higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$. We show that $\mathcal{O}^{\mathcal{H}}$ has enough projectives, and also initiate the study of BGG reciprocity, by proving it for all $\mathcal{O}^{\mathcal{H}}$ over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$. Finally, we provide a BGG resolution for the universal modules $\mathbb{M}(\lambda, \mathcal{H})$ in certain cases; this yields a Weyl-type character formula for them, and involves the action of a parabolic Weyl semigroup.

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1. INTRODUCTION

Throughout this paper, and unless otherwise specified, we work over \mathbb{C} , with \mathfrak{g} denoting an arbitrary Kac–Moody algebra,¹ $U\mathfrak{g}$ its universal enveloping algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a fixed Cartan subalgebra, and $\lambda \in \mathfrak{h}^*$ an arbitrary (highest) weight. As further notation: denote by Δ the root system, $\Pi = \{\alpha_i : i \in I\}$ a base of simple roots indexed by nodes I , $\{e_i, f_i, \alpha_i^\vee : i \in I\}$ a set of Chevalley generators, and W the Weyl group generated by the simple reflections $\{s_i : i \in I\}$. We will identify subsets $J \subseteq I$ with the corresponding Dynkin sub-diagrams of the diagram on I for \mathfrak{g} .

The reader immediately interested in the main results can skip directly to Section 2.

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¹More precisely, we work with any Lie algebra $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g} \twoheadrightarrow \bar{\mathfrak{g}}$ for a given generalized Cartan matrix. Thus \mathfrak{g} lies in between $\tilde{\mathfrak{g}}$ generated purely by the Chevalley–Serre relations, and the quotient $\bar{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}$ by the maximal ideal trivially intersecting \mathfrak{h} . Our results hold over all such intermediate Lie algebras \mathfrak{g} .

1.1. Characters. The study of semisimple, affine, and Kac–Moody Lie algebras \mathfrak{g} and their representations is a prominent theme in mathematics, from early work by Cartan, Killing, and Weyl to the Langlands program in recent times, with numerous other connections and applications. A central question involves understanding the structure of (simple) highest weight \mathfrak{g} -modules. In this work we focus on their characters and associated information.

We begin when \mathfrak{g} is simple and finite-dimensional. If the highest weight $\lambda \in \mathfrak{h}^*$ is dominant integral – denoted $\lambda \in P^+$ – the character of the corresponding simple module $L(\lambda)$ is given by the celebrated formula of Weyl [40] (and variants by Freudenthal and others). In standard notation:

$$\lambda \in P^+ \implies \text{ch } L(\lambda) = \sum_{w \in W} \frac{(-1)^{\ell(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}, \quad (1.1)$$

with \bullet the dot-action. In contrast, when λ is “generic”, the module itself is a Verma module $M(\lambda)$ [39], with a transparent character formula (related to the Kostant partition function [26]):

$$\lambda \in \mathfrak{h}^* \implies \text{ch } M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}. \quad (1.2)$$

For arbitrary highest weights, one uses Kazhdan–Lusztig theory [3, 8, 22, 33] to write down the character. For instance, if λ is dominant integral and $w \in W$, then we have the simple character

$$\text{ch } L(ww_\circ \bullet \lambda) = \sum_{x \leq w} (-1)^{\ell(w) - \ell(x)} P_{x,w}(1) \text{ch } M(xw_\circ \bullet \lambda), \quad (1.3)$$

where $P_{x,w}$ denotes the relevant Kazhdan–Lusztig polynomial. Notice that computing weight multiplicities – or even the easier question of which weights occur – using these formulas for $L(\lambda)$ is hard for two reasons: (a) the presence of signs, leading to cancellations, and (b) furthermore for non-integrable modules, the recursive nature of Kazhdan–Lusztig polynomials.

If \mathfrak{g} is of infinite type, less is known. For symmetrizable \mathfrak{g} , one uses the Weyl–Kac character formula, but character formulas are not known for *all* highest weights in affine type – and indeed, simple modules with highest weight λ at critical level behave very differently from those with λ at non-critical level (see e.g. [17]). For non-symmetrizable \mathfrak{g} , even the first step above is challenging, i.e. it remains open if the maximal integrable module (for λ dominant integral) is simple.

Clearly, understanding arbitrary highest weight modules (i.e., quotients of Verma modules) is harder – even for \mathfrak{g} of finite type, hence for arbitrary Kac–Moody \mathfrak{g} .

1.2. Weights. We now turn to the theme of the present work. Closely associated to the “quantitative” Weyl character formula is a “qualitative” picture, which was known from the outset – the easier question of determining the weights (i.e. ignoring multiplicities). As is folklore: the set of weights of a simple finite-dimensional highest weight module $L(\lambda)$ is W -invariant with convex hull the polytope with vertices $W(\lambda)$, and the weights are recovered by intersecting with the λ -translate of the root lattice. A similar statement holds for integrable $L(\lambda)$ over Kac–Moody \mathfrak{g} .

The uniformity of this description turns out to hold more generally. Recently in [16, 15, 23], Dhillon and Khare proved several positive formulas for the weights of $L(\lambda)$ for arbitrary (including non-integrable) simple modules over all Kac–Moody \mathfrak{g} . In contrast to the above story for characters, these weight-formulas hold uniformly, for all highest weights and across all types (for \mathfrak{g}). One of these formulas exactly generalizes the above result in terms of convex hulls (always in \mathfrak{h}^*): now one works with a W_J -invariant polyhedral shape rather than a W -invariant one, corresponding to the partial integrability $J \subseteq I$ of (the not necessarily fully-integrable module) $L(\lambda)$.

We now present one of these formulas; this serves to motivate our main result, as well as to introduce some of the necessary notation. As this result – and our paper – makes extensive use of parabolic Verma modules [18, 29], we begin by setting notation for them.

Definition 1.1. Given a subset $S \subseteq \mathbb{R}$ and subsets X, Y of a real vector space, SX will denote the set of finite S -linear combinations of elements of X , with the empty sum denoting 0. Moreover,

$$X \pm Y := \{x \pm y : x \in X, y \in Y\}, \quad X \setminus Y := \{x \in X : x \notin Y\}$$

will denote the Minkowski sum and difference, and set difference, respectively.

For $J \subseteq I$, define $\Pi_J := \{\alpha_j : j \in J\}$ and $\Delta_J = \Delta_J^+ \sqcup \Delta_J^-$ to be $\Delta \cap \mathbb{Z}\Pi_J$. Now let the Levi subalgebra $\mathfrak{l}_J := \mathfrak{h} + \bigoplus_{\alpha \in \Delta_J} \mathfrak{g}_\alpha$, and let $\mathfrak{g}_J := \mathfrak{g}(A_{J \times J})$, where $A = A_{I \times I}$ is the generalized Cartan matrix for \mathfrak{g} . Also fix a (non-canonical) realization of \mathfrak{g}_J as a subalgebra of \mathfrak{g} . Next, for an \mathfrak{h} -module V , denote by $\text{wt } V := \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$ its *set of weights*, where $V_\mu := \{v \in V : h \cdot v = \mu(h)v \ \forall h \in \mathfrak{h}\}$. E.g. for $\mathfrak{a} = \mathfrak{g}, \mathfrak{g}_J, \mathfrak{l}_J$ (over $\text{ad } \mathfrak{h}$) and $\alpha \in \Delta$, \mathfrak{a}_α denotes the α -root space.

Given $\lambda \in \mathfrak{h}^*$, define its *integrability* J_λ to be:

$$J_\lambda := \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}, \quad (1.4)$$

where $\langle \cdot, \cdot \rangle$ denotes the evaluation map: $\mathfrak{h}^* \times \mathfrak{h} \rightarrow \mathbb{C}$. For $J \subseteq J_\lambda$, define the *parabolic Verma module* $M(\lambda, J) := \text{Ind}_{\mathfrak{p}_J}^{\mathfrak{g}} L_J^{\max}(\lambda)$, where $\mathfrak{p}_J := \mathfrak{l}_J + \mathfrak{n}^+$ is a parabolic subalgebra of \mathfrak{g} , and $L_J^{\max}(\lambda)$ is the maximal integrable highest weight \mathfrak{l}_J -module, which is given a \mathfrak{p}_J -module structure via $(\mathfrak{p}_J)_\alpha \cdot L_J^{\max}(\lambda) = 0$ for $\alpha \notin \Delta_J$. Note that $M(\lambda, J) \cong U\mathfrak{g} \otimes_{U\mathfrak{p}_J} L_J^{\max}(\lambda)$.

Both the ‘‘quantitative’’ and ‘‘qualitative’’ pictures are well known for parabolic Verma modules. For the former, we mention a variant (see e.g. [16]) that extends the Atiyah–Bott version of the Weyl–Kac character formula [1] and hence subsumes Equations (1.1) and (1.2). Namely, for an arbitrary parabolic Verma module over Kac–Moody \mathfrak{g} , one has

$$\text{ch } M(\lambda, J) = \sum_{w \in W_J} \frac{(-1)^{\ell(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}, \quad \forall \lambda \in \mathfrak{h}^*, \ J \subseteq J_\lambda, \quad (1.5)$$

where W_J is the corresponding parabolic Weyl subgroup.

For the latter picture, one has Minkowski difference formulas obtained from parabolic induction:

$$\begin{aligned} \text{wt } M(\lambda, J) &= \text{wt } L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+) \\ &= ((\lambda - \mathbb{Z}_{\geq 0}\Delta^+) \cap \text{conv } W_J(\lambda)) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+). \end{aligned} \quad (1.6)$$

Resuming the above discussion, we write a positive weight-formula for simple modules:

Theorem 1.2 (Khare [23], Dhillon–Khare [16]). *Let \mathfrak{g} and $\lambda \in \mathfrak{h}^*$ be arbitrary. Then,*

$$\text{wt } L(\lambda) = \text{wt } M(\lambda, J_\lambda). \quad (1.7)$$

Given the uniformity of these weight-formulas for $L(\lambda)$, $\lambda \in \mathfrak{h}^*$ (via (1.6)), our original goal in this work was a more challenging result: a positive formula for the weights of an arbitrary highest weight module V . We make a few remarks here, addressing the treatment and proofs below.

First, a weight-formula for V was unknown beyond simple and parabolic Verma modules, even in finite type. We provide a uniform, positive formula for all highest weight modules over all \mathfrak{g} . Perhaps one ‘‘miracle’’ here is that such a formula exists in the first place, and it uses simply the Dynkin diagram and the images in V of some lines in $M(\lambda)$ killed by $\mathfrak{n}^+!$ See Theorem A.

Second is the ‘‘next’’ question, of characters. Our quest for a formula for $\text{wt } V$ also yields rewards on this side. Even more: we discovered a novel (to our knowledge) family of ‘‘higher order Verma modules’’ $\mathbb{M}(\lambda, \mathcal{H})$ which ‘‘cover’’ all highest weight modules – with the same weights – and subsume parabolic Verma modules $M(\lambda, J)$. (Over $\mathfrak{sl}_2^{\oplus n}$, these modules $\mathbb{M}(\lambda, \mathcal{H})$ comprise *all* highest weight modules.) A question of future interest is to study the family $\mathbb{M}(\lambda, \mathcal{H})$, starting with their characters and BGG-type resolutions (we begin this study by obtaining such a resolution in certain cases, in Section 7), as well as their geometric counterparts via localization.

A curious byproduct of our weight-based approach is a family of Minkowski difference formulas for all parabolic Verma modules. These subsume (1.6) as a special case, and interestingly, do not hold in general on the level of representations via parabolic induction (except (1.6)).

1.3. Holes. We lead up to our main results by introducing – by example – another key ingredient.

As mentioned above, Dhillon and Khare [16, 23] showed that for all $\lambda \in \mathfrak{h}^*$, the convex hull of weights of a simple highest weight \mathfrak{g} -module $L(\lambda)$ recovers $\text{wt } L(\lambda)$, by intersecting a suitable W_{J_λ} -invariant shape with the root lattice-translate $\lambda + \mathbb{Z}\Delta$. However, the same does not hold for all highest weight modules V . For the simplest example, if $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, then

$$V_{00} := M(0, 0)/M(-2, -2) \tag{1.8}$$

has a “hole” inside the hull: its weights are precisely the lattice points along the boundary of $\text{conv}(\text{wt } V_{00})$, i.e., $-\mathbb{Z}_{\geq 0}\alpha_1 \cup -\mathbb{Z}_{\geq 0}\alpha_2$, and all interior lattice points $(-2\mathbb{Z}_{\geq 0})^2$ lie in $\text{conv}(\text{wt } V_{00})$ but not in $\text{wt } V_{00}$. This simple example is at the heart of progress in multiple directions, below.

More generally, holes can occur in a \mathfrak{g} -module $M(\lambda) \rightarrow V$ as follows. Suppose $\{\alpha_h : h \in H\}$ is a set of pairwise “orthogonal” roots such that $\langle \lambda, \alpha_h^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all h – i.e., H is an independent subset of J_λ . If $f_h \in \mathfrak{g}_{-\alpha_h}$ denotes a Chevalley generator, then applying the lowering operator-product

$$\prod_{h \in H} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1}$$

to the highest weight line V_λ can *sometimes* yield zero.² Whenever this happens, letting $\mathfrak{l}_H := \mathfrak{sl}_2^{\oplus H} + \mathfrak{h}$ denote the corresponding Levi, the set $\text{wt } U(\mathfrak{l}_H)V_\lambda$ comprises the λ -shifted root-lattice points along a “thickening” of the boundary of its convex hull – i.e., there is a hole in the interior of the convex hull. Clearly, whether or not this happens depends on (a) the highest weight module V , and (b) the independent subset $H \subseteq J_\lambda$ of nodes. The latter also shows that there are at most finitely many holes in V , each corresponding to a one-dimensional weight space of $M(\lambda)$ that consists of maximal vectors (i.e., vectors annihilated by all raising operators e_i).

2. MAIN RESULTS

In all results below, V denotes a general nonzero highest weight \mathfrak{g} -module over an arbitrary Kac–Moody algebra \mathfrak{g} , with a general (fixed) highest weight $\lambda \in \mathfrak{h}^*$. The reader can go through the entire work assuming $\mathfrak{g} = \mathfrak{sl}_{n+1}$, or \mathfrak{g} semisimple, without losing (most of) the novelty in the work.

With the motivation and background given above, we begin by formalizing the above notion of holes. In what follows, recall $J_\lambda := \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$.

Definition 2.1. For a module $M(\lambda) \rightarrow V$ over Kac–Moody \mathfrak{g} , define the “set of holes” in $\text{wt } V$ as:

$$\mathcal{H}_V := \left\{ H \subseteq J_\lambda \mid \text{the Dynkin subdiagram on } H \text{ has no edge and } \left(\prod_{h \in H} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1} \right) V_\lambda = 0 \right\}. \tag{2.1}$$

We take the empty product to be $1 \in U\mathfrak{g}$ here, so that $\emptyset \in \mathcal{H}_V$ if and only if $V = 0$.

Remark 2.2. Explicitly, the set of holes \mathcal{H}_V in V consists of precisely those independent sets $H \subseteq J_\lambda$, for which defining $\lambda_H = \lambda - \sum_{h \in H} (\langle \lambda, \alpha_h^\vee \rangle + 1)\alpha_h = \prod_{h \in H} s_h \bullet \lambda$, one has: (i) the weight space $M(\lambda)_{\lambda_H}$ is one-dimensional (via Kostant’s function, this is equivalent to H having no edge), (ii) this weight space consists of maximal vectors (via \mathfrak{sl}_2 -theory), and (iii) the same weight space in V is $V_{\lambda_H} = 0$.

Now our first result is a positive formula for $\text{wt } V$. Recall, $M(\lambda, J)$ is a parabolic Verma module.

²We will index simple roots in such “holes” by $h \in H \subseteq I$. This index should not be confused with elements of \mathfrak{h} .

Theorem A. Fix a complex Kac–Moody Lie algebra \mathfrak{g} , and \mathfrak{h} , Δ , $\Pi = \{\alpha_i : i \in I\}$ as above. Let $\lambda \in \mathfrak{h}^*$ and let $M(\lambda) \rightarrow V$ be nonzero. Then

$$\text{wt } V = \bigcup_{J \subseteq J_\lambda : J \cap H \neq \emptyset \forall H \in \mathcal{H}_V} \text{wt } M(\lambda, J) \quad (2.2)$$

if \mathcal{H}_V is nonempty. Otherwise, $\text{wt } V = \text{wt } M(\lambda)$.

An immediate consequence of Theorem A is the following “geometric combinatorial” formula:

Corollary 2.3. For all $\lambda \in \mathfrak{h}^*$ and nonzero modules $M(\lambda) \rightarrow V$ with \mathcal{H}_V nonempty,

$$\text{wt } V = \bigcup_{J \subseteq J_\lambda : J \cap H \neq \emptyset \forall H \in \mathcal{H}_V} \text{wt } L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+). \quad (2.3)$$

Remark 2.4. We provide some intuition behind the formula (2.2). Suppose $\text{wt } M(\lambda, J) \subseteq \text{wt } V$ for some $J \subseteq J_\lambda$. Then one has an inclusion of holes (i.e., of the lack of one-dimensional weight spaces killed by \mathfrak{n}^+ – or *highest weight lines*): $\mathcal{H}_V \subseteq \mathcal{H}_{M(\lambda, J)}$. By the universality – or the $U(\mathfrak{n}_{\Delta^- \setminus \Delta_J^-}^-)$ -freeness – of $M(\lambda, J)$, the line $\prod_{h \in H} f_h^{(\lambda, \alpha_h^\vee) + 1} \cdot M(\lambda)_\lambda$ being quotiented out of $M(\lambda, J)$ implies some $h \in J$. Therefore $J \cap H \neq \emptyset \forall H \in \mathcal{H}_V$ – which explains the necessity of this condition in the union in (2.2). Theorem A says that firstly, this condition also guarantees $\text{wt } M(\lambda, J) \subseteq \text{wt } V$; and secondly, such considerations recover *all* weights of V .

Remark 2.5 (Alternate formulas). For computational purposes, one can work with the subset \mathcal{H}_V^{\min} consisting of the minimal sets in \mathcal{H}_V under inclusion – i.e., replace respectively in (2.2):

$$\mathcal{H}_V \rightsquigarrow \mathcal{H}_V^{\min}, \quad J_\lambda \rightsquigarrow \bigcup_{H \in \mathcal{H}_V^{\min}} H.$$

Indeed, we work with \mathcal{H}_V^{\min} in Sections 6 and 7; for now, observe that if $V = M(\lambda, J_0)$, then working with minimal holes yields exactly one term on the right-hand side in (2.2): $J = J_0$. Section 5.3 also provides two “ k th order” weight-formulas for $\text{wt } V$ (for all \mathfrak{g} , λ , V , and $k \geq 1$), which extend Theorem A above and Theorem B below. Furthermore, there is yet another formulation of Equation (2.2) in terms of the “higher order parabolic category” $\mathcal{O}^{\mathcal{H}_V}$ – see (6.4).

We make two observations before proceeding further. First, the formula in Theorem A is clearly uniform across all types for \mathfrak{g} and all highest weights λ . Second, in each such case, the formula is visibly positive as well as non-recursive. These are in contrast to the situation for characters, in which case one does not even have conjectural formulas in all cases, or weight multiplicities even for all integrable simple highest weight modules $L(\lambda)$ if \mathfrak{g} is non-symmetrizable.

Our second result shows the existence of a *finite* collection of “uniform” highest weight modules $M(0) \rightarrow \mathbb{M}_t$, such that for every pair – $\lambda \in \mathfrak{h}^*$ and a module $M(\lambda) \rightarrow V$ – there exists t such that

$$\text{wt } V = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \text{wt } \mathbb{M}_t.$$

In particular, $\text{wt } V$ combines $\text{wt } L_{J_\lambda}^{\max}(\lambda)$ (which is a fundamental object – in fact, a parabolic Verma \mathfrak{l}_{J_λ} -module) with the weights of some \mathbb{M}_t from a finite collection *that works for all* $\lambda \in \mathfrak{h}^*$ *and all* V . (That said, which \mathbb{M}_t to use does depend on (λ, V) , as we explain.)

To define these modules \mathbb{M}_t and state our result, additional notation is required.

Definition 2.6.

- (1) Given a subset of simple roots (indexed by) $J \subseteq I$, define $\text{Indep}(J)$ to comprise the collection of independent subsets $H \subseteq J$, i.e. whose induced Dynkin subgraph in the Dynkin diagram of \mathfrak{g} has no edges. Note that \emptyset and $\{j\}$ are in $\text{Indep}(J)$ for all $J \subseteq I$ and $j \in J$.

(2) Given $J \subseteq I$ and a subset \mathcal{H} of $\text{Indep}(I)$, define the highest weight \mathfrak{g} -module

$$\mathbb{M}(\mathcal{H}) := \frac{M(0)}{\sum_{H \in \mathcal{H}} U\mathfrak{g} \left(\prod_{h \in H} f_h \right) M(0)_0}, \quad (2.4)$$

where $M(0)_0$ is the highest weight line in the Verma module $M(0)$. If $H = \emptyset$, define the empty product $\prod_{h \in H} f_h$ to be 1. (Thus, if $H = \emptyset \in \mathcal{H}$ then $\mathbb{M}(\mathcal{H}) = 0$.)

(3) Recall, a subset \mathcal{H} of a poset (\mathcal{P}, \preceq) is *upper-closed* if $H \in \mathcal{H}$, $H \preceq H'$ in \mathcal{P} imply $H' \in \mathcal{H}$.

The modules $\mathbb{M}(\mathcal{H})$ are studied in the next section; they comprise the sought-for finite set $\{\mathbb{M}_t\}_t$:

Theorem B. *Fix $\lambda \in \mathfrak{h}^*$. Let $M(\lambda) \twoheadrightarrow V$ be nonzero, and \mathcal{H}_V be as in Theorem A. Then \mathcal{H}_V is a proper, upper-closed subset of $\text{Indep}(J_\lambda)$, and*

$$\text{wt } V = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \text{wt } \mathbb{M}(\mathcal{H}_V). \quad (2.5)$$

Moreover, the finite collection of upper-closed subsets of $\text{Indep}(J_\lambda)$ is in bijection with the set $\{\text{wt } V : M(\lambda) \twoheadrightarrow V\}$, via $\Psi_\lambda : \mathcal{H} \mapsto \text{wt } L_{J_\lambda}^{\max}(\lambda) + \text{wt } \mathbb{M}(\mathcal{H})$. In particular, for $M(0) \twoheadrightarrow V$ we have:

$$\text{wt } V = \text{wt } \mathbb{M}(\mathcal{H}_V). \quad (2.6)$$

In fact, we extend (2.6) to all modules $M(\lambda) \twoheadrightarrow V$ (for all $\lambda \in \mathfrak{h}^*$) in the next section. Thus, (2.6) and its extension reveal a second “miracle” about the sets $\text{wt } V$ (cf. a few lines below Theorem 1.2): the “obvious” (by Remark 3.8) holes \mathcal{H}_V in the weight-set obtained from the top – i.e. the line V_λ – are the *only* ones, for every highest weight module over every Kac–Moody algebra.

Remark 2.7. Akin to the explicit weight-formula in Theorem A, the finite number of weight-sets for each $\lambda \in \mathfrak{h}^*$ (in Theorem B) is also in stark contrast to the situation for characters. For instance, let \mathfrak{g} be of infinite type, and consider any sequence of increasing words in the Weyl group:

$$w_1 := s_{i_1}, \quad w_2 := s_{i_2} s_{i_1}, \quad \dots; \quad \ell(w_n) = n \quad \forall n \geq 1.$$

Then the modules $M(0)/M(w_n \bullet 0)$ have pairwise distinct characters. Theorem B nevertheless shows that they – and all other modules $M(0) \twoheadrightarrow V$ – collectively yield only finitely many weight-sets, $\text{wt } \mathbb{M}(\mathcal{H})$. (In particular, the weight-sets of $M(0)/M(w_n \bullet 0)$ eventually stabilize.)

Remark 2.8. It is natural to ask if (2.5) specializes to the Minkowski difference formula (1.6) for V a parabolic Verma module. This is not true on the nose; rather, in Proposition 4.1 we exhibit a *family* of Minkowski difference formulas for each parabolic Verma module, of which (1.6) is one extreme. An interesting feature is: these formulas (except (1.6)) hold for weights, but do *not* lift in general to the level of parabolic induction. Following this family of formulas in Proposition 4.1, we explain how (2.5) generalizes the Minkowski difference formula at the other extreme to (1.6).

Our next result is an application.

Theorem C. *Fix a nonzero \mathfrak{g} -module $M(\lambda) \twoheadrightarrow V$ and set $S := \text{wt } V$. There exist unique maximum and minimum modules $V^{\max}(S), V^{\min}(S)$ with highest weight λ , which satisfy the property:*

A module $M(\lambda) \twoheadrightarrow V'$ has $\text{wt } V' = S$, if and only if $V^{\max}(S) \twoheadrightarrow V' \twoheadrightarrow V^{\min}(S)$.

In particular, $\text{wt } V^{\max}(S) = \text{wt } V^{\min}(S) = S$.

Theorem C has several consequences that are explored in Section 5:

- (1) We define the k th order upper- and lower-approximations $\mathbb{M}_k(\lambda, \mathcal{H}_V)$ and $\mathbb{L}_k(\lambda, \mathcal{H}_V)$ of every highest weight module $M(\lambda) \twoheadrightarrow V$. See Definition 5.7.
- (2) We isolate the common universal property of these two approximations, i.e. the *k th order integrability* of V . See Proposition 5.11 and Definition 5.12.
- (3) This leads to a “repeated-stratification” of the poset (under quotienting) of all highest weight \mathfrak{g} -modules – not just of their sets of weights. See Remark 5.13.

Remark 2.9 (Working over quotient Kac–Moody algebras). Our results until Section 6 are valid independent of which Kac–Moody quotient algebra $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow \bar{\mathfrak{g}}$ (all associated to a given generalized Cartan matrix) is used. This is because (see e.g. [16]) $\text{wt } M(\lambda, J)$, hence $\text{wt } L(\lambda) \forall \lambda \in \mathfrak{h}^*$, does not change across all such \mathfrak{g} . Note, this extends the folklore result that $\text{wt } L(\lambda)$ does not change across all \mathfrak{g} , for $\lambda \in P^+$. Theorem A extends these facts to say that $\text{wt } \mathbb{M}(\lambda, \mathcal{H})$ does not change across all such \mathfrak{g} , for every \mathcal{H} . (Section 6 works in finite type, and for results in Section 7, see Remark 7.2.) Similarly, the computation of weights of highest weight modules over $\mathfrak{l}_J \subseteq \mathfrak{g}$ and over $\mathfrak{g}(A_{J \times J})$ yield the same results modulo identifying $\mathfrak{g}(A_{J \times J}) \cong \mathfrak{g}_J \subseteq \mathfrak{l}_J$.

Having discussed weights in detail, in the final two sections we initiate the study of two facets on the representation side. First, we introduce and study “higher order versions” of the parabolic category $\mathcal{O}^{\mathfrak{p}_J}$, corresponding to hole-sets \mathcal{H} with higher order holes:

Definition 2.10. Given a complex semisimple Lie algebra \mathfrak{g} with simple roots $\Pi = \{\alpha_i : i \in I\}$, and a subset $\mathcal{H} \subseteq \text{Indep}(I)$, the *higher order parabolic category* $\mathcal{O}^{\mathcal{H}}$ is the full subcategory of objects in \mathcal{O} on which the following lowering operator-products all act locally nilpotently:

$$\mathbf{f}_H = \mathbf{f}_H^{(0)} := \prod_{h \in H} f_h, \quad H \in \mathcal{H}.$$

Zeroth and first order special cases are \mathcal{O} and $\mathcal{O}^{\mathfrak{p}_J}$, respectively.

Now for the next main result. Notice, Theorem 1.2 says that the weights of a simple module $L(\lambda)$ agree with those of its universal highest weight cover $M(\lambda, J_\lambda)$ in the parabolic category $\mathcal{O}^{\mathfrak{p}_{J_\lambda}}$. We extend this to every highest weight module V , inside the higher order parabolic category $\mathcal{O}^{\mathcal{H}_V}$. We also show each $\mathcal{O}^{\mathcal{H}}$ has enough projectives, and in a special case, a variant of BGG reciprocity.

Theorem D. Fix a complex semisimple Lie algebra \mathfrak{g} and a subset $\mathcal{H} \subseteq \text{Indep}(I)$.

- (1) With notation from Definition 2.10, the category $\mathcal{O}^{\mathcal{H}}$ is an abelian subcategory of \mathcal{O} , which has enough projectives and enough injectives.
- (2) The weights of every highest weight module in \mathcal{O} , say $M(\lambda) \rightarrow V$, agree with those of the universal highest weight cover of $L(\lambda)$ in $\mathcal{O}^{\mathcal{H}_V}$.
- (3) If $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$, then every projective module in $\mathcal{O}^{\mathcal{H}}$ has a “standard filtration”, and a variant of BGG reciprocity holds. See Theorem 6.17 for the details.

That said, over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus 2}$ and over every \mathfrak{g} of rank ≥ 3 , we show that the category $\mathcal{O}^{\mathcal{H}}$ is not always a highest weight category [10]. The reason is that filtrations for different projectives can feature multiple standard objects with the same highest weight. See Section 6.3.

Second, in Section 7 we initiate the study of the *characters* of “higher order Verma modules”. These include parabolic Verma modules and the family $\mathbb{M}(\mathcal{H})$ above, for which we provide a BGG-type resolution and its Weyl character formula – over any Kac–Moody \mathfrak{g} – in two “dual” settings:

- (1) for λ with arbitrary integrable roots J_λ , and pairwise orthogonal minimal holes in \mathcal{H}^{\min} ;
- (2) for λ with pairwise orthogonal integrable roots J_λ , and arbitrary minimal holes in \mathcal{H}^{\min} .

The latter case includes (a BGG resolution of) every highest weight module over $\mathfrak{sl}_2^{\oplus n}$.

Theorem E. In the two settings just above, there exists a parabolic Weyl semigroup $(W_{\mathcal{H}}, \ell_{\mathcal{H}})$, and a BGG resolution of the module $\mathbb{M}(\mathcal{H})$ of the form

$$0 \longrightarrow M_k \xrightarrow{d_k} M_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \mathbb{M}(\mathcal{H}) \rightarrow 0, \quad (2.7)$$

with $M_t \cong \bigoplus_{w \in W_{\mathcal{H}} : \ell_{\mathcal{H}}(w)=t} M(w \bullet 0) \forall t$. This implies the Weyl–Kac character formula

$$\text{ch } \mathbb{M}(\mathcal{H}) = \sum_{w \in W_{\mathcal{H}}} \frac{(-1)^{\ell_{\mathcal{H}}(w)} e^{w \bullet 0}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}, \quad (2.8)$$

formulated in the spirit of the classical character formulas above.

In particular, if the holes in \mathcal{H}^{\min} are pairwise orthogonal, then the character of $\mathbb{M}(\mathcal{H})$ is “ $W_{\mathcal{H}}$ -invariant”:

$$w(\operatorname{ch} \mathbb{M}(\mathcal{H})) = (-1)^{\ell(w) - \ell_{\mathcal{H}}(w)} \operatorname{ch} \mathbb{M}(\mathcal{H}), \quad \forall w \in W_{\mathcal{H}}. \quad (2.9)$$

While this is for highest weight $\lambda = 0$, we show analogous results for all $\lambda \in \mathfrak{h}^*$. See Section 7.

We conclude this section on a philosophical note. The recent papers [16, 15, 23, 24, 37] obtained information about (i) the weights of simple modules $L(\lambda)$ (for all $\lambda \in \mathfrak{h}^*$), and (ii) the convex hull of $\operatorname{wt} V$ and its face lattice for all highest weight modules, using the “first order information” associated to every module V – namely, its *integrability*, defined as:

$$J_V := \{h \in J_{\lambda} \mid f_h^{\langle \lambda, \alpha_h^{\vee} \rangle + 1} V_{\lambda} = 0\}. \quad (2.10)$$

This first order information corresponds to precisely the “singleton holes” in \mathcal{H}_V . Moreover, our results on $\operatorname{wt} V$ specialize to their analogues in the above works when \mathcal{H}_V is the upper-closure of its singleton elements. However, a general module $M(\lambda) \twoheadrightarrow V$ can involve “higher order holes”. That is: the above papers operated using integrability, i.e., \mathfrak{sl}_2 -theory. In contrast, we use higher order integrability – see Definition 5.12 – and “ $\mathfrak{sl}_2 \oplus \dots \oplus \mathfrak{sl}_2$ ” theory (in that each hole $H \in \mathcal{H}_V$ corresponds to a line in the Verma submodule $U\mathfrak{g}_H \cdot M(\lambda)_{\lambda}$ killed by \mathfrak{n}^+ , and $\mathfrak{g}_H \simeq \mathfrak{sl}_2^{\oplus H}$). For another, “higher level” use of this theory, see the character formulas in Section 7.

2.1. Organization. In each of the next sections, we prove one of the theorems above. In Section 3 we show Theorem A as well as (2.6), and introduce the higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$.

Section 4 proves Theorem B and a set of Minkowski differences for each parabolic Verma module.

Section 5 shows Theorem C and provides formulas for the k th order upper- and lower-approximations $\mathbb{M}_k(\lambda, \mathcal{H}_V)$ and $\mathbb{L}_k(\lambda, \mathcal{H}_V)$ of each highest weight module V – these include the Verma module $M(\lambda)$ and simple module $L(\lambda)$ when $k = 0$, and the parabolic Verma cover $M(\lambda, J_V)$ when $k = 1$. We also identify in Section 5.2 the “higher order integrability” that is preserved by the interval of modules $[\mathbb{L}_k(\lambda, \mathcal{H}_V), \mathbb{M}_k(\lambda, \mathcal{H}_V)]$ for each $k \geq 0$ and each V . We end with two “ k th order” weight-formulas in Section 5.3, which specialize to Theorems A and B for $k = 1, \infty$ respectively.

In Section 6, which is over \mathfrak{g} of finite type, we study the higher order parabolic categories $\mathcal{O}^{\mathcal{H}}$. We identify the simples and their standard covers in $\mathcal{O}^{\mathcal{H}}$, and show $\mathcal{O}^{\mathcal{H}}$ has enough projectives. We also prove BGG reciprocity in all $\mathcal{O}^{\mathcal{H}}$ over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ – leading to a potential formula in general – and explain why $\mathcal{O}^{\mathcal{H}}$ is not always a highest weight category, over $\mathfrak{sl}_2^{\oplus 2}$ and each higher rank \mathfrak{g} .

In Section 7, we provide a BGG resolution and Weyl character formula for the higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$ in two special situations. That is, we prove the extension to general $\lambda \in \mathfrak{h}^*$ of Theorem E. We end by discussing a speculative BGG resolution over a dihedral group, and formulating a few avenues of future study.

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3. THEOREM A: THE WEIGHT-FORMULA, AND HIGHER ORDER VERMA MODULES

3.1. Higher order Verma modules: examples. We first show Theorem A. Begin with Theorem B, which says the weight-sets of all $M(\lambda) \twoheadrightarrow V$ are of the form $\operatorname{wt} L_{J_{\lambda}}^{\max}(\lambda) + \operatorname{wt} \mathbb{M}(\mathcal{H}_V)$. Associated to this formula, there is a universal module of highest weight λ that we now introduce.

Definition 3.1. Given $\lambda \in \mathfrak{h}^*$ and a subset $\mathcal{H} \subseteq \operatorname{Indep}(J_{\lambda})$, define the module

$$\mathbb{M}(\lambda, \mathcal{H}) := \frac{M(\lambda)}{\sum_{H \in \mathcal{H}} U\mathfrak{g} \left(\prod_{h \in H} f_h^{\langle \lambda, \alpha_h^{\vee} \rangle + 1} \right) M(\lambda)_{\lambda}}. \quad (3.1)$$

We term these objects *higher order Verma modules*. These are fundamental objects which include all $M(\lambda)$ and $M(\lambda, J)$, see below – and they are indispensable to understanding all highest weight modules, for several reasons: (a) In Theorem A, we show that $\text{wt } V = \mathbb{M}(\lambda, \mathcal{H}_V)$. (b) In Theorem B, we show $\text{wt } \mathbb{M}(\lambda, \mathcal{H}) = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \text{wt } \mathbb{M}(\mathcal{H}_V)$, and these exhaust all weight-sets $\text{wt } V$. (c) In Theorem C, given V , $\mathbb{M}(\lambda, \mathcal{H}_V)$ turns out to be the maximum module $V^{\max}(\text{wt } V)$ (shown below). Despite these attractive properties – and notwithstanding Example 3.5 over $\mathfrak{sl}_2^{\oplus n}$ – to the best of our knowledge the nontrivial among these modules have not been studied in the literature. The only ones that have been studied are the “easy” case – parabolic Verma modules $M(\lambda, J)$ – and the original inspiration for these modules: $M(0, 0)/M(-2, -2)$ over $\mathfrak{sl}_2^{\oplus 2}$ (in e.g. previous work [23, 16]). Thus we seek to understand these modules – their characters, resolutions, etc. – before general highest weight modules and others in \mathcal{O} . We begin their study in this paper.

Remark 3.2. Some clarifying observations: (1) $\mathbb{M}(\lambda, \mathcal{H})$ is unchanged if one replaces \mathcal{H} by its upper-closure in $\text{Indep}(J_\lambda)$, or by any set in between. (2) In “reverse”, replacing \mathcal{H} by its “minimal” elements \mathcal{H}^{\min} does not change $\mathbb{M}(\lambda, \mathcal{H})$. Thus the modules $\mathbb{M}(\lambda, \mathcal{H})$ subsume the usual parabolic Verma modules $M(\lambda, J)$, in the special case that \mathcal{H} is the upper-closure of the singleton sets in it. (3) The modules $\mathbb{M}(\lambda, \mathcal{H})$ specialize to $\mathbb{M}(\mathcal{H})$ in (2.4) via $\lambda \rightsquigarrow 0$. (4) The extreme cases are: (i) $\mathbb{M}(\lambda, \mathcal{H}) = 0 \iff H = \emptyset \in \mathcal{H}$, (ii) $\mathbb{M}(\lambda, \mathcal{H}) = M(\lambda) \iff \mathcal{H} = \emptyset$. (iii) As the simple module $L(\lambda)$ has maximum possible integrability J_λ , similarly it has maximum possible \mathcal{H} -set, $\text{Indep}(J_\lambda) \setminus \{\emptyset\}$.

Remark 3.3 (Weak Minkowski decomposition). Like their first order versions $M(\lambda, J)$, the modules $\mathbb{M}(\lambda, \mathcal{H})$ have a freeness property over $U\mathfrak{a}_\mathcal{H}$, for a Lie subalgebra $\mathfrak{a}_\mathcal{H} \subseteq \mathfrak{n}^-$. Namely, define

$$\mathfrak{a}_\mathcal{H} := \bigoplus_{\alpha \notin \Delta_{\tilde{H}}} \mathfrak{n}_\alpha^-, \quad \text{where} \quad \tilde{H} := \bigcup_{H \in \mathcal{H}^{\min}} H \subseteq I.$$

Also define $\mathbf{f}_H^{(\lambda)} := \prod_{h \in H} f_h^{(\lambda, \alpha_h^\vee) + 1}$ for $H \in \text{Indep}(J_\lambda)$. Then by the PBW theorem,

$$\mathbb{M}(\lambda, \mathcal{H}) \cong_{\mathfrak{a}_\mathcal{H}} \frac{U\mathfrak{a}_\mathcal{H} \otimes_{\mathbb{C}} U\mathfrak{n}_{\Delta_{\tilde{H}}}^-}{U\mathfrak{a}_\mathcal{H} \otimes_{\mathbb{C}} \sum_{H \in \mathcal{H}^{\min}} U\mathfrak{n}_{\Delta_{\tilde{H}}}^- \cdot \mathbf{f}_H^{(\lambda)}} \cong_{\mathfrak{a}_\mathcal{H}} U\mathfrak{a}_\mathcal{H} \otimes_{\mathbb{C}} \frac{U\mathfrak{n}_{\Delta_{\tilde{H}}}^-}{\sum_{H \in \mathcal{H}^{\min}} U\mathfrak{n}_{\Delta_{\tilde{H}}}^- \cdot \mathbf{f}_H^{(\lambda)}}.$$

Hence by Theorem A, the weights of every highest weight \mathfrak{g} -module $M(\lambda) \twoheadrightarrow V$ satisfy:

$$\text{wt } V = \text{wt}_J V - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+), \quad \forall J \supseteq \bigcup_{H \in \mathcal{H}_V^{\min}} H. \quad (3.2)$$

Note that the special case $J = J_\lambda$ was one of the main results in previous work [37, Theorem C].

Next, here are some examples of highest weight modules, including $\mathbb{M}(\lambda, \mathcal{H})$ (beyond the “obvious” cases $M(\lambda, J)$ in Remark 3.2(2)), in order to build more intuition.

Example 3.4. As above: the first nontrivial example is in (1.8). (This was originally used in [23] by the first author to observe that the convex hull of $\text{wt } V$ does not always yield $\text{wt } V$.) In this example, $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\lambda = (0, 0)$. Setting $\mathcal{H} = \{\{1, 2\}\} = \{J_\lambda\}$ yields $\mathbb{M}(\lambda, \mathcal{H}) = M(0, 0)/M(-2, -2)$. This is the “simplest” module whose weights are not determined by their convex hull. It is also the prototypical module for all such cases; see Theorem 5.10 and the subsequent lines. By Theorem A,

$$\text{wt } M(0, 0)/M(-2, -2) = \text{wt } M((0, 0), \{1\}) \cup \text{wt } M((0, 0), \{2\}) = -\mathbb{Z}_{\geq 0}\alpha_2 \cup -\mathbb{Z}_{\geq 0}\alpha_1.$$

Example 3.5 (All highest weight modules over $\mathfrak{sl}_2^{\oplus n}$). Let $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$. For $n = 1$, every highest weight \mathfrak{g} -module is either Verma or finite-dimensional – i.e. a parabolic Verma module. What about higher n ? We claim, every module $M(\lambda) \twoheadrightarrow V$ equals $\mathbb{M}(\lambda, \mathcal{H})$ for some $\mathcal{H} \subseteq \text{Indep}(J_\lambda) = 2^{J_\lambda}$ – adding to the fundamental nature of these modules. The claim follows by noting that if $0 \rightarrow N \rightarrow M(\lambda) \rightarrow V \rightarrow 0$, then N is generated by weight spaces (since $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$), and hence by maximal weight vectors. By $\mathfrak{sl}_2^{\oplus n}$ -theory, these are precisely $\prod_{h \in H} f_h^{(\lambda, \alpha_h^\vee) + 1} \cdot M(\lambda)_\lambda$ for $H \subseteq J_\lambda$.

Example 3.6 (Some rank-4 examples). Let $\mathfrak{g} = \mathfrak{sl}_5$, $I = \{1, 2, 3, 4\}$, and $\lambda = \varpi_1 - \varpi_4$, with ϖ_i the fundamental weights. Note that $J_\lambda = \{1, 2, 3\}$ and $\text{Indep}(J_\lambda) = \{\{1\}, \{2\}, \{3\}, \{1, 3\}\}$. Consider

$$\mathbb{M}(\lambda, \{\{2\}, \{1, 3\}\}) = \frac{M(\lambda)}{U(\mathfrak{g})f_2 \cdot M(\lambda)_\lambda + U(\mathfrak{g})f_1^2 f_3 \cdot M(\lambda)_\lambda}.$$

Notice, $\mathcal{H} = \{\{2\}, \{1, 3\}\}$ is upper-closed in $\text{Indep}(J_\lambda)$, and $\mathbb{M}(\lambda, \mathcal{H})$ is a quotient of the parabolic Verma module $M(\lambda, \{2\})$. Now Theorem A yields:

$$\text{wt } \mathbb{M}(\lambda, \{\{2\}, \{1, 3\}\}) = \text{wt } M(\lambda, \{1, 2\}) \cup \text{wt } M(\lambda, \{2, 3\}) \cup \text{wt } M(\lambda, \{1, 2, 3\}),$$

where we can omit the final set as it lies in the first two terms; see Remark 3.9.

Similarly, consider the following two modules:

$$V_1 = \frac{M(\lambda)}{U(\mathfrak{g})f_2^2 f_3 \cdot M(\lambda)_\lambda}, \quad V_2 = \frac{M(\lambda)}{U(\mathfrak{g})f_1^2 \cdot M(\lambda)_\lambda + U(\mathfrak{g})f_2^2 f_3 \cdot M(\lambda)_\lambda}.$$

Then $J_{V_1} = \emptyset$, $J_{V_2} = \{1\}$, and the sets $\mathcal{H}_{V_1} = \emptyset$, $\mathcal{H}_{V_2} = \{\{1\}, \{1, 3\}\}$ are upper-closed in $\text{Indep}(J_\lambda)$. As the nodes 2, 3 are adjacent in the Dynkin diagram, $\{2, 3\}$ does not contribute to a hole in \mathcal{H}_{V_1} and \mathcal{H}_{V_2} . Hence by Theorem A, $\text{wt } V_1 = \text{wt } M(\lambda)$ and $\text{wt } V_2 = \text{wt } M(\lambda, \{1\})$. \square

Following these examples, we add to the intuition behind Theorem A in Remark 2.4. First note an elementary lemma, which follows by considering the Kostant partition function.

Lemma 3.7. *Fix Kac-Moody \mathfrak{g} and $\lambda \in \mathfrak{h}^*$. The weight space $M(\lambda)_\mu$ of the Verma module is one-dimensional if and only if $\mu = \lambda - \sum_{h \in H} n_h \alpha_h$, where $H \subseteq I$ is independent and all $n_h \in \mathbb{Z}_{>0}$.*

Remark 3.8. Returning to Theorem A, suppose $0 \rightarrow N \rightarrow M(\lambda) \rightarrow V \rightarrow 0$. It is not clear if weights are lost upon quotienting $M(\lambda)$ by maximal vectors in N corresponding to non-independent nodes – e.g. they are not lost in Example 3.6 with $f_2^2 f_3 M(\lambda)_\lambda$ (modulo proving Theorem A). However, 1-dimensional weight spaces $M(\lambda)_\mu \subseteq N$ are clearly sensitive for $\text{wt } V$, since then $V_\mu = 0$ and weights are lost, by $\mathfrak{sl}_2^{\oplus n}$ -theory. In proving Theorem A, we show that $\text{wt } V = \text{wt } \mathbb{M}(\lambda, \mathcal{H}_V)$ (see (3.1)). This shows the converse to the above application of $\mathfrak{sl}_2^{\oplus n}$ -theory (and extends (2.6)): weights are lost when passing from $M(\lambda)$ to V (i.e. now, when passing to $\mathbb{M}(\lambda, \mathcal{H}_V)$) *only* if one proceeds as in Example 3.5. That is, only if one quotients out 1-dimensional weight spaces in $M(\lambda)$ spanned by *maximal* vectors for $\mathfrak{sl}_2^{\oplus H}$, for some independent set/hole – by Lemma 3.7 – $H \in \mathcal{H}_V$.

We continue with a “computational” remark and some examples.

Remark 3.9. One can work with fewer sets $J \subseteq J_\lambda$ in (2.2), as $M(\lambda, J) \rightarrow M(\lambda, J')$ for $J \subseteq J' \subseteq J_\lambda$. Also, J only needs to intersect the minimal holes $H \in \mathcal{H}_V^{\min}$, so one can just use the subsets $J \subseteq \cup_{H \in \mathcal{H}_V^{\min}} H$. Next, if the holes in \mathcal{H}_V^{\min} are pairwise disjoint, one uses exactly $\prod_{H \in \mathcal{H}_V^{\min}} |H|$ -many transversal sets J in (2.2) – as in the previous two examples – by selecting one node from each H in J . Of course, this does not always hold – e.g. for $\mathfrak{g} = \mathfrak{sl}_2^{\oplus 3}$ and $\mathcal{H}_V^{\min} = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, the three sets $J = \{1, 2\}, \{2, 3\}, \{1, 3\}$ suffice. (This example also features below, see (7.15).)

Example 3.10. Suppose $V = L(\lambda)$ is simple. Then for each integrable direction $j \in J_\lambda$, $\{j\} \in \mathcal{H}_{L(\lambda)}$ since $f_j^{\langle \lambda, \alpha_j^\vee \rangle + 1} \cdot L(\lambda)_\lambda = 0$ (as its preimage generates a proper submodule of $M(\lambda)$). Moreover, every hole $H \subseteq J_\lambda$. Thus there is a unique set J in the union in (2.2) for $V = L(\lambda)$, namely, $J = J_\lambda$. Hence Theorem 1.2 is an immediate consequence of Theorem A: $\text{wt } L(\lambda) = \text{wt } M(\lambda, J_\lambda)$.

Example 3.11 (Multiplicity-free character formula for $\mathbb{M}(\lambda, \mathcal{H})$). Let $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$, $\lambda \in \mathfrak{h}^*$, and let $\emptyset \neq \mathcal{H} \subseteq \text{Indep}(J_\lambda) = 2^{J_\lambda}$. We compute $\text{ch } \mathbb{M}(\lambda, \mathcal{H})$, or simply the weights of $\mathbb{M}(\lambda, \mathcal{H})$, since all highest weight $\mathfrak{sl}_2^{\oplus n}$ -modules have one-dimensional weight spaces. Enumerate the subsets of J_λ that intersect all holes in \mathcal{H} as $\{J_1, \dots, J_l\}$. (We may consider only the minimal such subsets.) Then

$$\text{wt } \mathbb{M}(\lambda, \mathcal{H}) = \bigcup_{i=1}^l (\text{wt } L_{J_i}^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{J_i}^+)) = \bigcup_{i=1}^l (\text{wt } L_{J_i}^{\max}(\lambda) - \mathbb{Z}_{\geq 0} \Pi_{J_i^c}).$$

by Theorem A. Over $\mathfrak{sl}_2^{\oplus n}$, it is not hard to show that

$$(\text{wt } L_{J_i}^{\max}(\lambda) - \mathbb{Z}_{\geq 0} \Pi_{J_i^c}) \cap (\text{wt } L_{J_j}^{\max}(\lambda) - \mathbb{Z}_{\geq 0} \Pi_{J_j^c}) = (\text{wt } L_{J_i \cup J_j}^{\max}(\lambda) - \mathbb{Z}_{\geq 0} \Pi_{(J_i \cup J_j)^c}).$$

By the inclusion-exclusion principle, and since all weight spaces of $\mathbb{M}(\lambda, \mathcal{H})$ are one-dimensional,

$$\text{ch } \mathbb{M}(\lambda, \mathcal{H}) = \sum_{\emptyset \neq S \subseteq \{1, \dots, l\}} (-1)^{|S|-1} \text{ch } M(\lambda, \cup_{i \in S} J_i), \quad \mathcal{H} \neq \emptyset. \quad (3.3)$$

This provides an alternating formula for $\text{ch } \mathbb{M}(\lambda, \mathcal{H})$ in terms of the sets J_i that are transversing the holes in \mathcal{H} (equivalently, in \mathcal{H}^{\min}) – over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$. This picture is “orthogonal” to the alternating formula (7.19) that we obtain below – that formula is alternating in terms of the holes in \mathcal{H}^{\min} , and follows from the BGG-type resolution (7.18) for the same module $\mathbb{M}(\lambda, \mathcal{H})$ over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$.

3.2. Proof of Theorem A: reverse inclusion. We now turn to the proof of Theorem A. It is clear that for any highest weight module V , the universal module $\mathbb{M}(\lambda, \mathcal{H}_V) \twoheadrightarrow V$, implying an inclusion of their weight-sets. We will show this inclusion is in fact an *equality* (extending (2.6)), so that $\mathbb{M}(\lambda, \mathcal{H}_V)$ is the “universal highest weight cover” of V . First, an observation useful below:

Lemma 3.12. *Fix λ and $\mathcal{H} \subseteq \text{Indep}(J_\lambda)$, and let $M := \mathbb{M}(\lambda, \mathcal{H})$. Then \mathcal{H}_M equals the upper-closure of \mathcal{H} . In particular, for all nonzero modules $M(\lambda) \twoheadrightarrow V$, one has $\mathcal{H}_V = \mathcal{H}_N$ for $N = \mathbb{M}(\lambda, \mathcal{H}_V)$.*

Returning to our present goal, in the course of proving Theorem A we study three sets of weights:

$$\text{wt } \mathbb{M}(\lambda, \mathcal{H}_V), \quad \text{wt } V, \quad \text{and} \quad S(\lambda, \mathcal{H}_V), \quad (3.4)$$

where for convenience we define for any subset $\mathcal{H} \subseteq \text{Indep}(J_\lambda)$:

$$S(\lambda, \mathcal{H}) := \begin{cases} \text{wt } M(\lambda), & \text{if } \mathcal{H} = \emptyset, \\ \emptyset, & \text{if } \emptyset \in \mathcal{H}, \\ \bigcup_{J \subseteq J_\lambda : J \cap H \neq \emptyset \ \forall H \in \mathcal{H}} \text{wt } M(\lambda, J), & \text{if } \mathcal{H} \neq \emptyset \text{ and } \emptyset \notin \mathcal{H}. \end{cases} \quad (3.5)$$

Theorem A asserts that the final two sets of weights in (3.4) coincide, but here we also claim the added equality $\text{wt } V = \text{wt } \mathbb{M}(\lambda, \mathcal{H}_V)$ for all nonzero modules $M(\lambda) \twoheadrightarrow V$ (which extends (2.6) and generalizes (1.7), and) which is repeatedly used in later sections. Since $\mathbb{M}(\lambda, \mathcal{H}_V) \twoheadrightarrow V$, the first set in (3.4) contains the second. The next step is:

Proposition 3.13. *If $\lambda \in \mathfrak{h}^*$ and $M(\lambda) \twoheadrightarrow V$ is nonzero, then $\text{wt } V \supseteq S(\lambda, \mathcal{H}_V)$.*

The proof appeals to a result from previous work, which is the special case $\mathcal{H}_V = \emptyset$ of Theorem A:

Theorem 3.14 ([37, Proposition 2.7]). *Let $\lambda \in \mathfrak{h}^*$, and $M(\lambda) \twoheadrightarrow V$. Suppose there are no holes in $\text{wt } V$, i.e., $\mathcal{H}_V = \emptyset$. Then $\text{wt } V = \text{wt } M(\lambda)$.*

Proof of Proposition 3.13. Notice $\emptyset \notin \mathcal{H}_V$ since $V \neq 0$. If $\mathcal{H}_V = \emptyset$, the result follows from Theorem 3.14. Thus, assume henceforth that $\emptyset \notin \mathcal{H}_V$ and $\emptyset \neq \mathcal{H}_V$. Pick $J \subseteq J_\lambda$ with the property that $J \cap H \neq \emptyset \ \forall H \in \mathcal{H}_V$; thus $J^c \not\supseteq H \ \forall H \in \mathcal{H}_V$. Hence by Theorem 3.14 over \mathfrak{g}_{J^c} ,

$$\lambda - \mathbb{Z}_{\geq 0} \Pi_{J^c} = \text{wt } U(\mathfrak{g}_{J^c}) V_\lambda \subseteq \text{wt } V.$$

Next for any $\xi \in \mathbb{Z}_{\geq 0} \Pi_{J^c}$, every nonzero vector $x \in V_{\lambda - \xi}$ is a maximal vector for the action of the Levi \mathfrak{l}_J , so it generates the highest weight \mathfrak{l}_J -module $U(\mathfrak{l}_J)x$ of highest weight $\lambda - \xi$. Hence $\text{wt } L_J^{\max}(\lambda - \xi) \subseteq \text{wt } (U(\mathfrak{l}_J)x) \subseteq \text{wt } V$. Now by the integrable slice decomposition (see Lemma 3.16),

$$\text{wt } M(\lambda, J) = \bigsqcup_{\xi \in \mathbb{Z}_{\geq 0} \Pi_{J^c}} \text{wt } L_J^{\max}(\lambda - \xi) \subseteq \text{wt } V.$$

As J is arbitrary, this yields $S(\lambda, \mathcal{H}_V) \subseteq \text{wt } V$. □

3.3. Proof of Theorem A: forward inclusion. Returning to the discussion preceding Proposition 3.13, the proof of Theorem A is completed by showing the other inclusion:

Theorem 3.15. *For all \mathfrak{g} , weights λ , and nonzero modules $M(\lambda) \twoheadrightarrow V$, $\text{wt } V \subseteq S(\lambda, \mathcal{H}_V)$.*

Recall that \mathcal{H}_V and $S(\lambda, \mathcal{H}_V)$ were defined in Definition 2.1 and Equation (3.5), respectively.

Theorem 3.15 not only implies Theorem A (given Proposition 3.13), but together with Lemma 3.12 it also implies the remaining sought-for inclusion in (3.4):

$$\text{wt } M \subseteq S(\lambda, \mathcal{H}_M) = S(\lambda, \mathcal{H}_V), \quad \text{for } M = \mathbb{M}(\lambda, \mathcal{H}_V).$$

In the rest of this section, we show Theorem 3.15. The proof uses a fundamental result from [16]:

Lemma 3.16 (Integrable slice decomposition, Dhillon–Khare [16]). *If $J \subseteq J_\lambda$, then*

$$\text{wt } M(\lambda, J) = \text{wt } L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+) = \bigsqcup_{\xi \in \mathbb{Z}_{\geq 0} \Pi_{I \setminus J}} \text{wt } L_J^{\max}(\lambda - \xi). \quad (3.6)$$

We will also need the following application of \mathfrak{sl}_3 - (or rank-2) theory, aka the Serre relations:

Lemma 3.17. *Fix Kac–Moody \mathfrak{g} , integers $k, M, N \geq 0$, and pairwise distinct nodes $h_1, \dots, h_k, h, i \in I$, such that h is adjacent to i but not to h_1, \dots, h_k . Say $Y \in U(\mathfrak{n}^-)$ is a linear combination of words in the Chevalley generators $f_i, f_{h_1}, \dots, f_{h_k}$, with N occurrences of f_i in each word. Then*

$$f_h^{M-N\langle \alpha_i, \alpha_h^\vee \rangle} \cdot Y = X \cdot f_h^M,$$

for $X \in U(\mathfrak{n}^-)$ again a linear combination of words with N occurrences of f_i in each word.

Proof. It suffices to work with Y a single word in the given alphabet. The next calculation (found in textbooks) holds in any associative algebra, and is specialized here to $U(\mathfrak{n}^-)$:

$$f_h^l f_i = \sum_{j=0}^{-\langle \alpha_i, \alpha_h^\vee \rangle} \binom{l}{j} (\text{ad } f_h)^j (f_i) f_h^{l-j} = X_1 f_h^{l+\langle \alpha_i, \alpha_h^\vee \rangle}, \quad \text{for all } l \geq -\langle \alpha_i, \alpha_h^\vee \rangle. \quad (3.7)$$

Here $X_1 \in U(\mathfrak{n}^-)$ is a linear combination of words in $\{f_l : l \in I\}$, each containing just one f_i – and the sum stops where it does due to the Serre relations. Now suppose $Y = Y_1 f_i Y_2 f_i \cdots f_i Y_{N+1}$, with each Y_j a word in the f_{h_t} . Successively applying (3.7) with $l = M - (N-j)\langle \alpha_i, \alpha_h^\vee \rangle$ for $j = 0, 1, \dots$,

$$\begin{aligned} f_h^{M-N\langle \alpha_i, \alpha_h^\vee \rangle} \cdot Y &= Y_1 \cdot X_1 f_h^{M-(N-1)\langle \alpha_i, \alpha_h^\vee \rangle} \cdot Y_2 f_i \cdots f_i Y_{N+1} \\ &= Y_1 X_1 Y_2 \cdot X_2 f_h^{M-(N-2)\langle \alpha_i, \alpha_h^\vee \rangle} \cdot Y_3 f_i \cdots f_i Y_{N+1} \\ &= \cdots \\ &= Y_1 X_1 \cdots Y_N \cdot X_N f_h^M \cdot Y_{N+1}. \end{aligned}$$

Setting $X := Y_1 X_1 \cdots Y_N X_N Y_{N+1}$, we are done, since each X_t contains exactly one f_i . \square

With Lemmas 3.16 and 3.17 in hand, we complete the proof of the remaining half of Theorem A.

Proof of Theorem 3.15. When $\mathcal{H}_V = \emptyset$, Theorem 3.14 implies $\text{wt } V = \text{wt } M(\lambda) = S(\lambda, \mathcal{H}_V)$. We now assume $\mathcal{H}_V \neq \emptyset$, and also $\emptyset \notin \mathcal{H}_V$ (else the result is trivial). Notice, for each $H \in \text{Indep}(J_\lambda)$ the vector $\left(\prod_{h \in H} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1} \right) m_\lambda$ is a maximal vector in the $(\prod_{h \in H} s_h) \bullet \lambda$ -weight space of $M(\lambda)$.

We now turn to the proof, with $\mathcal{H}_V \neq \emptyset$, $\emptyset \notin \mathcal{H}_V$. The idea is to work with *all* triples (λ', V', μ') , where $\lambda' \in \mathfrak{h}^*$, $M(\lambda') \twoheadrightarrow V'$ (with holes $\mathcal{H}_{V'}$), and $\mu' \in \text{wt } V'$ are arbitrary. We make the following

Claim. *For every triple (λ', V', μ') as above, $\mu' \in S(\lambda', \mathcal{H}_{V'})$.*

This claim – which implies the theorem – is now shown by induction on $\text{ht}(\lambda' - \mu') \geq 0$. In the base case, $\mu' = \lambda'$, and so the claim holds trivially.

Induction step: Fix an arbitrary (λ, V, μ) as above, with $\text{ht}(\lambda - \mu) > 0$ (and assume the result is true for all triples (λ', V', μ') with smaller $\text{ht}(\lambda' - \mu')$). We introduce notation for a set in the union in $S(\lambda, \mathcal{H}_V)$ in (3.5):

$$\mathfrak{J}(V) := \{J \subseteq J_\lambda \mid J \cap H \neq \emptyset \ \forall H \in \mathcal{H}_V\}. \quad (3.8)$$

Thus the goal is to find $J \in \mathfrak{J}(V)$ such that $\mu \in \text{wt } M(\lambda, J)$. This would imply $\mu \in S(\lambda, \mathcal{H}_V)$ and hence show the induction step.

We now break up the remainder of the induction step into (sub-)steps, in the interest of clarity. The reader may find it helpful to first read the plan for all steps, before reading the details.

Step 1: First choose an arbitrary element $K \in \mathfrak{J}(V)$ – for instance, $K = J_\lambda$. We produce a PBW monomial $F_1 \in U(\mathfrak{n}_K^-)$ such that $\mu \leq \nu \leq \lambda$, where $\nu := \lambda + \text{wt}(F_1) \in \text{wt } V$.

To do so, write \mathfrak{n}^- as the direct sum of the two Lie subalgebras $\mathfrak{n}' := \bigoplus_{\alpha \in \Delta^- \setminus \Delta_K^-} \mathfrak{n}_\alpha^-$ and \mathfrak{n}_K^- . Via

the PBW theorem, fix a basis for $U(\mathfrak{n}^-)$ consisting of monomials in negative root vectors such that in each monomial, elements from \mathfrak{n}' always occur to the left of those from \mathfrak{n}_K^- . Now fix a nonzero highest weight vector $v_\lambda \in V_\lambda$, and pick a nonzero weight vector

$$z := F_2 \cdot F_1 \cdot v_\lambda \in V_\mu, \quad \text{for PBW monomials } F_1 \in U(\mathfrak{n}_K^-) \text{ and } F_2 \in U(\mathfrak{n}')$$

with $\lambda + \text{wt}(F_1) + \text{wt}(F_2) = \mu$. Note that $\text{wt}(F_1) \in -\mathbb{Z}_{\geq 0} \Pi_K$ and $\text{wt}(F_2) \in \mathbb{Z}_{\geq 0} (\Delta^- \setminus \Delta_K^-)$. Now set $\nu := \lambda + \text{wt}(F_1)$, and note that $F_1 v_\lambda \in (V_K)_\nu = V_\nu$, where $V_K := U(\mathfrak{g}_K) v_\lambda$.

Step 2: There are now two cases. If $\nu \in \text{wt } L_K^{\max}(\lambda)$, then $\mu \in S(\lambda, \mathcal{H}_V)$.

Indeed, $\mu = \nu + \text{wt}(F_2) \in \text{wt } M(\lambda, K)$ (by (1.6)), and this is in $S(\lambda, \mathcal{H}_V)$ since $K \in \mathfrak{J}(V)$.

Step 3: Thus, henceforth $\nu \notin \text{wt } L_K^{\max}(\lambda)$. We claim there exists $i \in K \subseteq J_\lambda$ such that (i) $V_{s_i \bullet \lambda} = f_i^{m_i} V_\lambda \neq 0$, where $m_i := \langle \lambda, \alpha_i^\vee \rangle + 1$, and (ii) $V' := U(\mathfrak{g}) \cdot V_{s_i \bullet \lambda}$ has a nonzero μ -weight space.

To see why, choose and fix the highest weight vector $m_\lambda \in M(\lambda)_\lambda$ which maps to v_λ under $M(\lambda) \twoheadrightarrow V$. Also let $M_K(\lambda) \cong U(\mathfrak{n}_K^-)$ be the Verma \mathfrak{g}_K -module. Since

$$\nu \notin \text{wt } L_K^{\max}(\lambda) = \text{wt} \left[\frac{M_K(\lambda)}{\sum_{t \in K} U(\mathfrak{n}_K^-) f_t^{m_t} m_\lambda} \right],$$

the ν -weight space of $M_K(\lambda)$ equals that of its submodule $\sum_{t \in K} U(\mathfrak{n}_K^-) f_t^{m_t} m_\lambda$. Hence the ν -weight space of $V_K = U(\mathfrak{g}_K) v_\lambda$ equals that of $\sum_{t \in K} U(\mathfrak{n}_K^-) f_t^{m_t} v_\lambda$. Now $F_1 v_\lambda \in V_\nu = (V_K)_\nu$ is a linear combination of vectors $X_t f_t^{m_t} v_\lambda$, with $X_t \in U(\mathfrak{n}_K^-)$. Since $F_2 F_1 \cdot v_\lambda \neq 0$, it follows that there exist a node $i \in K$ and a PBW monomial F_3 in $U(\mathfrak{n}_K^-)$ such that

$$z' := F_2 \cdot F_3 \cdot f_i^{m_i} v_\lambda \in V_\mu$$

is nonzero. Defining $V' := U(\mathfrak{g}) \cdot f_i^{m_i} v_\lambda$, this proves both assertions (i) and (ii), since $z' \in V'_\mu$.

Step 4: If $H \in \mathcal{H}_V$ is a hole, then $\emptyset \neq H \setminus \{i\} \in \mathcal{H}_{V'}$, where $V' \neq 0$ is as in Step 3.

The previous three steps helped us arrive at $V' = U(\mathfrak{g}) f_i^{m_i} V_\lambda$, to which we now apply the induction hypothesis. The present step does not use the previous three steps, except the definition of V' . We begin with the definition of $\mathcal{H}_{V'}$, using the description of the highest weight line $V'_{s_i \bullet \lambda}$:

$$\mathcal{H}_{V'} = \left\{ H' \in \text{Indep}(J_{s_i \bullet \lambda}) \mid \left(\prod_{h \in H'} f_h^{(s_i \bullet \lambda, \alpha_h^\vee) + 1} \right) f_i^{m_i} \cdot V_\lambda = 0 \right\}.$$

Fix a hole $H \in \mathcal{H}_V$. Since V' (or its highest weight line) is nonzero, $\{i\} \notin \mathcal{H}_V$, and hence $H \setminus \{i\} \neq \emptyset$. Clearly, $H \setminus \{i\}$ is also independent, and it is easy to verify that

$$H \setminus \{i\} \subseteq J_\lambda \setminus \{i\} \subseteq J_{s_i \bullet \lambda}.$$

It remains to verify the final defining condition for $\mathcal{H}_{V'}$ (above), for $H' := H \setminus \{i\}$. If all nodes in H' are disconnected from i – which includes the case $i \in H$ – then all f_h commute with f_i and $\langle s_i \bullet \lambda, \alpha_h^\vee \rangle = \langle \lambda, \alpha_h^\vee \rangle$, and the desired equality follows from the definition of \mathcal{H}_V :

$$\left(\prod_{h \in H'} f_h^{\langle s_i \bullet \lambda, \alpha_h^\vee \rangle + 1} \right) f_i^{m_i} \cdot V_\lambda = 0 \quad (3.9)$$

Otherwise, $i \notin H$ (so $H' = H$) and at least one node $h \in H'$ is adjacent to i in the Dynkin diagram of \mathfrak{g} . In this case, it suffices to show that

$$\left(\prod_{h \in H'} f_h^{\langle s_i \bullet \lambda, \alpha_h^\vee \rangle + 1} \right) f_i^{m_i} \cdot V_\lambda \in U(\mathfrak{n}^-) \prod_{h \in H'} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1} \cdot V_\lambda.$$

In what follows, define and use

$$c_h := \langle s_i \bullet \lambda, \alpha_h^\vee \rangle + 1, \quad \forall h \in H' = H \setminus \{i\} = H.$$

(Recall, $m_h := \langle \lambda, \alpha_h^\vee \rangle + 1$.) Since $i \notin H$, one has

$$c_h = \langle \lambda, \alpha_h^\vee \rangle + 1 - m_i \langle \alpha_i, \alpha_h^\vee \rangle = m_h + m_i |\langle \alpha_i, \alpha_h^\vee \rangle|. \quad (3.10)$$

Now fix any ordering of $H' = H$, say $H' = \{h_1, \dots, h_k\}$, and apply Lemma 3.17, with $M = m_{h_1}$ and $N = m_i$. Then via (3.10),

$$f_{h_1}^{c_{h_1}} f_i^{m_i} \cdot V_\lambda = X_1 f_{h_1}^{m_{h_1}} \cdot V_\lambda,$$

with $X_1 \in U(\mathfrak{n}^-)$ a linear combination of words, each with exactly m_i occurrences of f_i . Next,

$$f_{h_2}^{c_{h_2}} f_{h_1}^{c_{h_1}} f_i^{m_i} \cdot V_\lambda = f_{h_2}^{c_{h_2}} X_1 f_{h_1}^{m_{h_1}} \cdot V_\lambda = X_2 f_{h_2}^{m_{h_2}} f_{h_1}^{m_{h_1}} \cdot V_\lambda,$$

for some $X_2 \in U(\mathfrak{n}^-)$ as above – again applying Lemma 3.17. Repeating this procedure,

$$\left(\prod_{h \in H'} f_h^{\langle s_i \bullet \lambda, \alpha_h^\vee \rangle + 1} \right) f_i^{m_i} \cdot V_\lambda \in U(\mathfrak{n}^-) \prod_{t=1}^k f_{h_t}^{m_{h_t}} \cdot V_\lambda,$$

and this vanishes, by the definition of $H = H' \in \mathcal{H}_V$.

Step 5: Concluding the proof.

By Step 3(ii) and the induction hypothesis for $(s_i \bullet \lambda, V', \mu)$, there exists $J' \in \mathfrak{J}(V')$ such that $\mu \in \text{wt } M(s_i \bullet \lambda, J')$. Define $J := J' \cap J_\lambda$; then $i \notin J$ since $i \notin J_{s_i \bullet \lambda}$. Now using the integrable slice decomposition (3.6) twice,

$$\begin{aligned} \text{wt } M(\lambda, J) &= \bigsqcup_{\xi \in \mathbb{Z}_{\geq 0} \Pi_{I \setminus J}} \text{wt } L_J^{\max}(\lambda - \xi) \supseteq \bigsqcup_{m_i \alpha_i \leq \xi \in \mathbb{Z}_{\geq 0} \Pi_{I \setminus J}} \text{wt } L_J^{\max}(\lambda - \xi) \\ &= \bigsqcup_{\xi' \in \mathbb{Z}_{\geq 0} \Pi_{I \setminus J}} \text{wt } L_J^{\max}((\lambda - m_i \alpha_i) - \xi') \\ &= \text{wt } M(s_i \bullet \lambda, J) \supseteq \text{wt } M(s_i \bullet \lambda, J'), \end{aligned}$$

noting that $J \subseteq J' \subseteq J_{s_i \bullet \lambda}$. Thus $\mu \in \text{wt } M(s_i \bullet \lambda, J') \subseteq \text{wt } M(\lambda, J)$. We now assert that $J \in \mathfrak{J}(V)$, which concludes the proof of the induction step in the claim at the beginning. Indeed, if $H \in \mathcal{H}_V$,

$$\emptyset \subsetneq J' \cap (H \setminus \{i\}) \subseteq J' \cap H = J' \cap (J_\lambda \cap H) = J \cap H,$$

where the first inclusion follows from Step 4, since $J' \in \mathfrak{J}(V')$. \square

Remark 3.18. In the spirit of Remark 2.9, the above proof should have worked with quadruples $(\mathfrak{g}, \lambda', V', \mu')$, where $\tilde{\mathfrak{g}} \rightarrowtail \mathfrak{g} \rightarrowtail \bar{\mathfrak{g}}$ for a fixed generalized Cartan matrix. We have suppressed the additional variable \mathfrak{g} , since the proof only uses weight-sets of parabolic Verma modules in the proof (and in the formula for $S(\lambda, \mathcal{H})$), and these remain invariant across \mathfrak{g} .

4. THEOREM B: MINKOWSKI DIFFERENCE FORMULA FOR HIGHEST WEIGHT MODULES

Following the proof of Theorem A, this short section quickly shows Theorem B. We begin by isolating a key step in the proof, which is interesting in its own right: a family of Minkowski difference formulas for parabolic Verma modules $M(\lambda, J)$, of which (1.6) is one extremal case.

Proposition 4.1. *Suppose $\lambda \in \mathfrak{h}^*$ and $J \subseteq J_\lambda$. Then for all subsets J' between J and J_λ ,*

$$\text{wt } M(\lambda, J) = \text{wt } L_{J'}^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+). \quad (4.1)$$

Before proving (4.1), notice that if $L_{J'}^{\max}(\lambda) \not\cong L_J^{\max}(\lambda)$ then (4.1) does not extend to the level of representations via parabolic induction. For instance, in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $0 \neq \lambda \in P^+$ (so $J_\lambda = I = \{\alpha_1\}$ by abuse of notation), $J = \emptyset$, and $J' = J_\lambda$, one has

$$U\mathfrak{g} \otimes_{U\mathfrak{p}_J} L_J^{\max}(\lambda) = M(\lambda) \cong M(0) \otimes \mathbb{C}_\lambda,$$

where \mathbb{C}_λ is a one-dimensional \mathfrak{h} -module with eigenvalue λ . In contrast,

$$U\mathfrak{g} \otimes_{U\mathfrak{p}_J} L_{J_\lambda}^{\max}(\lambda) = M(0) \otimes L(\lambda),$$

and this has a strictly larger character than $M(\lambda)$. Thus, the family (4.1) of Minkowski difference formulas appears to be a novel one, and is valid on the level of weights but not for characters.

Proof of Proposition 4.1. The formula for all J' follows from the ones for $J' = J$ in (1.6) and for $J' = J_\lambda$, by sandwiching. Thus it suffices to prove (4.1) for $J' = J_\lambda$, i.e. that

$$\text{wt } L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+) = \text{wt } L_{J_\lambda}^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+).$$

The forward inclusion is obvious, since $J \subseteq J_\lambda$. Next, the parabolic Verma module over \mathfrak{g}_{J_λ} for (λ, J) surjects onto the maximal integrable \mathfrak{g}_{J_λ} -module $L_{J_\lambda}^{\max}(\lambda)$. Hence by (1.6),

$$\text{wt } L_{J_\lambda}^{\max}(\lambda) \subseteq \text{wt } L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta_{J_\lambda}^+ \setminus \Delta_J^+) \subseteq \text{wt } L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+).$$

Subtracting $\mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+)$ from both sides proves the reverse inclusion. \square

Remark 4.2. We take a moment to explain how Theorem B generalizes one case in Proposition 4.1. Let $V = M(\lambda, J)$ in Theorem A. Then \mathcal{H}_V is the upper-closure in $\text{Indep}(J_\lambda)$ of $\{\{j\} : j \in J\}$. Hence $\mathbb{M}(\mathcal{H}_V) = M(0, J)$ in Theorem B, and so by (1.6) we recover the $J' = J_\lambda$ case of (4.1):

$$\text{wt } M(\lambda, J) = \text{wt } V = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \text{wt } M(0, J) = \text{wt } L_{J_\lambda}^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+).$$

Proof of Theorem B. By Theorem A, and Proposition 4.1 for $J' = J_\lambda$, and recalling $\mathfrak{J}(V)$ from (3.8),

$$\text{wt } V = \bigcup_{J \in \mathfrak{J}(V)} (\text{wt } L_{J_\lambda}^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+)) = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \bigcup_{J \in \mathfrak{J}(V)} -\mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+).$$

Next, consider the highest weight module $M = \mathbb{M}(\mathcal{H}_V)$. Here $\lambda = 0$, and $\mathcal{H}_M = \mathcal{H}_V$ by Lemma 3.12. Again applying Theorem A, this time to the right-hand side of (2.5),

$$\text{wt } L_{J_\lambda}^{\max}(\lambda) + \text{wt } \mathbb{M}(\mathcal{H}_V) = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \bigcup_{J \in \mathfrak{J}(V)} \text{wt } M(0, J), \quad (4.2)$$

and via (1.6), this equals the final expression in the previous computation.

This shows (2.5). For the penultimate assertion, Theorem A yields $\text{wt } \mathbb{M}(\lambda, \mathcal{H}_V) = \text{wt } V$ (which also implies the final assertion of course), so the map Ψ_λ is surjective. Also by Remark 3.2(4), the upper-closed subset $\mathcal{H} \subseteq \text{Indep}(J_\lambda)$ is proper, if and only if $\emptyset \not\in \mathcal{H}$, if and only if $\mathbb{M}(\lambda, \mathcal{H}) \neq 0$. Now to show injectivity, suppose $\mathcal{H}_1 \neq \mathcal{H}_2$ are proper upper-closed subsets of $\text{Indep}(J_\lambda)$. Choose a minimal set H in their symmetric difference, say $H \in \mathcal{H}_1 \setminus \mathcal{H}_2$. Then the one-dimensional weight space

$$\prod_{h \in H} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1} \cdot M(\lambda)_\lambda$$

is easily seen to be quotiented in $\mathbb{M}(\lambda, \mathcal{H}_1)$ but not in $\mathbb{M}(\lambda, \mathcal{H}_2)$. Thus (e.g. by Remark 2.2),

$$\lambda - \sum_{h \in H} (\langle \lambda, \alpha_h^\vee \rangle + 1) \alpha_h \in \text{wt } \mathbb{M}(\lambda, \mathcal{H}_2) \setminus \text{wt } \mathbb{M}(\lambda, \mathcal{H}_1),$$

and so the map Ψ_λ is injective as well. \square

5. THEOREM C; HIGHER ORDER APPROXIMATIONS, INTEGRABILITY, AND WEIGHT-FORMULAS

This section begins by proving our next main result in somewhat greater detail than may be necessary, to help understand the subsequent examples. Following the examples, the three subsections discuss the other parts of the section-title, for every highest weight \mathfrak{g} -module.

Proof of Theorem C. Given two highest weight modules V, V' with common highest weight λ , Theorem B asserts: (1) $\text{wt } V = \text{wt } V'$ if and only if $\mathcal{H}_V = \mathcal{H}_{V'}$; and (2) $\mathbb{M}(\lambda, \mathcal{H}_V) \twoheadrightarrow V$, with equality of weights. This reasoning implies that $V^{\max}(\text{wt } V) = \mathbb{M}(\lambda, \mathcal{H}_V)$.

To construct V^{\min} , we adopt a more “natural” notation. By Theorem B, instead of $S = \text{wt } V$, one can equivalently work via Ψ_λ^{-1} with proper upper-closed subsets \mathcal{H}' of $\text{Indep}(J_\lambda)$. (Thus $V^{\max}(\mathcal{H}') = \mathbb{M}(\lambda, \mathcal{H}')$.) Now the construction of $V^{\min}(\mathcal{H}')$ is a familiar one: first define $N(\lambda, \mathcal{H}') \subseteq M(\lambda)$ to be the sum of all submodules N of $M(\lambda)$ for which the weight space

$$N_{\lambda_H} = 0, \quad \text{where } \lambda_H = \lambda - \sum_{h \in H} (\langle \lambda, \alpha_h^\vee \rangle + 1) \alpha_h, \quad \forall H \in \text{Indep}(J_\lambda) \setminus \mathcal{H}'. \quad (5.1)$$

Next, define the highest weight module

$$\mathbb{L}(\lambda, \mathcal{H}') := M(\lambda)/N(\lambda, \mathcal{H}'). \quad (5.2)$$

We claim this is precisely $V^{\min}(\mathcal{H}')$. Indeed, note $\mathcal{H}_{\mathbb{L}(\lambda, \mathcal{H}')} = \mathcal{H}'$, which shows via Ψ_λ that $\text{wt } \mathbb{L}(\lambda, \mathcal{H}') = \text{wt } \mathbb{M}(\lambda, \mathcal{H}')$, proving one implication in the desired property. Next, if $\text{wt } V = \text{wt } \mathbb{M}(\lambda, \mathcal{H}')$, consider the exact sequence $0 \rightarrow N_V \rightarrow M(\lambda) \rightarrow V \rightarrow 0$. Since $\mathcal{H}_V = \mathcal{H}'$ (via Ψ_λ), the definition of \mathcal{H}_V implies (5.1) for $N = N_V$. But then $N_V \subseteq N(\lambda, \mathcal{H}')$, so $V \twoheadrightarrow \mathbb{L}(\lambda, \mathcal{H}')$. \square

Remark 5.1. Since $\text{wt } \mathbb{L}(\lambda, \mathcal{H}_V) = \text{wt } V = \text{wt } \mathbb{M}(\lambda, \mathcal{H}_V)$, the finite collection $\{\text{wt } \mathbb{L}(\lambda, \mathcal{H}_V)\}_V = \{\text{wt } \mathbb{L}(\lambda, \mathcal{H})\}_{\mathcal{H}}$ also exhausts all weight-sets of highest weight modules $M(\lambda) \twoheadrightarrow V$.

Example 5.2. As a special case of Remark 5.1: say $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ for some $n \geq 1$. Then for every highest weight module $M(\lambda) \twoheadrightarrow V$, we have $\mathbb{M}(\lambda, \mathcal{H}_V) = V = \mathbb{L}(\lambda, \mathcal{H}_V)$, since their weights agree and all weight spaces are 1-dimensional. (In particular, $\mathbb{M}(\lambda, \mathcal{H}) = \mathbb{L}(\lambda, \mathcal{H})$ for all $\lambda \in \mathfrak{h}^*$ and \mathcal{H} .)

As mentioned in the proof, the construction of $\mathbb{L}(\lambda, \mathcal{H}')$ should sound familiar to the reader. We illustrate with three special cases from previous literature, classical and modern; the third provides an alternate proof/solution to a question (unpublished) posed by Lepowsky, as we explain below.

Example 5.3. The original “zeroth order” construction (as is explained presently) along these lines is that of the simple module $L(\lambda)$. Indeed, that is the special case where one quotients $M(\lambda)$ by the sum of all proper submodules N , i.e. submodules N for which $N_\lambda = 0$. Thus,

$$L(\lambda) = \mathbb{L}(\lambda, \mathcal{H}'), \quad \text{where } \mathcal{H}' = \text{Indep}(J_\lambda) \setminus \{\emptyset\}. \quad (5.3)$$

Example 5.4. The second-named author recently showed in [37, Theorem B] the existence of $V^{\max}(S), V^{\min}(S)$ for $S = \text{wt } M(\lambda, J)$. Clearly, this is a special case of Theorem C.

The third example is slightly different in flavor (and motivates the next two subsections); it comes from [16]. The authors first explain that associated to every module is its *first order information*:

Theorem 5.5 (Dhillon–Khare, [16]). *Given $\lambda \in \mathfrak{h}^*$ and a highest weight \mathfrak{g} -module $M(\lambda) \twoheadrightarrow V$, the following “first order data” are equivalent, i.e., can each be recovered from the others:*

- (1) *The integrability, $J_V := \{i \in J_\lambda \mid f_i^{\langle \lambda, \alpha_i^\vee \rangle + 1} V_\lambda = 0\}$ (as in (2.10)).*

- (2) *The Weyl group symmetry of $\text{wt } V$.*
- (3) *The convex hull $\text{conv}(\text{wt } V)$.*

Thus the parabolic Verma module $M(\lambda, J_V) \twoheadrightarrow V$, and they have the same convex hull of weights.

Moreover, for simple or parabolic Verma modules V , these data are further equivalent to (4) the weights of V .

Dhillon–Khare next write down what is our third example. Notice in the definition of \mathcal{H}_V in Theorem A, the highest weight λ is the “0th order hole” in V . Given Theorem 5.5, we would similarly like to call the integrable simple directions J_V the “1st order holes”. This is supported by

Example 5.6. Dhillon–Khare [16] (following Khare [23] in some cases) showed the existence of unique largest and smallest modules $M(\lambda, J)$ and $L(\lambda, J)$, respectively, with a prescribed integrability $J \subseteq J_\lambda$ – or by Theorem 5.5, with a prescribed shape of the convex hull of weights. The construction is as above: $M(\lambda, J) = \mathbb{M}(\lambda, \{\{j\} : j \in J\}) = \mathbb{M}(\lambda, \{H \in \mathcal{H}_{M(\lambda, J)} : |H| \leq 1\})$, and the authors introduced $L(\lambda, J)$. In the language of this paper,

$$L(\lambda, J) = \mathbb{L}(\lambda, \mathcal{H}'), \quad \text{where } \mathcal{H}' = \text{Indep}(J_\lambda) \setminus (\{\emptyset\} \sqcup \{\{i\} : i \in J_\lambda \setminus J\}). \quad (5.4)$$

Note that \mathcal{H}' is indeed upper-closed in $\text{Indep}(J_\lambda)$.

5.1. Universal modules approximating a highest weight module. Example 5.6 refines the familiar chain of surjections $M(\lambda) \twoheadrightarrow V \twoheadrightarrow L(\lambda)$ to

$$M(\lambda) \twoheadrightarrow M(\lambda, J_V) \twoheadrightarrow V \twoheadrightarrow L(\lambda, J_V) \twoheadrightarrow L(\lambda).$$

The “zeroth”/outermost boundary-terms share the same highest weight as V , while the “first”/next inner terms share the same integrability as well. Above, we have now produced a refinement of this chain, by replacing the innermost V by the surjections

$$\cdots \twoheadrightarrow \mathbb{M}(\lambda, \mathcal{H}_V) \twoheadrightarrow V \twoheadrightarrow \mathbb{L}(\lambda, \mathcal{H}_V) \twoheadrightarrow \cdots$$

These refinements, and the comments after Theorem 5.5, motivate us to define the following chain of highest weight modules:

Definition 5.7. Fix \mathfrak{g}, λ , and an upper-closed subset $\mathcal{H} \subseteq \text{Indep}(J_\lambda)$, and denote by \mathcal{H}^c its complement. Given an integer $0 \leq k \leq \infty$, define the universal “upper” and “lower” modules

$$\mathbb{M}_k(\lambda, \mathcal{H}) := \mathbb{M}(\lambda, \{H \in \mathcal{H} : |H| \leq k\}), \quad (5.5)$$

$$\mathbb{L}_k(\lambda, \mathcal{H}) := \mathbb{L}(\lambda, \{H \in \mathcal{H}^c : |H| \leq k\}^c). \quad (5.6)$$

(Note, the \mathcal{H} -set on the right in (5.6) is upper-closed.) Now given a module $M(\lambda) \twoheadrightarrow V$, define its k th order upper- and lower-approximations to be $\mathbb{M}_k(\lambda, \mathcal{H}_V)$ and $\mathbb{L}_k(\lambda, \mathcal{H}_V)$, respectively.

Once the definitions are in place, the following is straightforward.

Proposition 5.8. *For any nonzero module $M(\lambda) \twoheadrightarrow V$,*

$$\mathbb{M}_0(\lambda, \mathcal{H}_V) = M(\lambda), \quad \mathbb{M}_1(\lambda, \mathcal{H}_V) = M(\lambda, J_V), \quad (5.7)$$

$$\mathbb{L}_0(\lambda, \mathcal{H}_V) = L(\lambda), \quad \mathbb{L}_1(\lambda, \mathcal{H}_V) = L(\lambda, J_V). \quad (5.8)$$

This explains the precise sense in which Verma modules and parabolic Verma modules are the 0th and 1st order upper-approximations, respectively, of every highest weight module.

Remark 5.9. The modules $\mathbb{M}_k, \mathbb{L}_k$ clearly refine the above chain of surjections, since the \mathcal{H} -sets in (5.5) are increasing in k , and in (5.6) are decreasing in k . That is, the \mathcal{H} -sets in all terms in

$$M(\lambda) \twoheadrightarrow M(\lambda, J_V) \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{M}(\lambda, \mathcal{H}_V) \twoheadrightarrow V \twoheadrightarrow \mathbb{L}(\lambda, \mathcal{H}_V) \twoheadrightarrow \cdots \twoheadrightarrow L(\lambda, J_V) \twoheadrightarrow L(\lambda)$$

(except the central V) increase from \emptyset at the left, to $\text{Indep}(J_\lambda) \setminus \{\emptyset\}$ at the right. Moreover, the above chain of the \mathbb{M}_k stabilizes, in that $\mathbb{M}(\lambda, \mathcal{H}_V) = \mathbb{M}_\infty(\lambda, \mathcal{H}_V) = \mathbb{M}_K(\lambda, \mathcal{H}_V)$, where K is the size of any largest (in size) independent set in J_λ . Similarly for the chain of \mathbb{L}_k .

As an **application** of these modules, we (re-)solve when the integrability of a highest weight module determines its weights. Here is one of the main results of Dhillon–Khare [16]:

Theorem 5.10. *Fix $(\mathfrak{g}$ and) a highest weight λ . The integrability $J \subseteq J_\lambda$ of a highest weight module determines its weights, if and only if the Dynkin diagram of $J_\lambda \setminus J$ is a complete graph.*

This affirmatively answers a question asked by Lepowsky [30] (to Khare) in connection with Theorem 5.5. Namely, Lepowsky asked (in the language of this paper) whether or not the holes in the weight-sets obtained from “potential integrability” as in Example 3.4, are the only obstructions to determining the set of weights. Theorem 5.10 is now transparent from our main results and higher order Verma modules – indeed, the calculation now reduces to one of set-theory:

Proof. By the construction of $L(\lambda, J)$ (in [16], or see (5.4)), it suffices to understand if $\text{wt } M(\lambda, J) = \text{wt } L(\lambda, J)$ (or from above, when $\text{wt } \mathbb{M}_1(\lambda, \mathcal{H}) = \text{wt } \mathbb{L}_1(\lambda, \mathcal{H})$). Via Theorem B and Ψ_λ^{-1} , this is if and only if the upper-closure of the singleton sets in J equals the right-hand side of (5.4), i.e.,

$$\text{Indep}(J_\lambda) \setminus \text{Indep}(J_\lambda \setminus J) = \text{Indep}(J_\lambda) \setminus (\{\emptyset\} \sqcup \{\{i\} : i \in J_\lambda \setminus J\}).$$

Taking complements, this happens if and only if $J_\lambda \setminus J$ is complete. \square

5.2. Higher order integrability, and stratification of the set of highest weight modules. The vigilant reader may have noticed that for the titular “universal” modules $\mathbb{M}_k(\lambda, \mathcal{H}), \mathbb{L}_k(\lambda, \mathcal{H})$, while Theorem C describes how they are individually universal, we have not mentioned the sense in which (for each fixed k) they are jointly so! (And indeed, the individual universalities disagree, since the upper-closed – equivalently, minimal – \mathcal{H} -sets for $\mathbb{M}_k(\lambda, \mathcal{H})$ and $\mathbb{L}_k(\lambda, \mathcal{H})$ are unequal.)

We now explain the sought-for common universality, for each $k > 0$ (for $k = 0$ it is simply the identification of the highest weight). The next result shows in particular that this universality refines the common universal property for the equal-weighted modules $\mathbb{M}(\lambda, \mathcal{H}), \mathbb{L}(\lambda, \mathcal{H})$.

Proposition 5.11 (kth order universal property). *Fix Kac–Moody \mathfrak{g} , a weight $\lambda \in \mathfrak{h}^*$, and an integer $k \geq 1$. For a subset $\mathcal{X} \subseteq \text{Indep}(J_\lambda)$, write $\mathcal{X}_{\leq k}$ for the subset $\{H \in \mathcal{X} : |H| \leq k\}$.*

- (1) *Given an upper-closed subset $\mathcal{H} \subseteq \text{Indep}(J_\lambda)$, there exist unique smallest and largest upper-closed subsets $\underline{\mathcal{H}}, \bar{\mathcal{H}}$ respectively, such that $\underline{\mathcal{H}}_{\leq k} = \mathcal{H}_{\leq k} = \bar{\mathcal{H}}_{\leq k}$. Moreover, these sets are precisely the upper-closures of the ones occurring in the definitions of the modules $\mathbb{M}_k(\lambda, \mathcal{H}), \mathbb{L}_k(\lambda, \mathcal{H})$ respectively, i.e. in Equations (5.5) and (5.6).*
- (2) *In particular, for each upper-closed subset \mathcal{H} , $\mathbb{M}_k(\lambda, \mathcal{H})$ and $\mathbb{L}_k(\lambda, \mathcal{H})$ are the unique largest and smallest highest weight modules with “kth order integrability data” $\mathcal{H}_{\leq k}$. For $|J_\lambda| \leq k \leq \infty$, this specializes to Theorem C, i.e. the common universal property for $\mathbb{M}(\lambda, \mathcal{H}), \mathbb{L}(\lambda, \mathcal{H})$.*
- (3) *Any two intervals $[\mathbb{L}_k(\lambda, \mathcal{H}), \mathbb{M}_k(\lambda, \mathcal{H})]$ and $[\mathbb{L}_k(\lambda, \mathcal{H}'), \mathbb{M}_k(\lambda, \mathcal{H}')]$ are disjoint or equal.*

Proof. Briefly: once there is a claimed formula for the sets $\underline{\mathcal{H}}_{\leq k}, \bar{\mathcal{H}}_{\leq k}$, it is easily verified. This shows (1). The first claim in (2) is now the common universal property of the pair $(\mathbb{M}_k(\lambda, \mathcal{H}), \mathbb{L}_k(\lambda, \mathcal{H}))$, and follows from Theorem C. The second claim follows from the definitions, since $\underline{\mathcal{H}}_{\leq k} = \bar{\mathcal{H}}_{\leq k} = \mathcal{H}$ if $k \geq |J_\lambda|$. Finally, (3) is immediate: if $M(\lambda) \twoheadrightarrow V$ is such that $(\mathcal{H}_V)_{\leq k} = \mathcal{H}_{\leq k} = \mathcal{H}'_{\leq k}$, then

$$\mathbb{M}_k(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}_{\leq k}) = \mathbb{M}(\lambda, \mathcal{H}'_{\leq k}) = \mathbb{M}_k(\lambda, \mathcal{H}')$$

where $\mathcal{H}, \mathcal{H}'$ are upper-closed without loss of generality. A similar proof works for the \mathbb{L}_k . \square

If the restriction of upper-closedness is removed from Proposition 5.11(1), then $\bar{\mathcal{H}}_{\leq k}$ remains unchanged. Moreover, the \mathbb{M}_k -module would remain unchanged even if $\underline{\mathcal{H}}_{\leq k}$ is reduced to the subsets of \mathcal{H}^{\min} of size $\leq k$, where \mathcal{H}^{\min} denotes the subset of \mathcal{H} of minimal (hence pairwise incomparable) elements. As a consequence of this and Proposition 5.11, we now introduce

Definition 5.12. Given \mathfrak{g}, λ , and a module $M(\lambda) \twoheadrightarrow V$, define the *kth order integrability* of V for an integer $k \geq 0$, to be $(\mathcal{H}_V^{\min})_{\leq k}$ if $k > 0$ and λ if $k = 0$.

This is precisely what is captured by the Verma module $M(\lambda)$ when $k = 0$, and by the parabolic Verma module $M(\lambda, J_V)$ when $k = 1$.

Remark 5.13. The modules $\mathbb{M}_k, \mathbb{L}_k$ serve to “stratify” the poset (under surjection) $\mathfrak{X} := \text{HW}(\mathfrak{g})$ of highest weight \mathfrak{g} -modules with all highest weights. Clearly, $U\mathfrak{g}/\sum_{i \in I}(U\mathfrak{g})e_i$ is the “infinity” element, surjecting onto all of \mathfrak{X} . At the zeroth level, $\mathfrak{X} = \bigsqcup_{\lambda \in \mathfrak{h}^*} \mathfrak{X}_\lambda$, with $\mathfrak{X}_\lambda = [L(\lambda), M(\lambda)]$ the interval of modules with highest weight λ . (Moreover, Theorem B shows passing to weights yields a finite set from each \mathfrak{X}_λ .) At the next level, the modules in \mathfrak{X}_λ are partitioned by integrability:

$$\mathfrak{X}_\lambda = \bigsqcup_{J \subseteq J_\lambda} [L(\lambda, J), M(\lambda, J)] = \bigcup_{\substack{\mathcal{H} \subseteq \text{Indep}(J_\lambda) \\ \text{upper closed}}} [\mathbb{L}_1(\lambda, \mathcal{H}), \mathbb{M}_1(\lambda, \mathcal{H})]$$

One can continue sub-stratifying each stratum, via Proposition 5.11, with the following results. (i) At each stage, one obtains a partition into intervals $[\mathbb{L}_k(\lambda, \mathcal{H}), \mathbb{M}_k(\lambda, \mathcal{H})]$. (ii) One sub-stratifies \mathfrak{X}_λ only K -many times, with K the size of any largest (in size) independent set in J_λ . (iii) At the final, innermost level, every partitioned set $[\mathbb{L}(\lambda, \mathcal{H}), \mathbb{M}(\lambda, \mathcal{H})]$ comprises highest weight modules with the same set of weights (and hence, the same apex λ , convex hull/integrability $(\mathcal{H}_V^{\min})_{\leq 1}, \dots$).

Remark 5.13 leads to the following concrete recipe. To successively better approximate the set $\text{wt } V = \text{wt } \mathbb{M}(\lambda, \mathcal{H}_V) = \text{wt } \mathbb{M}_\infty(\lambda, \mathcal{H}_V)$, one starts with $U\mathfrak{g}/\sum_{i \in I}(U\mathfrak{g})e_i$. The 0th order hole, i.e. λ , uniquely fixes the apex of the highest weight cone containing $\text{wt } V$. Next, the 1st order upper-approximation refines the dimension 1 faces, i.e. edges, of the 0th order shape $\text{conv}(\text{wt } M(\lambda))$ by truncating the semi-infinite rays at λ along the J_V directions. And so on.

Viewed representation-theoretically, the integrability refines the weights of some of the rank-1 highest weight submodules $\{U\mathfrak{g}_i \cdot M(\lambda)_\lambda : i \in J_V\}$. Similarly, the k th order approximation/integrability refines the submodules $\{U\mathfrak{g}_H \cdot M(\lambda)_\lambda : H \in (\mathcal{H}_V^{\min})_{\leq k}\}$ for *minimal* holes H of size at most k , by truncating an interior portion of the corresponding faces of $\text{conv}(\text{wt } V)$ containing λ (and more from the interior of $\text{conv}(\text{wt } V)$). This refining is transferred to other vertices / faces of $\text{conv}(\text{wt } V)$ via the Weyl group symmetries of $\text{wt } V$.

5.3. k th order weight-formulas. In the above spirit, we end this section with “ k th order weight-formulas” that specialize to Theorems A and B. First recall from the proof of Theorem A that

$$\text{wt } V = \bigcup_{J \subseteq J_\lambda : J \cap H \neq \emptyset \forall H \in \mathcal{H}_V} \text{wt } M(\lambda, J) = \text{wt } \mathbb{M}(\lambda, \mathcal{H}_V). \quad (5.9)$$

Given the above discussion in this section, the first equality shows $\text{wt } V$ to be a union of weights of many “first order Verma modules”, i.e. parabolic Vermas $M(\lambda, J) = \mathbb{M}(\lambda, \{\{j\} : j \in J\})$. The second shows $\text{wt } V$ to be the weight-set of exactly one “ ∞ -order Verma module”, i.e. $\mathbb{M}(\lambda, \mathcal{H}_V)$.

Remark 5.14. Just as a 0th order Verma module $M(\lambda)$ is also a 1st order Verma module $M(\lambda, J)$, we adopt the convention that $\mathbb{M}(\lambda, \mathcal{H})$ is a k th order Verma module (for $0 \leq k \leq \infty$) if $|H| \leq k$ for all $H \in \mathcal{H}^{\min}$. This is compatible with the k th order approximations $\mathbb{M}_k, \mathbb{L}_k$ earlier in this section.

Given these weight-formulas and remarks, it is natural to ask if there exist “intermediate” k th order formulas for each $k \geq 1$. Namely, formulas that show $\text{wt } V$ is a finite union of sets of the form $\text{wt } \mathbb{M}(\lambda, \mathcal{H}_i^{(k)})$, $i \geq 1$ where each hole in each $\mathcal{H}_i^{(k)} \subseteq \text{Indep}(J_\lambda)$ has size at most k .

Our next result provides a positive answer for each k . These are positive weight-formulas for $\text{wt } V$ that naturally interpolate from only singletons in each $\mathcal{H}_i^{(1)}$ (i.e. parabolic Vermas $\text{wt } M(\lambda, J)$ in (2.2)) to the unique set $\mathcal{H}^{(\infty)} = \mathcal{H}_V$ (i.e. $\text{wt } V = \text{wt } \mathbb{M}(\lambda, \mathcal{H}_V)$). We need the following notion.

Definition 5.15. Enumerate a subset $\mathcal{H}_0 = \{H_1, \dots, H_N\} \subseteq \text{Indep}(I)$. Given an integer $1 \leq k \leq \infty$, we say a set $\mathcal{H}^{(k)} \subseteq \text{Indep}(I)$ is \mathcal{H}_0 -admissible of order k if there are subsets $H'_t \subseteq H_t$ of size $\min(k, |H_t|)$, such that $\mathcal{H}^{(k)}$ consists of the distinct sets among H'_1, \dots, H'_N .

Example 5.16. Here are two examples where $\mathcal{H}_0 = \mathcal{H}_V$ for a highest weight \mathfrak{g} -module $M(\lambda) \twoheadrightarrow V$. First, if $k = 1$ then an order 1 \mathcal{H}_V -admissible set $\mathcal{H}^{(1)}$ consists of singleton sets $\{j\}$ for $j \in J_\lambda$, such that every hole in \mathcal{H}_V contains one or more of these j , and every j is in at least one $H \in \mathcal{H}_V$. The universal module for this first order hole-set is indeed the “first order” parabolic Verma module: $\mathbb{M}(\lambda, \mathcal{H}^{(1)}) = M(\lambda, J)$, where $J = \{j : \{j\} \in \mathcal{H}^{(1)}\}$.

The second example is when $k = \infty$ (or $k \gg 0$). Then there is only one \mathcal{H}_V -admissible set of order k : $\mathcal{H}^{(\infty)} = \mathcal{H}_V$ itself. The corresponding higher order Verma module is $\mathbb{M}(\lambda, \mathcal{H}_V)$.

Proposition 5.17. *Given a \mathfrak{g} -module $M(\lambda) \twoheadrightarrow V$ (for arbitrary \mathfrak{g}, λ), and an integer $1 \leq k \leq \infty$,*

$$\text{wt } V = \bigcup_{\mathcal{H}^{(k)}} \text{wt } \mathbb{M}(\lambda, \mathcal{H}^{(k)}), \quad (5.10)$$

where the union runs over all \mathcal{H}_V -admissible sets of order k .

Notice that the weight-formulas (5.9) in (the proof of) Theorem A are the $k = 1, \infty$ special cases of Proposition 5.17, in light of Example 5.16.

Proof. Denote by $\Psi(k)$ the right-hand side of (5.10), which the result claims is independent of k :

$$\Psi(k) := \bigcup_{\mathcal{H}^{(k)}} \text{wt } \mathbb{M}(\lambda, \mathcal{H}^{(k)}), \quad 1 \leq k \leq \infty. \quad (5.11)$$

We now claim the inclusions $\Psi(1) \subseteq \Psi(2) \subseteq \dots \subseteq \Psi(\infty)$. The result then follows from (5.9) (shown while proving Theorem A), which says: $\Psi(\infty) = \text{wt } V = \Psi(1)$. To be precise, (5.9) does not explicitly say $\text{wt } V = \Psi(1)$, but one then notes that defining $\text{supp } \mathcal{H}_V := \bigcup_{H \in \mathcal{H}_V} H$, the set of J in the union in (5.9) may be reduced to only the $J \subseteq \text{supp } \mathcal{H}_V$ – equivalently, every J may be replaced by $J \cap \text{supp } \mathcal{H}_V$, since $\text{wt } M(\lambda, J) \subseteq \text{wt } M(\lambda, J \cap \text{supp } \mathcal{H}_V)$. From this one checks: $\text{wt } V = \Psi(1)$.

To show the claim, fix $k' < k''$ in $[1, \infty]$, and list all order k' and order k'' \mathcal{H}_V -admissible sets as

$$\mathcal{H}_1^{(k')}, \dots, \mathcal{H}_{m'}^{(k')}; \quad \text{respectively,} \quad \mathcal{H}_1^{(k'')}, \dots, \mathcal{H}_{m''}^{(k'')}.$$

Also list $\mathcal{H}^{(\infty)} = \mathcal{H}_V = \{H_1, \dots, H_N\}$. Now given $i' \in [1, m']$, use Definition 5.15 to write the elements of $\mathcal{H}_{i'}^{(k')}$ as a multiset, $\mathcal{H}_{i'}^{(k')} \longleftrightarrow H'_1, \dots, H'_N$ with $H'_t \subseteq H_t \forall t \in [1, N]$. Now arbitrarily choose H''_t of size $\min(k'', |H_t|)$ such that $H'_t \subseteq H''_t \subseteq H_t \forall t \in [1, N]$. Then the distinct sets among the H''_t comprise $\mathcal{H}_{\varphi(i')}^{(k'')}$ for some function $\varphi : [1, m'] \rightarrow [1, m'']$. This implies

$$\mathbb{M}(\lambda, \mathcal{H}_{\varphi(i')}^{(k'')}) \twoheadrightarrow \mathbb{M}(\lambda, \mathcal{H}_{i'}^{(k')}) \quad \Rightarrow \quad \text{wt } \mathbb{M}(\lambda, \mathcal{H}_{i'}^{(k')}) \subseteq \text{wt } \mathbb{M}(\lambda, \mathcal{H}_{\varphi(i')}^{(k'')}).$$

As this holds for all $1 \leq k' < k'' \leq \infty$ and $i' \in [1, m']$,

$$\Psi(k') = \bigcup_{i'=1}^{m'} \text{wt } \mathbb{M}(\lambda, \mathcal{H}_{i'}^{(k')}) \subseteq \bigcup_{i'=1}^{m'} \text{wt } \mathbb{M}(\lambda, \mathcal{H}_{\varphi(i')}^{(k'')}) \subseteq \bigcup_{i''=1}^{m''} \text{wt } \mathbb{M}(\lambda, \mathcal{H}_{i''}^{(k'')}) = \Psi(k'').$$

This shows the claim, and completes the proof. \square

A consequence of Proposition 5.17 is a family of weight-formulas that generalizes Theorem B. We begin with the extreme cases, which were shown in (4.2):

$$\text{wt } V = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \text{wt } \mathbb{M}(0, \mathcal{H}_V) = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \bigcup_{J \subseteq J_\lambda : J \cap H \neq \emptyset \forall H \in \mathcal{H}_V} \text{wt } M(0, J). \quad (5.12)$$

These turn out to be the $k = \infty$ and $k = 1$ cases of the following result:

Corollary 5.18. *Given a \mathfrak{g} -module $M(\lambda) \twoheadrightarrow V$ (for arbitrary \mathfrak{g}, λ), and an integer $1 \leq k \leq \infty$,*

$$\text{wt } V = \text{wt } L_{J_\lambda}^{\max}(\lambda) + \bigcup_{\mathcal{H}^{(k)}} \text{wt } \mathbb{M}(0, \mathcal{H}^{(k)}), \quad (5.13)$$

where the union runs over all \mathcal{H}_V -admissible sets of order k .

This follows from (5.12) and Proposition 5.17, using $(\lambda, V) \rightsquigarrow (0, \mathbb{M}(0, \mathcal{H}_V))$ (via e.g. Lemma 3.12).

6. THEOREM D: HIGHER ORDER PARABOLIC CATEGORY \mathcal{O} , ENOUGH PROJECTIVES, AND BGG RECIPROCITY

We next discuss refinements of the BGG Category \mathcal{O} [6]. In keeping with the perspective of higher order approximations and integrability, we begin by noting how the usual Category \mathcal{O} and the parabolic Category $\mathcal{O}^{\mathfrak{p}_J}$ are zeroth and first order special cases of the categories

$$\mathcal{O}^{\mathcal{H}}, \quad \mathcal{H} \subseteq \text{Indep}(I)$$

which we introduced in Definition 2.10: the objects in \mathcal{O} on which all \mathbf{f}_H , $H \in \mathcal{H}$ act locally nilpotently. As mentioned there, **in this section \mathfrak{g} is assumed to be of finite type**.

Indeed, Definition 2.10 of $\mathcal{O}^{\mathcal{H}}$ specializes to the usual and parabolic categories as follows:

- (1) If \mathcal{H} is empty, then \mathcal{O}^{\emptyset} is just Category \mathcal{O} , and contains all Verma modules $\mathbb{M}(\lambda, \emptyset) = M(\lambda)$.
- (2) More generally, if $\mathcal{H}_J = \{\{j\} : j \in J\}$ for $J \subseteq I$, then (see e.g. [20, Section 9.3] and use that \mathfrak{g} is semisimple, to show that) $\mathcal{O}^{\mathcal{H}_J}$ is precisely the parabolic Category $\mathcal{O}^{\mathfrak{p}_J}$, and it contains the parabolic Verma modules $M(\lambda, J) = \mathbb{M}(\lambda, \mathcal{H}_J)$ for all J -dominant integral weights λ .

To the best of our knowledge, these subcategories of \mathcal{O} have not been studied beyond the parabolic categories $\mathcal{O}^{\mathfrak{p}_J}$. Yet, they naturally generalize $\mathcal{O}^{\mathfrak{p}_J}$, and are intimately linked with higher order Verma modules – having higher order holes/integrability. This section initiates their study.

6.1. Enough projectives. We show the first half of Theorem D – $\mathcal{O}^{\mathcal{H}}$ has enough projectives – following several intermediate results. In this section and the next, Remark 3.2(2) is useful: we will often work not with \mathcal{H} , but instead with \mathcal{H}^{\min} , the collection of minimal holes in \mathcal{H} , which form a pairwise incomparable collection. Thus, e.g. the final assertion in Theorem B says that the finite collection of “sets of incomparable subsets of $\text{Indep}(J_\lambda)$ ” is in bijection with $\{\text{wt } V : M(\lambda) \twoheadrightarrow V\}$.

Coming to properties of $\mathcal{O}^{\mathcal{H}}$, a first “sanity check” is that Definition 2.10 of $\mathcal{O}^{\mathcal{H}}$ fits well with minimal elements and upper-closures, just like the modules $\mathbb{M}(\lambda, \mathcal{H})$ do (see Remark 3.2):

Lemma 6.1. *If $\mathcal{H} \subseteq \text{Indep}(I)$, $\mathcal{O}^{\mathcal{H}} = \mathcal{O}^{\mathcal{H}^{\min}} = \mathcal{O}^{\overline{\mathcal{H}}}$, where $\overline{\mathcal{H}}$ is the upper-closure of \mathcal{H} in $\text{Indep}(I)$.*

Proof. Here is a proof for completeness. From the definitions, $\mathcal{O}^{\overline{\mathcal{H}}} \subseteq \mathcal{O}^{\mathcal{H}} \subseteq \mathcal{O}^{\mathcal{H}^{\min}}$. Now it suffices to show the reverse inclusion to the second one, since $(\overline{\mathcal{H}})^{\min} = \mathcal{H}^{\min}$. Let $M \in \mathcal{O}^{\mathcal{H}^{\min}}$, $H \in \mathcal{H}$, and let v be a weight vector in M . It suffices to show that \mathbf{f}_H acts nilpotently on v . Choose a minimal hole $H_0 \subseteq H$ with $H_0 \in \mathcal{H}^{\min}$; now $\mathbf{f}_{H_0}^n v = 0$ for some n . But then $\mathbf{f}_H^n v = 0$. \square

Next, these generalizations of the parabolic category $\mathcal{O}^{\mathfrak{p}_J}$ share the same basic properties as it:

Lemma 6.2. *Fix a subset $\mathcal{H} \subseteq \text{Indep}(I)$. Then $\mathcal{O}^{\mathcal{H}}$ is an abelian subcategory of $\mathcal{O} \subseteq \mathfrak{g}\text{-Mod}$ that is closed under: taking submodules, quotients, finite direct sums, extensions in \mathcal{O} , restricted duals, and tensor products with finite dimensional \mathfrak{g} -modules. In particular, if $M \in \mathcal{O}^{\mathcal{H}}$ splits in \mathcal{O} according to the action of the center $Z(U\mathfrak{g})$ into $M = \bigoplus_\chi M^\chi$, then each M^χ also lies in $\mathcal{O}^{\mathcal{H}}$.*

Proof. We only outline the proof for restricted duals $M \mapsto M^\vee$ and tensoring $M \mapsto M \otimes L(\lambda)$ for λ dominant integral, assuming the closure of $\mathcal{O}^{\mathcal{H}}$ under the other operations is shown.

First, duals. Since \mathfrak{g} is of finite type, every module $M \in \mathcal{O}^{\mathcal{H}}$ has finite length, and via the other operations listed, every simple subquotient is in $\mathcal{O}^{\mathcal{H}}$. Construct $M^\vee \in \mathcal{O}$ by dualizing a Jordan–Hölder series for M ; every simple factor is $L(\mu)^\vee \cong L(\mu)$. Now if \mathbf{f}_H acts locally finitely on each such factor, then it does so on extensions in \mathcal{O} between them, and hence on M^\vee as desired.

Next, tensoring. Given weights $\mu, \nu \in \mathfrak{h}^*$ and nonzero weight vectors $m_\mu, v_\nu \in M_\mu, v_\nu \in L(\lambda)_\nu$, it suffices to show that \mathbf{f}_H acts nilpotently on $m_\mu \otimes v_\nu$ for every hole $H \in \mathcal{H}$. By definition, there exists $K > 0$ such that $\mathbf{f}_H^K m_\mu = 0$; similarly, there exists $N > 0$ such that $f_h^N v_\nu = 0 \forall h \in H$. Now,

$$\mathbf{f}_H^n(m_\mu \otimes v_\nu) = \sum_{k_h \in [0, n] \forall h \in H} \prod_{h \in H} \binom{n}{k_h} \cdot \left(\prod_{h \in H} f_h^{k_h} \cdot m_\mu \otimes \prod_{h \in H} f_h^{n-k_h} \cdot v_\nu \right), \quad \forall n \geq 0.$$

Hence, every term on the right vanishes if $n \geq K + N - 1$. \square

We next identify the highest weight modules, simple objects and their universal covers in $\mathcal{O}^{\mathcal{H}}$.

Definition 6.3. Given $\lambda \in \mathfrak{h}^*$ and $\mathcal{H} \subseteq \text{Indep}(I)$, extend Definition 3.1 and define \mathcal{H}'_λ as follows:

$$\mathbb{M}(\lambda, \mathcal{H}) := \frac{M(\lambda)}{\sum_{H \in \mathcal{H}^{\min}} U\mathfrak{g} \left(\prod_{h \in J_\lambda \cap H} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1} \right) M(\lambda)_\lambda}, \quad \mathcal{H}'_\lambda := \{J_\lambda \cap H : H \in \mathcal{H}^{\min}\}. \quad (6.1)$$

Thus $\mathbb{M}(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}'_\lambda)$ for all λ and \mathcal{H} ; we use this fact below without reference. In particular, if J_λ does not intersect some $H \in \mathcal{H}^{\min}$, then $\mathbb{M}(\lambda, \mathcal{H}) = 0$.

Proposition 6.4. Fix a nonempty subset $\mathcal{H} \subseteq \text{Indep}(I)$ and a weight $\lambda \in \mathfrak{h}^*$.

- (1) The module $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$ if and only if $J_\lambda \cap H \neq \emptyset \forall H \in \mathcal{H}^{\min}$, if and only if $\mathbb{M}(\lambda, \mathcal{H}) \neq 0$.
- (2) In this case, the universal highest weight cover in $\mathcal{O}^{\mathcal{H}}$ of $L(\lambda)$ is $\mathbb{M}(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}'_\lambda)$.
- (3) Suppose $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$. A highest weight module $M(\lambda) \rightarrowtail V$ belongs to $\mathcal{O}^{\mathcal{H}}$ if and only if $\mathcal{H}'_\lambda \subseteq \mathcal{H}_V$, if and only if $\mathbb{M}(\lambda, \mathcal{H}_V) \in \mathcal{O}^{\mathcal{H}}$.

Note that the condition in part (1) is reminiscent of the set $\mathfrak{J}(V)$ used in proving Theorem A. This is made precise in Equation (6.4).

As a special case, recall that the simples in $\mathcal{O}^{\mathfrak{p}_J}$ are $L(\lambda)$ for λ J -dominant integral; and their universal covers are the parabolic Verma modules $M(\lambda, J)$. This follows from Proposition 6.4: set

$$\mathcal{H} = \mathcal{H}_J = \{\{j\} : j \in J\} = \mathcal{H}^{\min},$$

in which case $\mathcal{H}'_\lambda = \mathcal{H}_J$ as well, and so $\mathbb{M}(\lambda, \mathcal{H}'_\lambda) = M(\lambda, J)$.

Definition 6.5. Given a subset $\mathcal{H} \subseteq \text{Indep}(I)$, and a weight λ such that $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$, define its *universal cover* in $\mathcal{O}^{\mathcal{H}}$ to be $\mathbb{M}(\lambda, \mathcal{H}'_\lambda)$ (see Proposition 6.4(2)). Also define *standard objects* to be all modules $\mathbb{M}(\lambda, \mathcal{H}_0) \in \mathcal{O}^{\mathcal{H}}$.

The proof of Proposition 6.4 requires one last lemma, in addition to the two above:

Lemma 6.6. Suppose $\lambda \in \mathfrak{h}^*$, $H \in \text{Indep}(J_\lambda)$, and $M(\lambda) \rightarrowtail V$ is a nonzero highest weight module. If \mathbf{f}_H is nilpotent on the highest weight line V_λ , then $H \in \mathcal{H}_V$.

This result and Proposition 6.4 repeatedly use the following fact, as useful in each $\mathcal{O}^{\mathcal{H}}$ as it was in \mathcal{O} . Namely, if v_λ is a maximal vector for a raising operator e_h , with weight λ , then

$$e_h^n \cdot f_h^n v_\lambda \in \mathbb{C}^\times v_\lambda, \quad \text{whenever } \langle \lambda, \alpha_h^\vee \rangle \notin \mathbb{Z}_{\geq 0} \ni n \text{ or } 0 \leq n \leq \langle \lambda, \alpha_h^\vee \rangle \in \mathbb{Z}. \quad (6.2)$$

Proof. Let $n > 0$ denote the smallest power such that $\mathbf{f}_H^n \cdot v_\lambda = 0$, where we fix a nonzero highest weight vector $v_\lambda \in V_\lambda$. Also define $m_h := \langle \lambda, \alpha_h^\vee \rangle + 1 \in \mathbb{Z}_{>0}$ for $h \in J_\lambda$. Now if $n < m_h \forall h \in H$, then applying e_h^n for all h to the equation $\mathbf{f}_H^n v_\lambda = 0$ yields $v_\lambda = 0$, which is false. Thus $H_1 := \{h \in H : n \geq m_h\}$ is nonempty. Applying $\prod_{h \in H_1} e_h^{n-m_h} \prod_{h \in H \setminus H_1} e_h^n$ to $\mathbf{f}_H^n V_\lambda = 0$ – via (6.2) – yields $\prod_{h \in H_1} f_h^{m_h} \cdot V_\lambda = 0$. Thus H_1 lies in the upper-closed set \mathcal{H}_V , hence so does H . \square

Proof of Proposition 6.4.

(1) First suppose J_λ intersects every hole in \mathcal{H}^{\min} . Then $\mathbf{f}_H = \mathbf{f}_{J_\lambda \cap H} \mathbf{f}_{H \setminus J_\lambda} \forall H \in \mathcal{H}$, where the two factors on the right commute and $\mathbf{f}_{J_\lambda \cap H}$ acts nilpotently on the highest weight line $L(\lambda)_\lambda$. Hence it acts nilpotently on other vectors in $L(\lambda)$ as well, using arguments similar to the proof of Lemma 3.17. But then so does \mathbf{f}_H . Hence $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$.

Conversely, suppose $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$, and say there exists $H \in \mathcal{H}^{\min}$ which is disjoint from J_λ . If $\mathbf{f}_H^n \cdot L(\lambda)_\lambda = 0$, then applying $\prod_{h \in H} e_h^n$ via (6.2) yields: $L(\lambda)_\lambda = 0$, a contradiction.

(2) We claim the upper-closure of \mathcal{H}'_λ – defined in (6.1) – is the smallest upper-closed subset $\mathcal{H}_0 \subseteq \text{Indep}(I)$ such that $\mathbb{M}(\lambda, \mathcal{H}_0) \in \mathcal{O}^{\mathcal{H}}$. To see why, first fix a nonzero highest weight vector $v_\lambda \in \mathbb{M}(\lambda, \mathcal{H}'_\lambda)_\lambda$. For $H \in \mathcal{H}^{\min}$, write $\mathbf{f}_H = \mathbf{f}_{J_\lambda \cap H} \mathbf{f}_{H \setminus J_\lambda}$ as above. Then $J_\lambda \cap H \in \mathcal{H}'_\lambda$, so \mathbf{f}_H acts nilpotently on v_λ , hence acts locally nilpotently on $\mathbb{M}(\lambda, \mathcal{H}'_\lambda)$ – for all $H \in \mathcal{H}^{\min}$. Thus, $\mathbb{M}(\lambda, \mathcal{H}'_\lambda) \in \mathcal{O}^{\mathcal{H}^{\min}} = \mathcal{O}^{\mathcal{H}}$ (by Lemma 6.1).

Now suppose $\mathbb{M}(\lambda, \mathcal{H}_0) \in \mathcal{O}^{\mathcal{H}}$, and assume henceforth that \mathcal{H}_0 is upper-closed. We claim that $\mathcal{H}'_\lambda \subseteq \mathcal{H}_0$. To see why, fix a nonzero highest weight vector $v_\lambda \in \mathbb{M}(\lambda, \mathcal{H}_0)_\lambda$, and let $H \in \mathcal{H}^{\min}$. Then $\mathbf{f}_H = \mathbf{f}_{J_\lambda \cap H} \mathbf{f}_{H \setminus J_\lambda}$ acts nilpotently on v_λ . Say its n th power annihilates v_λ . Applying $\prod_{h \in H \setminus J_\lambda} e_h^n$ as above via (6.2), $\mathbf{f}_{J_\lambda \cap H}$ acts nilpotently on v_λ . Now applying Lemma 6.6 with $J_\lambda \cap H$ in place of H shows that $J_\lambda \cap H \in \mathcal{H}_0$. As this holds for all $H \in \mathcal{H}^{\min}$, the desired conclusion follows: $\mathcal{H}'_\lambda \subseteq \mathcal{H}_0$.

(3) If $\mathcal{H}'_\lambda \subseteq \mathcal{H}_V$ then by the definitions, $\mathbb{M}(\lambda, \mathcal{H}'_\lambda) \rightarrowtail \mathbb{M}(\lambda, \mathcal{H}_V) \rightarrowtail V$, and the first of these lies in $\mathcal{O}^{\mathcal{H}}$, which is closed under quotienting. Conversely, say $V \in \mathcal{O}^{\mathcal{H}}$, and $H \in \mathcal{H}^{\min}$ (so $J_\lambda \cap H \neq \emptyset$ by part (1)). Now \mathbf{f}_H is nilpotent on the highest weight line V_λ , hence so is $\mathbf{f}_{J_\lambda \cap H}$ by (6.2). By Lemma 6.6, $J_\lambda \cap H \in \mathcal{H}_V$ for all $H \in \mathcal{H}^{\min}$, which finishes the proof. \square

With these results now shown, we begin proving our next main theorem, on $\mathcal{O}^{\mathcal{H}}$.

Proof of Theorem D, first part. The third assertion – involving BGG reciprocity – is shown in Theorem 6.17. Here we prove the rest, starting by showing that $\mathcal{O}^{\mathcal{H}}$ has enough projectives. Recall that the BGG Category \mathcal{O} decomposes as a direct sum over twisted W -orbits: $\mathcal{O} = \bigoplus \mathcal{O}^{W \bullet \lambda}$, using central characters and Harish-Chandra's theorem. Hence so does the subcategory $\mathcal{O}^{\mathcal{H}} \subseteq \mathcal{O}$, via Lemma 6.2. It suffices to work in one such intersection

$$\mathcal{A} := \mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}, \quad (6.3)$$

where we fix $\lambda \in \mathfrak{h}^*$ satisfying: $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$. Indeed, if we show there exist enough projectives P in each such category, and run over all dot-orbits $W \bullet \lambda$, then all such P are in fact projectives in $\mathcal{O}^{\mathcal{H}}$.

Now one shows – using Proposition 6.4 – that \mathcal{A} has enough projectives. This is via the sufficient criterion in [4, Theorem 3.2.1], wherein one verifies five conditions (not six, by Ringel's subsequent remark in [4]). As the verification is mostly standard, it is deferred to Appendix A – we do include it because for some simples, there are multiple standard objects that get used in BGG reciprocity – and because the proof of BGG reciprocity also uses similar arguments, see Section 6.4.

This proves the first assertion; we now turn to the second. We claim the following equalities for every highest weight module $M(\lambda) \rightarrowtail V$:

$$\text{wt } \mathbb{M}(\lambda, (\mathcal{H}_V)'_\lambda) = \text{wt } V = \bigcup_{K \subseteq J_\lambda : L(w_{J_\lambda} \setminus K \bullet \lambda) \in \mathcal{O}^{\mathcal{H}_V}} \text{wt } M(\lambda, K), \quad (6.4)$$

where w_J is the longest element of W_J for any $J \subseteq I$. Note that the second equality yields an alternate weight-formula to Theorem A.

We begin with the first equality, which proves the second assertion in the theorem via Proposition 6.4(2). This equality follows because as shown above, $\text{wt } V = \text{wt } \mathbb{M}(\lambda, \mathcal{H}_V)$; now apply Proposition 6.4(3) using that $V \in \mathcal{O}^{\mathcal{H}_V}$.

This completes the proof of Theorem D(2). We conclude with the proof of the second equality in (6.4); this follows from the claim that

$$\mathfrak{J}(V) = \{K \subseteq J_\lambda \mid L(w_{J_\lambda \setminus K} \bullet \lambda) \in \mathcal{O}^{\mathcal{H}_V}\},$$

where $\mathfrak{J}(V)$ is as in (3.8). To see the claim, first note that $H \subseteq J_\lambda$ if $H \in \mathcal{H}_V^{\min}$. By this and Proposition 6.4(1), $L(w_{J_\lambda \setminus K} \bullet \lambda) \in \mathcal{O}^{\mathcal{H}_V}$ if and only if $J_{w_{J_\lambda \setminus K} \bullet \lambda} \cap (J_\lambda \cap H) \neq \emptyset$ for all $H \in \mathcal{H}_V^{\min}$. Now by (3.8), it suffices to show for $K \subseteq J_\lambda$ that $J_{w_{J_\lambda \setminus K} \bullet \lambda} \cap J_\lambda = K$. One inclusion is because λ is $J_\lambda \setminus K$ -dominant integral, so $-w_{J_\lambda \setminus K} \bullet \lambda$ is strictly dominant integral for $J_\lambda \setminus K$. The reverse inclusion follows from writing $w_{J_\lambda \setminus K} \bullet \lambda = \lambda - \sum_{j \in J_\lambda \setminus K} l'_j \alpha_j$ for some $l'_j \in \mathbb{Z}_{\geq 0}$, and evaluating against $\langle -, \alpha_k^\vee \rangle$ for $k \in K$. \square

We conclude with a well known consequence of standard facts on finite length abelian categories \mathcal{A} with finitely many simple objects and enough projectives.

Corollary 6.7. *Every simple object $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$ has a projective cover $P^{\mathcal{H}}(\lambda) \in \mathcal{O}^{\mathcal{H}}$, and so*

$$P^{\mathcal{H}}(\lambda) \twoheadrightarrow \mathbb{M}(\lambda, \mathcal{H}'_\lambda) \twoheadrightarrow L(\lambda).$$

Moreover, for all objects $M \in \mathcal{O}^{\mathcal{H}}$, one has $\dim \text{Hom}_{\mathcal{O}^{\mathcal{H}}}(P^{\mathcal{H}}(\lambda), M) = [M : L(\lambda)]$, the number of Jordan–Hölder factors of M isomorphic to $L(\lambda)$.

6.2. Properties of standard filtrations. Having proved that the categories $\mathcal{O}^{\mathcal{H}}$ all have enough projectives, it is natural to ask if these projectives have “standard filtrations”; and if they do, then does BGG reciprocity hold in some form. We begin by defining the former notion.

Definition 6.8. Fix a subset $\mathcal{H} \subseteq \text{Indep}(I)$. An object $M \in \mathcal{O}^{\mathcal{H}}$ is said to have a *standard filtration* in $\mathcal{O}^{\mathcal{H}}$ if there exists a subcategory $\mathcal{O}^{\mathcal{H}'}$ of $\mathcal{O}^{\mathcal{H}}$ that contains M , and a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M,$$

with each subquotient $M_i/M_{i-1} \cong \mathbb{M}(\lambda_i, \mathcal{H}') = \mathbb{M}(\lambda_i, \mathcal{H}'_{\lambda_i})$ for some $\lambda_i \in \mathfrak{h}^*$ (by Proposition 6.4(2)).

Remark 6.9. It is also possible to define a weaker notion: an object $M \in \mathcal{O}^{\mathcal{H}}$ has a *weakly standard filtration* in $\mathcal{O}^{\mathcal{H}}$ if there exists a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$, with each subquotient $M_i/M_{i-1} \cong \mathbb{M}(\lambda_i, \mathcal{H}_i)$ for some $\lambda_i \in \mathfrak{h}^*$, $\mathcal{H}_i \subseteq \text{Indep}(J_{\lambda_i})$. However, we work with the above, stronger notion – which we prove holds for all projectives in $\mathcal{O}^{\mathcal{H}}$, over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ below.

The result in this part that will be useful in showing BGG reciprocity, is the natural one:

Proposition 6.10. *Suppose \mathfrak{g} is semisimple and $\mathcal{H} \subseteq \text{Indep}(I)$. Given objects $M', M'' \in \mathcal{O}^{\mathcal{H}}$, their direct sum $M' \oplus M''$ has a standard filtration in $\mathcal{O}^{\mathcal{H}}$, if and only if both M' and M'' do.*

That said, this result, and indeed the treatment of category $\mathcal{O}^{\mathcal{H}}$ for general \mathcal{H} , differ from its “zeroth order” and “first order” (parabolic) special cases in the literature, in that now one is no longer working with $U\mathfrak{n}'$ -free modules (for a nonzero Lie subalgebra $\mathfrak{n}' \subseteq \mathfrak{n}^-$) if \mathcal{H} is more general. In particular, “standard objects” $\mathbb{M}(\lambda, \mathcal{H})$ are not always obtained by induction from $U\mathfrak{n}'$ to $U\mathfrak{n}^-$. E.g. over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus 2}$ (see (1.8)), the universal cover

$$V_{00} = M(0, 0)/M(-2, -2) = \mathbb{M}((0, 0), \{(1, 2)\}) \cong \mathbb{C}[f_1, f_2]/(f_1 f_2).$$

This is due to non-singleton sets in \mathcal{H} , and it makes the proofs in this section diverge from the literature – including for the next lemma.

Lemma 6.11. *Suppose $\mathcal{H} \subseteq \text{Indep}(I)$ is such that $0 \rightarrow N \rightarrow M \rightarrow \mathbb{M}(\lambda, \mathcal{H}) \rightarrow 0$ is a short exact sequence in the category $\mathcal{O}^{\mathcal{H}}$. If λ is maximal in $\text{wt } M$, then the extension M splits.*

Proof. Note that $M_\lambda \neq 0$, so $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$, so $\mathbb{M}(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}'_\lambda) \neq 0$ by Proposition 6.4. Pick a weight vector $0 \neq v_\lambda \in \mathbb{M}(\lambda, \mathcal{H})_\lambda$ and its preimage $m_\lambda \in M_\lambda$. Then $V := U\mathfrak{g} \cdot m_\lambda \in \mathcal{O}^{\mathcal{H}}$, and $M(\lambda) \rightarrow V$ because $\mathfrak{n}^+ m_\lambda = 0$ by assumption. We claim that $M = V \oplus N$. Indeed, by Proposition 6.4(3), $\mathcal{H}'_\lambda \subseteq \mathcal{H}_V$, which gives a sequence of surjections whose composite is an isomorphism:

$$\mathbb{M}(\lambda, \mathcal{H}'_\lambda) \rightarrow \mathbb{M}(\lambda, \mathcal{H}_V) \rightarrow V \rightarrow \mathbb{M}(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}'_\lambda).$$

Thus the final map is an isomorphism, so $V \cap N = 0$, yielding the desired splitting. \square

Lemma 6.12. *Suppose $\mathcal{H} \subseteq \text{Indep}(I)$, and $M \in \mathcal{O}^{\mathcal{H}}$ has a standard filtration in $\mathcal{O}^{\mathcal{H}}$. If λ is maximal in $\text{wt } M$, then there exists a submodule M' of M satisfying: (i) $M' \cong \mathbb{M}(\lambda, \mathcal{H})$, and (ii) M/M' has a standard filtration in $\mathcal{O}^{\mathcal{H}}$.*

Lemma 6.12 is proved using Lemma 6.11 and similar arguments to the classical case of \mathcal{O} . In turn, it implies Proposition 6.10. The proofs are similar to e.g. those in [20, Section 3.7].

6.3. BGG reciprocity – subtlety in the higher order case, over all \mathfrak{g} of rank ≥ 3 . With the above machinery and results at hand, we turn to the remainder of Theorem D – i.e., BGG reciprocity (6.6) in all categories $\mathcal{O}^{\mathcal{H}}$ over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$. That BGG reciprocity holds in the zeroth/first order cases (i.e., the usual/parabolic categories \mathcal{O}) is well known, see [6], [32], [20, Chapters 3, 9].

Before working over $\mathfrak{sl}_2^{\oplus n}$, we first show that the situation in $\mathcal{O}^{\mathcal{H}}$ has a subtlety when \mathcal{H} has higher order holes – over *any* \mathfrak{g} of rank at least 3. This is because *multiple* “standard objects” $V = \mathbb{M}(\lambda, \mathcal{H}_0)$ exist over a given $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$ (for certain \mathcal{H}) – recall, these were classified in Proposition 6.4(3). It turns out that standard filtrations for different projectives in a block can feature more than one such standard object $\mathbb{M}(\lambda, \mathcal{H}_0)$, but only one of these is the universal cover $\mathbb{M}(\lambda, \mathcal{H}'_\lambda)$ (see Proposition 6.4 and Definition 6.5). This is already a break from the parabolic case [32, Theorem 6.1]. We begin by illustrating this in an even simpler case – in rank two. The key object is again $V_{00} = M(0, 0)/M(-2, -2)$ from (1.8), and its generalization $\mathbb{M}(\lambda, \{\{1, 2\}\})$.

Example 6.13 ($\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$). We present the complete picture over this algebra \mathfrak{g} , to provide familiarity before tackling the case of $\mathfrak{sl}_2^{\oplus n}$ for general n . By Lemma 6.1, one needs to consider the subcategories $\mathcal{O}^{\mathcal{H}}$, with \mathcal{H} from among the following five upper-closed subsets of $\text{Indep}(I) = 2^I$:

$$\mathcal{H} = \emptyset, \quad \{\{1\}, \{1, 2\}\}, \quad \{\{2\}, \{1, 2\}\}, \quad \{\{1\}, \{2\}, \{1, 2\}\}, \quad \{\{1, 2\}\}.$$

The first case is that of the usual category \mathcal{O} , and the next three cases are of its parabolic subcategories. These were addressed in [6] and [32], respectively.

Thus, henceforth fix $\mathcal{H} = \{\{1, 2\}\} = \mathcal{H}^{\min}$. If $J_\lambda = \{1\}$ and $\langle \lambda, \alpha_2^\vee \rangle + 1$ is either zero or a non-integer, then the linkage class is $\{\lambda > s_1 \bullet \lambda\}$, and $L(s_1 \bullet \lambda) \notin \mathcal{O}^{\mathcal{H}}$, so the block of $\mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}$ containing $L(\lambda)$ has only one simple object – which is also parabolic Verma and projective in that block. (The analogous story for $J_\lambda = \{2\}$ and $\langle \lambda, \alpha_1^\vee \rangle + 1$ as above, also holds.)

The only remaining case is when $\mathcal{H} = \{\{1, 2\}\} = \mathcal{H}^{\min}$ and λ lies in a block with a dominant integral element – which we can set to be λ . Then the block $\mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}$ has three simples: $L(\lambda)$, $L(s_1 \bullet \lambda)$, $L(s_2 \bullet \lambda)$. Their universal covers in $\mathcal{O}^{\mathcal{H}}$ are, respectively:

$$\mathbb{M}(\lambda, \{\{1, 2\}\}) = \frac{M(\lambda)}{M(s_1 s_2 \bullet \lambda)} \rightarrow L(\lambda), \quad \mathbb{M}(s_1 \bullet \lambda, \{\{2\}\}) = L(s_1 \bullet \lambda), \quad \mathbb{M}(s_2 \bullet \lambda, \{\{1\}\}) = L(s_2 \bullet \lambda).$$

In particular, there is a unique standard object of highest weight $s_i \bullet \lambda$ for $i = 1, 2$. However, there are four standard objects in $\mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}$ of highest weight λ : $\mathbb{M}(\lambda, \{J\})$ for $\emptyset \neq J \subseteq \{1, 2\}$, and $L(\lambda) \cong \mathbb{M}(\lambda, \{\{1, 2\}\})$. Moreover, the projective cover of the “highest” simple $L(\lambda)$ is its “Verma cover” $\mathbb{M}(\lambda, \{\{1, 2\}\})$ – which has length 3 – while those of the other two simple modules turn out to be their projective covers in smaller categories – in fact, in parabolic categories $\mathcal{O}^{\mathfrak{p}_J} = \mathcal{O}^{\mathcal{H}_J}$:

$$\begin{aligned} 0 \rightarrow M(\lambda, \{1\}) \rightarrow P^{\mathcal{H}_{\{1\}}}(s_2 \bullet \lambda) \rightarrow M(s_2 \bullet \lambda, \{1\}) \rightarrow 0 \\ 0 \rightarrow M(\lambda, \{2\}) \rightarrow P^{\mathcal{H}_{\{2\}}}(s_1 \bullet \lambda) \rightarrow M(s_1 \bullet \lambda, \{2\}) \rightarrow 0 \end{aligned} \tag{6.5}$$

(this follows from Theorem 6.17 below). Now one can easily verify “BGG-type” reciprocity in $\mathcal{O}^{\mathcal{H}}$.

Remark 6.14. Thus, BGG reciprocity is more subtle even in rank 2, when $\mathcal{H} = \{\{1, 2\}\}$. Specifically, using multiple standard objects for some of the weights in the block is now necessary, as can already be seen here via our original motivating example (1.8). Namely, in Example 6.13, the standard filtrations for the projectives involve three standard objects with highest weight λ : $\mathbb{M}(\lambda, \{J\})$ for $\emptyset \neq J \subseteq \{1, 2\}$ (even for $\lambda = (0, 0)$).

The previous sentence suggests $\mathcal{O}^{\mathcal{H}}$ is not a highest weight category for $\mathcal{H} = \{\{1, 2\}\}$. To see why, we discuss if any of the three standard objects $\mathbb{M}(\lambda, \{J\})$ for $\emptyset \neq J \subseteq \{1, 2\}$ can be “avoided”, via alternate standard filtrations for the projective objects in $\mathcal{O}^{\{\{1, 2\}\}}$. However, this is not possible:

- (1) First, $\mathbb{M}(\lambda, \mathcal{H}'_{\lambda}) = \mathbb{M}(\lambda, \{\{1, 2\}\})$ cannot occur in any filtration of $P^{\mathcal{H}}(s_i \bullet \lambda)$ in (6.5) – because even the simpler statement $\text{wt } \mathbb{M}(\lambda, \mathcal{H}'_{\lambda}) \subseteq \text{wt } P^{\mathcal{H}}(s_i \bullet \lambda)$ is false.
- (2) One can ask if the above notion of standard filtration could be broadened to require the top quotient to merely be a standard object $\mathbb{M}(\nu, \mathcal{H}')$ rather than the universal cover $\mathbb{M}(\nu, \mathcal{H}'_{\nu})$. By Proposition 6.4(3), such a set \mathcal{H}' could be an upper-closed subset containing \mathcal{H}'_{ν} . This weakening could enable using $\mathbb{M}(\lambda, \mathcal{H}'')$ for $\mathcal{H}'' \neq \mathcal{H}'_{\lambda}$, in the standard filtration for $P^{\mathcal{H}}(\lambda)$.

Unfortunately, this hope is also in vain, in that even in the above example with $\mathcal{H} = \{\{1, 2\}\}$, it leads to violating the requirement that the remaining standard factors in the filtration of $P^{\mathcal{H}}(\nu)$ should have highest weights $\mu > \nu$. Indeed, set $\mu = \nu = \lambda$; now there are four upper-closed subsets \mathcal{H}' containing $\mathcal{H}'_{\lambda} = \{\{1, 2\}\}$:

$$\mathcal{H}' = \{\{1, 2\}\}, \quad \{\{1\}, \{1, 2\}\}, \quad \{\{2\}, \{1, 2\}\}, \quad \{\{1\}, \{2\}, \{1, 2\}\}.$$

So if any other module $\mathbb{M}(\lambda, \mathcal{H}'')$ is used in the standard filtration for $P^{\mathcal{H}}(\lambda) = \mathbb{M}(\lambda, \mathcal{H}'_{\lambda})$, then as λ is maximal in its dot-orbit, the kernel of $P^{\mathcal{H}}(\lambda) \rightarrow \mathbb{M}(\lambda, \mathcal{H}'')$ has all factors with highest weight $\mu < \lambda$ – but highest weight categories and BGG reciprocity require $\mu > \lambda$.

Remark 6.15. The above discussion shows that $\mathcal{O}^{\mathcal{H}}$ is not a highest weight category for general \mathcal{H} in the sense of Cline–Parshall–Scott [10] – as early as $\mathfrak{g} = \mathfrak{sl}_2^{\oplus 2}$ and $\mathcal{H} = \{\{1, 2\}\}$. In this sense, the category $\mathcal{O}^{\mathcal{H}}$ diverges in higher order, from the zeroth and first order (parabolic) category \mathcal{O} .

We end this part by showing the same fact over every \mathfrak{g} of higher rank, as promised above.

Proposition 6.16. *Suppose \mathfrak{g} is semisimple of rank at least 3. Then there exist $\lambda \in \mathfrak{h}^*$ and an upper-closed set $\mathcal{H} \subseteq \text{Indep}(I)$ such that $\mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}$ is not a highest weight category.*

Proof. Since the Dynkin diagram contains at least two leaves (in particular, it is not complete), choose two “orthogonal” simple roots and label them by α_1, α_2 . Let $J := I \setminus \{1, 2\}$ and fix a generic weight $\lambda \in \text{span}_{\mathbb{C}}\{\varpi_j : j \in J\}$ such that $W \bullet \lambda \cap (\lambda + \mathbb{Z}\Pi) = W_{\{1, 2\}} \bullet \lambda$; here ϖ_j denotes the fundamental weight corresponding to $j \neq 1, 2$. Now the integrabilities are computed as:

$$J_{\lambda} = \{1, 2\}, \quad J_{s_1 \bullet \lambda} = \{2\}, \quad J_{s_2 \bullet \lambda} = \{1\}, \quad J_{s_1 s_2 \bullet \lambda} = \emptyset.$$

Let $\mathcal{H} = \{\{1, 2\}\}$; then there are only three simple objects in the block $\mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}$, and this reduces to the above situation over $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. \square

6.4. Proof of BGG reciprocity over $\mathfrak{sl}_2^{\oplus n}$. The above remarks explain why one needs to refine the “usual” notion of BGG reciprocity. We now do so over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ for all n , thereby completing the proof of Theorem D.

Theorem 6.17. *Suppose $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ for some $n \geq 1$, and $\lambda \in \mathfrak{h}^*$. Define $w_K := \prod_{k \in K} s_k$ for $K \subseteq I = \{1, \dots, n\}$ and also fix $\mathcal{H} \subseteq \text{Indep}(I) = 2^I$.*

- (1) *If $L(w_K \bullet \lambda) \in \mathcal{O}^{\mathcal{H}}$ for $K \subseteq J_{\lambda}$, its projective cover $P^{\mathcal{H}}(w_K \bullet \lambda)$ has a “standard filtration” by objects $\mathbb{M}(\mu, \mathcal{H}'_{w_K \bullet \lambda})$, with topmost quotient the “maximal” standard object $\mathbb{M}(w_K \bullet \lambda, \mathcal{H}'_{w_K \bullet \lambda})$ over $L(w_K \bullet \lambda)$, and all other subquotients of highest weight $\mu \in (W \bullet \lambda)_{> w_K \bullet \lambda}$.*

(2) For all highest weights μ “in” this filtration, a modified form of BGG reciprocity holds:

$$\sum_{\substack{\mathcal{H}_0 \supseteq \mathcal{H}'_\mu, \\ \mathcal{H}_0 \text{ upper-closed} \\ \text{in } \text{Indep}(J_\mu)}} [P^{\mathcal{H}}(w_K \bullet \lambda) : \mathbb{M}(\mu, \mathcal{H}_0)] = [P^{\mathcal{H}}(w_K \bullet \lambda) : \mathbb{M}(\mu, \mathcal{H}'_{w_K \bullet \lambda})] = [\mathbb{M}(\mu, \mathcal{H}'_\mu) : L(w_K \bullet \lambda)]. \quad (6.6)$$

Note the presence of the summation on the left side in (6.6), in contrast to the BGG reciprocity formulas in the zeroth and first order parabolic categories. This summation – i.e. using multiple standard objects over a given simple object – is indeed needed when discussing BGG reciprocity for general $\mathcal{O}^{\mathcal{H}}$, as was explained above over $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and all higher rank \mathfrak{g} .

The proof of Theorem 6.17 will require computing the integrability of the highest weights of the simples in the block \mathcal{O}^λ . This is achieved by the following lemma, which also bounds the integrabilities of all weights in the interval $[w_K \bullet \lambda, \lambda]$, in greater generality than $\mathfrak{sl}_2^{\oplus n}$.

Lemma 6.18. *Fix semisimple \mathfrak{g} , a subset $\mathcal{H} \subseteq \text{Indep}(I)$, and a weight $\lambda \in \mathfrak{h}^*$ that is maximal in its block in \mathcal{O} . Suppose (a) the integrability J_λ is an independent set of nodes, that further satisfies (b) the integrability of $w_{J_\lambda} \bullet \lambda$ is empty, where $w_K = \prod_{k \in K} s_k$ for $K \subseteq J_\lambda$.*

- (1) For all $K \subseteq J_\lambda$ one has $J_{w_K \bullet \lambda} = J_\lambda \setminus K$.
- (2) For all subsets $K' \subseteq K \subseteq J_\lambda$ and weights $w_K \bullet \lambda \leq \mu \leq w_{K'} \bullet \lambda$, one has the inclusion of integrabilities: $J_{w_K \bullet \lambda} \subseteq J_\mu \subseteq J_{w_{K'} \bullet \lambda}$. In particular, if $L(w_K \bullet \lambda) \in \mathcal{O}^{\mathcal{H}}$ then $L(\mu) \in \mathcal{O}^{\mathcal{H}}$.
- (3) Suppose $L(w_K \bullet \lambda) \in \mathcal{O}^{\mathcal{H}}$ for some $K \subseteq J_\lambda$. Then $\mathbb{M}(w_K \bullet \lambda, \mathcal{H}'_{w_K \bullet \lambda})$ has a subquotient $L(w_{K'} \bullet \lambda)$ for some $K' \subseteq J_\lambda$, if and only if (i) $L(w_{K'} \bullet \lambda) \in \mathcal{O}^{\mathcal{H}}$ and (ii) $K' \supseteq K$.

Here and below, we use $\mathbf{f}_{H'}^{(\lambda)} := \prod_{h \in H'} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1}$ for $\lambda \in \mathfrak{h}^*$ and $H' \subseteq J_\lambda$ an independent subset. Also note that the hypothesis “ $w_{J_\lambda} \bullet \lambda$ has empty integrability” does not follow from the remaining hypotheses in the lemma – consider e.g. $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = \alpha_2$.

Proof. Recall the hypothesis $J_{w_{J_\lambda} \bullet \lambda} = \emptyset$, which is now used extensively without further reference. Also note that $W_{J_\lambda} \bullet \lambda$ is in bijection with W_{J_λ} .

- (1) If $k \in K$, then $\langle w_K \bullet \lambda, \alpha_k^\vee \rangle = \langle (w_K s_k) \bullet (s_k \bullet \lambda), \alpha_k^\vee \rangle = \langle s_k \bullet \lambda, \alpha_k^\vee \rangle = -\langle \lambda, \alpha_k^\vee \rangle - 2 < 0$. Now write $w_K \bullet \lambda = w_{J_\lambda} \bullet \lambda + \sum_{j \in J_\lambda \setminus K} l_j \alpha_j$, with all $l_j \in \mathbb{Z}_{\geq 0}$. Given $i \in I \setminus J_\lambda$,

$$\langle w_K \bullet \lambda, \alpha_i^\vee \rangle = \langle w_{J_\lambda} \bullet \lambda, \alpha_i^\vee \rangle + \sum_{j \in J_\lambda \setminus K} l_j \langle \alpha_j, \alpha_i^\vee \rangle \notin \mathbb{Z}_{\geq 0}.$$

This shows one inclusion; the reverse inclusion is shown similarly.

- (2) Begin by writing: $\mu = w_K \bullet \lambda + \sum_{k \in K \setminus K'} l_k \alpha_k = \lambda - \sum_{k \in K} l'_k \alpha_k$, where $l_k, l'_k \in \mathbb{Z}_{\geq 0}$. Now suppose $i \notin J_{w_{K'} \bullet \lambda}$, so $i \notin J_{w_K \bullet \lambda}$ by (1). Then

$$\langle \mu, \alpha_i^\vee \rangle = \langle w_K \bullet \lambda, \alpha_i^\vee \rangle + \sum_{k \in K \setminus K'} l_k \langle \alpha_k, \alpha_i^\vee \rangle \notin \mathbb{Z}_{\geq 0}.$$

This shows one inclusion. Next if $i \in J_\lambda \setminus K$, then $\langle \mu, \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle - \sum_{k \in K} l'_k \langle \alpha_k, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$. This shows the other inclusion – and also implies that if $H \in \mathcal{H}^{\min}$ and $L(w_K \bullet \lambda) \in \mathcal{O}^{\mathcal{H}}$, then $J_\mu \cap H \supseteq J_{w_K \bullet \lambda} \cap H \neq \emptyset$, so $L(\mu) \in \mathcal{O}^{\mathcal{H}}$ by Proposition 6.4(1).

- (3) The result is straightforward if $\mathcal{H} = \emptyset$, since one now works in the Verma module $M(w_K \bullet \lambda)$. Thus, assume henceforth that $\mathcal{H} \neq \emptyset$. Set $M := \mathbb{M}(w_K \bullet \lambda, \mathcal{H}'_{w_K \bullet \lambda})$. The necessity of (i), (ii) easily follows from Lemma 6.1 and the $\mathcal{H} = \emptyset$ case. Conversely, if K' satisfies (i), (ii) then it suffices to show the weight space $L := \mathbf{f}_{K' \setminus K}^{(w_K \bullet \lambda)} M_{w_K \bullet \lambda}$ is nonzero. As the preimage in $M(w_K \bullet \lambda) \cong U(\mathfrak{n}^-)$ of L is a line, it suffices to show that $\mathbf{f}_{K' \setminus K}^{(w_K \bullet \lambda)}$ is not in the left-ideal

$$\mathbf{f}_{K' \setminus K}^{(w_K \bullet \lambda)} \notin U(\mathfrak{n}^-) \cdot \langle \mathbf{f}_{(J_\lambda \setminus K) \cap H}^{(w_K \bullet \lambda)} : H \in \mathcal{H}^{\min} \rangle. \quad (6.7)$$

To show this, work with a “PBW basis” of \mathfrak{n}^- in which $f_k, k \in K$ occur to the right, preceded by $f_j, j \in J_\lambda \setminus K$, then preceded by all other root vectors in \mathfrak{n}^- . Since the roots indexed by $K' \setminus K$ are pairwise orthogonal, no nontrivial Lie brackets among them exist.

Thus if (6.7) is false, then $\mathbf{f}_{(K' \setminus K) \cap H}^{(w_K \bullet \lambda)} = \mathbf{f}_{(J_\lambda \setminus K) \cap H}^{(w_K \bullet \lambda)}$ for some $H \in \mathcal{H}^{\min}$. Hence by part (1), $J_{w_{K'} \bullet \lambda} \cap H = (J_\lambda \setminus K') \cap H = \emptyset$, which contradicts $L(w_{K'} \bullet \lambda) \in \mathcal{O}^{\mathcal{H}}$ by Proposition 6.4(1). \square

With Lemma 6.18 at hand, we now have:

Proof of Theorem 6.17. It suffices to work with the objects in a block / linkage class $[\lambda] = W \bullet \lambda \cap (\lambda + \mathbb{Z}\Pi)$ that moreover lie in $\mathcal{O}^{\mathcal{H}}$. Note that we may take $\lambda \in \mathfrak{h}^*$ to be maximal in the block $[\lambda]$; now the simples in $[\lambda]$ (in \mathcal{O} , not $\mathcal{O}^{\mathcal{H}}$) are indexed by $W_{J_\lambda} \bullet \lambda$. Define

$$[\lambda]_{\mathcal{H}} := \{\mu \in [\lambda] : L(\mu) \in \mathcal{O}^{\mathcal{H}}\}. \quad (6.8)$$

(Thus, $[\lambda] = [\lambda]_\emptyset$.) Throughout the rest of this proof, we will work in the category

$$\mathcal{O}^{\mathcal{H}, [\lambda]} := \mathcal{O}^{[\lambda]_{\mathcal{H}}} = \mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{[\lambda]} \subseteq \mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}. \quad (6.9)$$

We also use without reference that the hypotheses of Lemma 6.18 hold over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ for all $\lambda \in \mathfrak{h}^*$.

We first show BGG reciprocity at λ , by **claiming** that the projective cover of $L(\lambda)$ in the block $\mathcal{O}^{\mathcal{H}, [\lambda]}$ is $P^{\mathcal{H}}(\lambda) = \mathbb{M}(\lambda, \mathcal{H}'_\lambda)$, where

$$\mathcal{H}'_\lambda = \{J_\lambda \cap H : H \in \mathcal{H}^{\min}\}$$

as in Proposition 6.4(2). Indeed, this object in $\mathcal{O}^{\mathcal{H}, [\lambda]}$ is indecomposable and surjects onto $L(\lambda)$, and a relatively standard argument (see e.g. the proof of Step 4 in showing that $\mathcal{O}^{\mathcal{H}, [\lambda]}$ has enough projectives, in the Appendix) shows the functorial isomorphism

$$\text{Hom}_{\mathcal{O}^{\mathcal{H}}}(\mathbb{M}(\lambda, \mathcal{H}'_\lambda), M) \cong M_\lambda, \quad \forall M \in \mathcal{O}^{\mathcal{H}, [\lambda]},$$

i.e., that $\text{Hom}_{\mathcal{O}^{\mathcal{H}}}(\mathbb{M}(\lambda, \mathcal{H}'_\lambda), -)$ co-represents the λ -weight space in $\mathcal{O}^{\mathcal{H}, [\lambda]}$. This proves the claim, and BGG reciprocity (6.6) involving $P^{\mathcal{H}}(\lambda), L(\lambda)$ now follows e.g. by Lemma 6.18(3).

BGG reciprocity at $w_K \bullet \lambda$, for $\emptyset \subsetneq K \subseteq J_\lambda$:

Next, we work with every other simple in $[\lambda]$ that occurs in $\mathcal{O}^{\mathcal{H}}$. From above, we call it $L(w_K \bullet \lambda)$, where $\emptyset \subsetneq K \subseteq J_\lambda$ is fixed and $w_K = \prod_{k \in s_k}$. Proving reciprocity requires working with those simples in $\mathcal{O}^{\mathcal{H}} \cap \mathcal{O}^{W \bullet \lambda}$ which lie above $w_K \bullet \lambda$ in the standard ordering. These simples are indexed by $W_K \bullet \lambda$. We identify $W_K \xrightarrow{\psi} (\mathbb{Z}/2\mathbb{Z})^{\oplus K} \simeq \{0, 1\}^K$, and so list $W_K = \{w_{K'} = \prod_{k \in K'} s_k : K' \subseteq K\}$.

We now prove BGG reciprocity at the weight $\lambda_K := w_K \bullet \lambda$. For ease of reading, the remainder of this proof is split into steps.

Step 1: The BGG construction of a cyclic module, and its standard filtration. Recalling that $m_i = \langle \lambda, \alpha_i^\vee \rangle + 1$, we first define and study – akin to [6] for \mathcal{O} – the cyclic module $P := U\mathfrak{g}/I_K$, where

$$I_K := U\mathfrak{g} \cdot \left(\{h - \lambda_K(h) : h \in \mathfrak{h}\}, \{e_{\alpha_k}^{m_k+1} : k \in K\}, \{e_\alpha : \alpha \in \Delta^+ \setminus \Delta_K^+\}, \{\mathbf{f}_{J_{\lambda_K} \cap H}^{(\lambda_K)} : H \in \mathcal{H}^{\min}\} \right). \quad (6.10)$$

(Note that $\Delta^+ \setminus \Delta_K^+ = \Pi_{I \setminus K}$.) Let $\mathbf{p} = \mathbf{p}_{\lambda_K}$ denote the image of $1_{U\mathfrak{g}}$ in P . Then there is a lattice of $\prod_{k \in K} (m_k + 1)$ -many submodules of P , indexed by integer tuples $\mathbf{l} = (l_k)_{k \in K}$:

$$P = P_0 := U\mathfrak{g} \cdot \mathbf{p} \supseteq P_1 := U\mathfrak{g} \cdot \prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \mathbf{p} \supseteq P_{\mathbf{m}} := U\mathfrak{g} \cdot \prod_{k \in K} e_{\alpha_k}^{m_k} \cdot \mathbf{p} \supseteq 0, \quad 0 \leq l_k \leq m_k \forall k. \quad (6.11)$$

This yields $\prod_{k \in K} (m_k + 1)$ -many subquotients, each of the form

$$Q_{\mathbf{l}} := \frac{U\mathfrak{g} \cdot \prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \mathbf{p}}{\sum_{k' \in K} U\mathfrak{g} \cdot e_{\alpha_{k'}} \prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \mathbf{p}}, \quad \mathbf{l} \in \times_{k \in K} [0, m_k], \text{ i.e., } \mathbf{0} \leq \mathbf{l} \leq \mathbf{m},$$

where the inequalities are coordinatewise. Each subquotient Q_1 is generated by a maximal vector

$$\prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \mathbf{p}, \quad \text{of weight } \mu_1 := w_K \bullet \lambda + \sum_{k \in K} l_k \alpha_k,$$

so Q_1 is a highest weight module. We now show that the filtration by the Q_1 is indeed standard:

Lemma 6.19. *Given a tuple $\mathbf{l} \in [\mathbf{0}, \mathbf{m}]$, with μ_1 as above, $Q_1 \cong \mathbb{M}(\mu_1, \mathcal{H}'_{w_K \bullet \lambda}) \in \mathcal{O}^{\mathcal{H}}$.*

In particular, by Lemma 6.2, the cyclic module $P \in \mathcal{O}^{\mathcal{H}}$, and this lemma provides a “standard filtration” for it, albeit in $\mathcal{O}^{\mathcal{H}}$ and not in a single block $\mathcal{O}^{\mathcal{H}, [\lambda]}$.

Proof. Note that Q_1 has highest weight $\mu_1 := w_K \bullet \lambda + \sum_{k \in K} l_k \alpha_k$, and so by Lemma 6.18(2), its integrability $J_{\mu_1} \supseteq J_{w_K \bullet \lambda}$. It follows that $\mathbb{M}(\mu_1, \mathcal{H}'_{w_K \bullet \lambda}) \in \mathcal{O}^{\mathcal{H}}$, by applying Proposition 6.4(3) with $\lambda \rightsquigarrow \mu_1$ and $\mathcal{H}_V \rightsquigarrow$ the upper-closure of $\mathcal{H}'_{w_K \bullet \lambda}$.

We now show $Q_1 \cong \mathbb{M}(\mu_1, \mathcal{H}'_{w_K \bullet \lambda})$, starting with an upper bound on Q_1 , via the generator-coset

$$\mathbf{p} = 1 + \text{span}_{U\mathfrak{g}}\{h - (w_K \bullet \lambda)(h), e_{\alpha_k}^{m_k+1}, e_{\alpha}, \mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}\}$$

(see (6.10) and Lemma 6.18(2)). Now note that $e_k^{l_k}$ (for $k \in K$) commutes with all $\mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}$, and

$$e_{\alpha_k}^{l_k}(h - \nu(h)) = (h - (\nu + l_k \alpha_k)(h))e_{\alpha_k}^{l_k}, \quad \forall h \in \mathfrak{h}, \nu \in \mathfrak{h}^*.$$

Using these relations and that

$$\mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(\mu_1)} = \mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}, \quad \forall \mathbf{0} \leq \mathbf{l} \leq \mathbf{m} \quad (6.12)$$

which follows from the independence of the set J_{λ} , it follows that Q_1 is a quotient of

$$\frac{U\mathfrak{g}}{U\mathfrak{g}\left(\{h - \mu_1(h) : h \in \mathfrak{h}\}, \{e_i : i \in I\}, \{\mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)} : H \in \mathcal{H}^{\min}\}\right)}.$$

But this is precisely $\mathbb{M}(\mu_1, \mathcal{H}'_{w_K \bullet \lambda})$.

This provides an upper bound on (the character of) Q_1 ; we now show it is also a lower bound, which will prove the lemma.

Claim. *Fix an ordering on I , hence the ordered basis $\{f_i : i \in I\}$ of \mathfrak{n}^- , and denote by \mathcal{B} the corresponding PBW / monomial basis of $U\mathfrak{n}^- \simeq \mathbb{C}[f_i : i \in I]$. Now let \mathcal{B}' denote the subset of monomials which are not divisible by $\mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}$ for any $H \in \mathcal{H}^{\min}$. Then the vectors $\{b' \cdot \prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \mathbf{p} \mid b' \in \mathcal{B}'\}$ are independent in Q_1 .*

(By (6.12), \mathcal{B}' is in bijection with a basis of $\mathbb{M}(\mu_1, \mathcal{H}'_{w_K \bullet \lambda})$, yielding the desired lower bound on Q_1 .)

The claim will follow via the filtration of P by the Q_1 (so we no longer fix \mathbf{l} in the proof of this lemma), from the statement that:

$$\tilde{\mathcal{B}} := \{b' \cdot \prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \mathbf{p} : 0 \leq l_k \leq m_k, k \in K, b' \in \mathcal{B}'\}$$

is a basis of the module $P = U\mathfrak{g}/I_K$ (where the left ideal I_K was defined in (6.10)).

To show this statement, write using the PBW theorem and changing variables:

$$U\mathfrak{g} \simeq \mathbb{C}[f_i : i \in I] \otimes \mathbb{C}[e_i : i \in I] \otimes \mathbb{C}[\alpha_i^{\vee} : i \in I].$$

Quotient by I_K in stages: first quotienting by the relations $\{\alpha_i^{\vee} - \langle w_K \bullet \lambda, \alpha_i^{\vee} \rangle : i \in I\}$ yields

$$U\mathfrak{g}/(\alpha_i^{\vee} : i \in I) \simeq \mathbb{C}[f_i : i \in I] \otimes \mathbb{C}[e_i : i \in I]. \quad (6.13)$$

Next, further quotienting this by all $e_{\alpha_k}^{m_k+1}$ and e_{α_i} for $k \in K, i \notin K$ yields

$$\mathbb{C}[f_i : i \in I] \otimes \text{span}_{\mathbb{C}} \{ \prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \overline{1_{U\mathfrak{g}}} \}. \quad (6.14)$$

More precisely, one needs to quotient (6.13), by the image in it of the space of vectors

$$\sum_{k \in K} X_k \cdot e_{\alpha_k}^{m_k+1} + \sum_{i \notin K} X'_i \cdot e_{\alpha_i}, \quad X_k, X'_i \in U\mathfrak{g}.$$

Writing each X_k, X'_i as linear combinations of PBW monomials in the above ordered basis of f_i, e_i, α_i^\vee , one obtains (6.14).

Finally, one obtains $P = U\mathfrak{g}/I_K$ by quotienting (6.14) by the image in it of the space of vectors

$$\sum_{H \in \mathcal{H}^{\min}} X_H \cdot \mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}, \quad X_H \in U\mathfrak{g}. \quad (6.15)$$

Write $X_H = \sum_t p_{H,t}(\{f_i\}) \cdot q_{H,t}(\{e_i\}) \cdot r_{H,t}(\{\alpha_i^\vee\})$, and note that the polynomial $r_{H,t}(\{\alpha_i^\vee\})$ “goes past” $\mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}$ yielding scalars, in the quotient space (6.14). Thus, we may suppose all $r_{H,t} \in \mathbb{C}$. Next, use a standard \mathfrak{sl}_2 -calculation with $m'_i := \langle w_K \bullet \lambda, \alpha_i^\vee \rangle + 1 \in \mathbb{Z}_{>0}$ for $i \in J_{w_K \bullet \lambda} \cap H$:

$$e_i f_i^{m'_i} = f_i^{m'_i} e_i + m'_i f_i^{m'_i-1} (\alpha_i^\vee - \langle w_K \bullet \lambda, \alpha_i^\vee \rangle) = f_i^{m'_i} e_i + m'_i f_i^{m'_i-1} (\alpha_i^\vee - \langle \lambda, \alpha_i^\vee \rangle), \quad (6.16)$$

where the final equality is because $i \in J_{w_K \bullet \lambda} \cap H = (J_\lambda \cap H) \setminus K$.

It follows from (6.16) that if e_{α_i} divides any $q_{H,t}$ for $i \in J_{w_K \bullet \lambda} \cap H$, then that monomial vanishes in (6.14). The same happens if $e_i | q_{H,t}$ for some $i \notin J_{w_K \bullet \lambda} \cap H$ and $i \in I \setminus K$, or if $e_{\alpha_k}^{m_k+1} | q_{H,t}$ for some $k \in K$. Thus, we reduce to computing the quotient of (6.14) by the subspace of vectors

$$\sum_{H \in \mathcal{H}^{\min}} X_H \cdot \mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}, \quad X_H = \sum_{0 \leqslant \mathbf{l} \leqslant \mathbf{m}} p_{H,\mathbf{l}}(\{f_i\}) \cdot \prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \overline{1_{U\mathfrak{g}}}.$$

As the second tensor factors in X_H and (6.14) coincide, and since $\mathbf{f}_{J_{w_K \bullet \lambda} \cap H}^{(w_K \bullet \lambda)}$ commutes with all e_{α_k} , this yields precisely $\text{span}_{\mathbb{C}}(\mathcal{B}') \otimes \text{span}_{\mathbb{C}}\{\prod_{k \in K} e_{\alpha_k}^{l_k} \cdot \overline{1_{U\mathfrak{g}}}\}$, which is indeed spanned by $\widetilde{\mathcal{B}}$. \square

Step 2: *The cyclic module is projective, and its $[\lambda]_{\mathcal{H}}$ -summand co-represents the $(w_K \bullet \lambda)$ -weight space.* The previous step shows that the module $P = U\mathfrak{g}/I_K \in \mathcal{O}^{\mathcal{H}}$ has a standard filtration by the modules $Q_1 \cong \mathbb{M}(\mu_1, \mathcal{H}'_{w_K \bullet \lambda}) \in \mathcal{O}^{\mathcal{H}}$ for $\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{m}$. Moreover, since $K \neq \emptyset$ here, $w_K \bullet \lambda$ has strictly smaller integrability than λ by Lemma 6.18(2) (and $L(w_K \bullet \lambda) \in \mathcal{O}^{\mathcal{H}}$). Thus the upper-closure in $\text{Indep}(I)$ of \mathcal{H} is contained in that of $\mathcal{H}'_{w_K \bullet \lambda}$, which we denote henceforth as $\overline{\mathcal{H}'}$ for convenience.

Now $P \in \mathcal{O}^{\overline{\mathcal{H}'}} \subseteq \mathcal{O}^{\mathcal{H}}$. We claim that P is projective in the truncated subcategory of objects with all weights $\leqslant \lambda$, which we denote by $\mathcal{O}_{\leqslant \lambda}^{\mathcal{H}}$. The claim is shown (as above) by noting that $\text{Hom}_{\mathcal{O}_{\leqslant \lambda}^{\mathcal{H}}}(P, -)$ co-represents the $(w_K \bullet \lambda)$ -weight space. Now use the block decomposition to write

$$P = \bigoplus_{[\mu]} P^{[\mu]}, \quad \text{with} \quad P^{[\mu]} \in \mathcal{O}^{\overline{\mathcal{H}'}, [\mu]} \subseteq \mathcal{O}^{\mathcal{H}, [\mu]}.$$

Then $P^{[\lambda]}$ is projective in $\mathcal{O}^{\mathcal{H}, [\lambda]} \subseteq \mathcal{O}_{\leqslant \lambda}^{\mathcal{H}}$, and it co-represents in $\mathcal{O}^{\mathcal{H}, [\lambda]}$ the $(w_K \bullet \lambda)$ -weight space.

Step 3: *The projective cover is the $[\lambda]_{\mathcal{H}}$ -summand – hence, BGG reciprocity at $w_K \bullet \lambda$.* Given the standard filtration (above) of the module $P = \bigoplus_{[\mu]} P^{[\mu]}$ in $\mathcal{O}^{\overline{\mathcal{H}'}}$, Proposition 6.10 now applies (with $\mathcal{H} \rightsquigarrow \overline{\mathcal{H}'}$) to show that $P^{[\lambda]}$ has a standard filtration in $\mathcal{O}^{\overline{\mathcal{H}'}} \subseteq \mathcal{O}^{\mathcal{H}}$. Moreover, the subquotients occur from among the Q_1 , hence each have highest weight $> w_K \bullet \lambda$; and the “topmost” quotient is indeed $Q_0 \cong \mathbb{M}(w_K \bullet \lambda, \mathcal{H}'_{w_K \bullet \lambda})$.

This concludes the proof of the assertions on the standard filtration of $P^{[\lambda]}$. We now turn to the proof of BGG reciprocity at $w_K \bullet \lambda$. Note that the subquotient Q_1 belongs to the block $\mathcal{O}^{[\lambda]}$, if and

only if every l_k is either 0 or m_k , in which case one obtains a highest weight of $w_{K'} \bullet \lambda$ for some $\emptyset \subseteq K' \subseteq K$ – moreover, for each such K' it is the highest weight of a unique subquotient Q_1 .

We now **claim** that the summand $P^{[\lambda]}$ of P is precisely the (indecomposable) projective cover of $L(w_K \bullet \lambda)$ in $\mathcal{O}^{\overline{\mathcal{H}}}$, hence in the larger category $\mathcal{O}^{\mathcal{H}}$. Now BGG reciprocity follows, because

$$\sum_{\mathcal{H}_0 \text{ upper-closed}} [P^{\mathcal{H}}(w_K \bullet \lambda) : \mathbb{M}(w_{K'} \bullet \lambda, \mathcal{H}_0)] = \mathbf{1}_{K \supseteq K'} = [\mathbb{M}(w_{K'} \bullet \lambda, \mathcal{H}'_{w_{K'} \bullet \lambda}) : L(w_K \bullet \lambda)].$$

(The first equality is from the claim just above, and the second is by Lemma 6.18(3).)

Thus, it remains to prove the claim. The following argument is due to Gurbir Dhillon. As explained above, the summand $P^{[\lambda]}$ co-represents the $(w_K \bullet \lambda)$ -weight space in $\mathcal{O}^{\overline{\mathcal{H}}, [\lambda]}$. On the other hand, the projective cover $P' := P^{\overline{\mathcal{H}}}(w_K \bullet \lambda)$ is characterized by the equation

$$\dim \text{Hom}_{\mathcal{O}^{\overline{\mathcal{H}}, [\lambda]}}(P', L(\nu)) = \delta_{w_K \bullet \lambda, \nu}, \quad \forall \nu \in [\lambda]_{\mathcal{H}}.$$

It is also standard that $P^{[\lambda]} \twoheadrightarrow P'$, and that the kernel is a sum of (copies of) $P^{\overline{\mathcal{H}}}(\nu)$ for $\nu > w_K \bullet \lambda$. Thus, $P^{[\lambda]} \cong P'$ if and only if no other indecomposable projective summands occur, i.e. if and only if $L(\nu)$ has no $(w_K \bullet \lambda)$ -weight space if $\nu \in [\lambda]_{> w_K \bullet \lambda}$ (the contrapositive of this may be easier to see). This indeed holds by $\mathfrak{sl}_2^{\oplus K}$ -theory in $[\lambda]_{\mathcal{H}}$, and shows BGG reciprocity at $w_K \bullet \lambda$. \square

6.5. Kazhdan–Lusztig combinatorics in $\mathcal{O}^{\mathcal{H}}$. As a related discussion, we study a *quotient space* of the regular representation of the Iwahori–Hecke algebra $\mathcal{H}(W)$, which relates naturally to the Grothendieck group of $\mathcal{O}^{\mathcal{H}}$. To begin, consider the regular block $\mathcal{O}^{\mathcal{H}, W \bullet 0}$ over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$.

We adopt standard convention, found e.g. in Equation (1.3). Thus, the Weyl group is $W = \{w_K : K \subseteq \{1, \dots, n\}\} \cong S_2^n$, and the simples in the block are $L(w_K w_{\circ} \bullet 0)$. Here w_{\circ} is the longest element in W and $w_K = \prod_{k \in K} s_k$ as above. Also note that

$$\mathcal{H}(W) \cong \otimes_{i=1}^n \mathcal{H}(\langle 1, s_i \rangle) = \otimes_{i=1}^n (R1 \oplus RT_i)$$

for a suitable Laurent polynomial ring R over \mathbb{Z} . In particular, there is a monomial basis:

$$\mathcal{H}(W) = \text{span}_R \left\{ T_K := \prod_{k \in K} T_k \mid K \subseteq \{1, \dots, n\} \right\}. \quad (6.17)$$

Moreover, each factor Hecke algebra has a Kazhdan–Lusztig basis $\{1, C_i\}$, so $\mathcal{H}(W)$ has the corresponding Kazhdan–Lusztig basis

$$\mathcal{H}(W) = \text{span}_R \left\{ C_K := \prod_{k \in K} C_k \mid K \subseteq \{1, \dots, n\} \right\}. \quad (6.18)$$

As is well known, in \mathcal{O} one can specialize to $q = 1$ and interpret the change-of-basis relations between the C_K and T_K in the Grothendieck group $K_0(\mathcal{O}^{W \bullet \lambda})$, via:

$$C_K \rightsquigarrow [L(w_K w_{\circ} \bullet 0)], \quad T_K \rightsquigarrow [M(w_K w_{\circ} \bullet 0)]. \quad (6.19)$$

These relations are precisely the ones in (1.3) with $\lambda \rightsquigarrow 0$; since one is working over $\mathfrak{sl}_2^{\oplus n}$, all Kazhdan–Lusztig polynomials $P_{x,w} \equiv \mathbf{1}_{x \leq w}$, with \leq the Bruhat order.

Our goal here is to explain that this decategorification phenomenon holds more generally, in $\mathcal{O}^{\mathcal{H}}$ for all \mathcal{H} over $\mathfrak{sl}_2^{\oplus n}$. We first illustrate this in the special case of Example 6.13 (above) over $\mathfrak{sl}_2^{\oplus 2}$.

Example 6.20. Let $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathcal{H} = \{\{1, 2\}\}$. The block $\mathcal{O}^{\mathcal{H}, W \bullet 0}$ has three simple objects $L(w_K w_{\circ} \bullet 0)$ for $K \neq \emptyset$, and their universal covers $\mathbb{M}(w_K w_{\circ} \bullet 0, \mathcal{H}'_{w_K w_{\circ} \bullet 0})$ were worked out in Example 6.13. We now write down the characters of the simple objects in $K_0(\mathcal{O}^{\{\{1, 2\}\}})$ via (6.19):

$$C_{\{1\}} = T_1, \quad C_{\{2\}} = T_2, \quad C_{\{1, 2\}} = T_1 T_2 - T_1 - T_2.$$

Notice, these relations are not correct on the nose in $\mathcal{H}(W)$, since there are no coefficients of the unit $1 = T_\emptyset$ in any of them. However, these relations are the images of the usual Kazhdan–Lusztig relations (1.3) in the quotient free R -module $\mathcal{H}(W)/(R \cdot T_\emptyset)$ – which reflects in $K_0(\mathcal{O}^{\mathcal{H}, W \bullet 0})$. \square

The story is similar over $\mathfrak{sl}_2^{\oplus n}$. One can compute the character of $\mathbb{M}(\lambda, \mathcal{H}) \in \mathcal{O}^{\mathcal{H}}$ in two ways:

- (1) Via a BGG-type resolution in terms of Verma modules in the usual Category \mathcal{O} . This is worked out in slightly greater generality in the next section – see Theorem 7.12.
- (2) Alternately, one works internally inside $\mathcal{O}^{\mathcal{H}}$ itself. In this case, one needs to compute the character (or the image in K_0) of $\mathbb{M}(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}'_\lambda)$, whenever $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$. This is worked out in the next result.

Proposition 6.21. *Fix $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ and a nonempty subset $\mathcal{H} \subseteq \text{Indep}(I) = 2^I$. Given $\lambda \in \mathfrak{h}^*$, define $K_* := \{i \in I : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$, and suppose λ is K_* -dominant integral. Let $[\lambda]_{\mathcal{H}} \subseteq [\lambda] = W_{K_*} \bullet \lambda$ index the set of simples in $\mathcal{O}^{\mathcal{H}}$ as in (6.9). Define*

$$\mathcal{H}^{\mathcal{H}}(W_{K_*}) := \frac{\mathcal{H}(W_{K_*})}{\sum_{J \not\subseteq [\lambda]_{\mathcal{H}}} R \cdot T_J}, \quad C_K^{\mathcal{H}} := [L(w_K w_\circ \bullet \lambda)], \quad T_K^{\mathcal{H}} := [\mathbb{M}(w_K w_\circ \bullet \lambda, \mathcal{H}'_{w_K w_\circ \bullet \lambda})] \quad (6.20)$$

(the last two definitions extend (6.19)), where $w_\circ = w_{K_*}$ and $K \in [\lambda]_{\mathcal{H}}$. Then the “truncated” Kazhdan–Lusztig relations over W_{K_*} hold in $K_0(\mathcal{O}^{\mathcal{H}})$, i.e. in the space $\mathcal{H}^{\mathcal{H}}(W_{K_*})$ with $q = 1$:

$$T_K^{\mathcal{H}} = \sum_{K' \subseteq K : w_{K'} w_\circ \bullet \lambda \in [\lambda]_{\mathcal{H}}} C_{K'}^{\mathcal{H}}, \quad C_K^{\mathcal{H}} = \sum_{K' \subseteq K : w_{K'} w_\circ \bullet \lambda \in [\lambda]_{\mathcal{H}}} (-1)^{|K| - |K'|} T_{K'}^{\mathcal{H}}. \quad (6.21)$$

Proof. The first equation follows directly from Lemma 6.18(3). The subtlety here is that one is now working in the finite poset $[\lambda]_{\mathcal{H}}$ rather than the full block $[\lambda]$. Set $W_{K_*} \simeq S_2^{K_*}$ to be the Weyl group of the block $[\lambda]$, where one identifies

$$K \subseteq K_* \iff w_K = \prod_{k \in K} s_k \iff w_K w_\circ \bullet \lambda.$$

As the regular representation of $\mathcal{H}(W_{K_*})$ is the Grothendieck ring of the full block $\mathcal{O}^{[\lambda]} = \mathcal{O}^{\emptyset, [\lambda]}$, as above one works in the quotient space $\mathcal{H}^{\mathcal{H}}(W_{K_*})$ in (6.20).

With this modification in place, the rest is standard. The incidence algebra of functions $f : [\lambda]_{\mathcal{H}} \times [\lambda]_{\mathcal{H}} \rightarrow \mathbb{Z}$ (with $f(x, w) = 0$ if $x \not\leq w$) acts on the space of functions $\text{Fun}([\lambda]_{\mathcal{H}}, K_0(\mathcal{O}^{\mathcal{H}, [\lambda]}))$ via convolution. To show the second equation in (6.21), note that the first says: $\text{id}_C * \zeta = \text{id}_T$, with

$$\text{id}_C(K) := C_K^{\mathcal{H}}, \quad \text{id}_T(K) := T_K^{\mathcal{H}},$$

and $\zeta(K', K) = \mathbf{1}_{K' \subseteq K}$ the zeta function of the incidence algebra (whose convolution-inverse is precisely the Möbius function). The second equation in (6.21) now follows by Möbius inversion, noting that since $[\lambda]_{\mathcal{H}} \subseteq [\lambda]$ is upper-closed, it inherits the Möbius function $(-1)^{|K| - |K'|}$ of $[\lambda]$. \square

7. THEOREM E: CHARACTERS AND BGG RESOLUTIONS VIA THE PARABOLIC WEYL SEMIGROUP

In this concluding section, we initiate the study of characters of some of the modules in this work. Given that the characters of $M(\lambda)$ and $M(\lambda, J)$ are well understood – in fact, at the level of BGG resolutions – it is natural to seek the same for the more general class of higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$. We obtain such resolutions in two settings.

Fix a Kac–Moody \mathfrak{g} and a (highest) weight $\lambda \in \mathfrak{h}^*$ for this section. Given an independent subset $H \subseteq J_\lambda$, recall the weight λ_H and the lowering operator-product $\mathbf{f}_H^{(\lambda)}$ defined above:

$$\lambda_H := \lambda - \sum_{h \in H} (\langle \lambda, \alpha_h^\vee \rangle + 1) \alpha_h = (\prod_{h \in H} s_h) \bullet \lambda, \quad \mathbf{f}_H^{(\lambda)} := \prod_{h \in H} f_h^{(\lambda, \alpha_h^\vee) + 1}. \quad (7.1)$$

7.1. Setting 1: Pairwise orthogonal holes, “parabolic” Weyl group. We now turn to the first setting in which we compute $\text{ch } \mathbb{M}(\lambda, \mathcal{H})$: when the elements of \mathcal{H}^{\min} are pairwise orthogonal.

Theorem 7.1. *Fix Kac–Moody \mathfrak{g} , a weight $\lambda \in \mathfrak{h}^*$, and an upper-closed set $\mathcal{H} \subseteq \text{Indep}(J_\lambda)$ such that $\mathcal{H}^{\min} \subseteq \text{Indep}(J_\lambda)$ consists of pairwise orthogonal subsets, say H_1, \dots, H_k . Then the module $\mathbb{M}(\lambda, \mathcal{H}) = \mathbb{M}(\lambda, \mathcal{H}^{\min})$ has a BGG resolution*

$$0 \longrightarrow M_k \xrightarrow{d_k} M_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \mathbb{M}(\lambda, \mathcal{H}) \rightarrow 0. \quad (7.2)$$

Here, $M_k = M(\lambda_{H_1 \sqcup \dots \sqcup H_k})$ and $M_0 = M(\lambda)$ (so $\lambda_\emptyset = \lambda$), and more generally, M_t is the direct sum of the Verma modules $M(\lambda_{H_{i_1} \sqcup \dots \sqcup H_{i_t}})$ over all t -tuples of indices $1 \leq i_1 < \dots < i_t \leq k$.

As a consequence, the character of $\mathbb{M}(\lambda, \mathcal{H})$ is given by the “Weyl character formula”

$$\text{ch } \mathbb{M}(\lambda, \mathcal{H}) = \sum_{S \subseteq \{1, \dots, k\}} (-1)^{|S|} \text{ch } M(\lambda_{\sqcup_{i \in S} H_i}). \quad (7.3)$$

Remark 7.2. In the spirit of Remark 2.9, note that Section 6 worked over \mathfrak{g} of finite type, while the results before it were independent of which Kac–Moody quotient algebra $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g} \twoheadrightarrow \bar{\mathfrak{g}}$ (fixing a generalized Cartan matrix) was used. The formulas in this section, while true for each quotient \mathfrak{g} , do not necessarily give the same answers *across* varying \mathfrak{g} . This is because the character of the Verma module can depend on \mathfrak{g} , whereas its weights do not.

Example 7.3. In the fundamental example in this regard, $V_{00} = M(0, 0)/M(-2, -2)$ as in (1.8), the resolution (7.2) specializes to

$$0 \rightarrow M(-2, -2) \rightarrow M(0, 0) \rightarrow V_{00} \rightarrow 0.$$

□

As the proof of Theorem 7.1 reveals, the complex in (7.2) is a BGG resolution [5, 27], in which one is working with the finite type Weyl group $W(\mathfrak{sl}_2^{\oplus k}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus k}$. In fact the differentials d_t are defined (below) using the Bruhat order in this group. Moreover, if one considers the words

$$\mathbf{s}_{H_j} := \prod_{h \in H_j} s_h, \quad 1 \leq j \leq k \quad (7.4)$$

as “order 2 Coxeter generators”, then the dot-action of the “parabolic” Weyl subgroup

$$W_{\mathcal{H}} := \langle \mathbf{s}_{H_1}, \dots, \mathbf{s}_{H_k} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus k} \quad (\text{with “natural” length function } \ell_{\mathcal{H}} : W_{\mathcal{H}} \rightarrow \{0, \dots, k\}) \quad (7.5)$$

on λ yields precisely the highest weights λ_H that occur in the BGG resolution (7.2). And indeed, the final equation (7.5) above, brings us back full circle to the first equations in this paper – the Weyl–Kac character formulas (1.1) (1.2), (1.5) – via their $W_{\mathcal{H}}$ -analogue:

Corollary 7.4. *Given Kac–Moody \mathfrak{g} , $\lambda \in \mathfrak{h}^*$, and an upper-closed subset $\mathcal{H} \subseteq \text{Indep}(J_\lambda)$, suppose $\mathcal{H}^{\min} = \{H_1, \dots, H_k\}$ consists of pairwise orthogonal independent subsets of J_λ . Then*

$$\text{ch } \mathbb{M}(\lambda, \mathcal{H}) = \sum_{w \in W_{\mathcal{H}}} \frac{(-1)^{\ell_{\mathcal{H}}(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \quad (7.6)$$

Proof of Theorem 7.1. Briefly, (7.2) is the BGG resolution in the restricted (simpler) case of $\mathfrak{g} = \mathfrak{sl}_2^{\oplus k}$, and so is the Koszul resolution of $R/(Ry_1 + \dots + Ry_k)$ for $R = \mathbb{C}[y_1, \dots, y_k]$ – where $y_j := \mathbf{f}_{H_j}^{(\lambda)} \forall j$ – subsequently tensored with the free R -module $M = U\mathfrak{n}^-$. (Notice, this case is therefore easier than the proof for arbitrary Weyl groups in [5, 27], given the simpler underlying Weyl group).

We give details for the interested reader. Let $W_{\mathcal{H}} = (\mathbb{Z}/2\mathbb{Z})^{\oplus k} \xrightarrow{\psi} 2^{\{1, \dots, k\}}$, and via ψ write

$$W_{\mathcal{H}} = \{w_J := \prod_{j \in J} \mathbf{s}_{H_j} \mid J \subseteq \{1, \dots, k\}\}, \quad w_J w_K = w_{J \Delta K}.$$

Next, we note the unique (up to scalar) embeddings $\iota(J', J)$ of the various Verma modules in the resolution above, according to the poset structure of the subsets of $\{1, \dots, k\}$ under inclusion. Namely, if $J \subseteq J' \subseteq \{1, \dots, k\}$, and if we denote $H_J := \sqcup_{j \in J} H_j$, then

$$M(\lambda_{H_{J'}}) \hookrightarrow M(\lambda_{H_J}) \hookrightarrow M(\lambda)$$

with λ_H in (7.1). Concretely, choosing a highest weight vector $m_\lambda \in M(\lambda)_\lambda$, these embeddings are:

$$\iota(J', J) : U\mathfrak{g} \left(\mathbf{f}_{H_{J'} \setminus H_J}^{(\lambda)} \cdot \mathbf{f}_{H_J}^{(\lambda)} \cdot m_\lambda \right) \hookrightarrow U\mathfrak{g} \left(\mathbf{f}_{H_J}^{(\lambda)} \cdot m_\lambda \right) = U\mathfrak{g}(m_{\lambda_{H_J}}) \hookrightarrow U\mathfrak{g}(m_\lambda), \quad (7.7)$$

where we **define** $m_{\lambda_{H_J}} := \mathbf{f}_{H_J}^{(\lambda)} \cdot m_\lambda$ for all $J \subseteq \{1, \dots, k\}$. (Thus, $w_J \bullet \lambda_{H_K} = \lambda_{H_{J \Delta K}}$ for $J, K \subseteq \{1, \dots, k\}$.) Moreover, the above Verma submodules have the expected intersection:

$$J, K \subseteq \{1, \dots, k\} \implies M(\lambda_{H_J}) \cap M(\lambda_{H_K}) = M(\lambda_{H_{J \cup K}}) \quad (7.8)$$

as submodules of $M(\lambda)$, by using weight space decompositions and the PBW theorem. The elements $y_j := \mathbf{f}_{H_j}^{(\lambda)}$ commute pairwise, and will be used to define – via the formulas as in the Koszul resolution for $R/(Ry_1 + \dots + Ry_k)$ above – the differentials d_t , or more precisely, their coordinates. Namely,

$$d_1 \left(\sum_{j=1}^k X_j m_{\lambda_{H_j}} \right) := \sum_{j=1}^k X_j \mathbf{f}_{H_j}^{(\lambda)} \cdot m_\lambda = \sum_{j=1}^k X_j y_j \cdot m_\lambda, \quad X_j \in U\mathfrak{n}^- . \quad (7.9)$$

Next if $t > 1$, then $d_t(J', J) : M(\lambda_{H_{J'}}) \rightarrow M(\lambda_{H_J})$ is zero unless $J \subseteq J'$ with $t = |J'| = |J| + 1$, in which case

$$\begin{aligned} J' &= \{i_1, \dots, i_t : 1 \leq i_1 < \dots < i_t \leq k\}, \quad X \in U\mathfrak{n}^- \\ \implies d_t \left(X m_{\lambda_{H_{J'}}} \right) &:= X \sum_{j=1}^t (-1)^{j-1} \mathbf{f}_{H_{i_j}}^{(\lambda)} m_{\lambda_{H_{J' \setminus \{i_j\}}}} = X \sum_{j=1}^t (-1)^{j-1} y_{i_j} m_{\lambda_{H_{J' \setminus \{i_j\}}}} . \end{aligned} \quad (7.10)$$

Observe that (7.9), (7.10) are precisely the formulas for the differentials in the Koszul complex

$$0 \longrightarrow R \xrightarrow{d_k} R^{\binom{k}{k-1}} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_3} R^{\binom{k}{2}} \xrightarrow{d_2} R^k \xrightarrow{d_1} R \longrightarrow 0, \quad (7.11)$$

with $R = \mathbb{C}[y_1, \dots, y_k]$, and under the identification wherein the free module $R^{\binom{k}{t}}$ has basis

$$\left\{ m_{\lambda_{H_{J'}}} : J' = \{i_1, \dots, i_t : 1 \leq i_1 < \dots < i_t \leq k\} \right\} .$$

Also note the same intersection property as (7.8):

$$J, K \subseteq \{1, \dots, k\} \implies R \prod_{t \in J} y_t \cap R \prod_{t \in K} y_t = R \prod_{t \in J \cup K} y_t .$$

Finally, transfer the Koszul complex from (7.11) to (7.2), in the usual manner. Define the R -module $M := U\mathfrak{n}^-$; by the PBW theorem this is free over the polynomial algebra $\mathbb{C}[\{f_h : h \in \sqcup_j H_j\}]$, which is in turn free over $R = \mathbb{C}[\{\mathbf{f}_{H_j}^{(\lambda)} : 1 \leq j \leq k\}]$. Thus, one tensors the resolution (7.11) with M to obtain the BGG complex in (7.2), with the specified differential maps. (Strictly speaking, one obtains (7.11) with R replaced by $M = U\mathfrak{n}^-$, and this is isomorphic as free $U\mathfrak{n}^-$ -modules to the complex in (7.2).) The R -freeness implies that (7.2) is indeed the desired resolution for $\mathbb{M}(\lambda, \mathcal{H})$. \square

7.2. Higher order Weyl group action on characters. As is well known, in a parabolic category $\mathcal{O}^{\mathbf{p}_J}$, the character of any object is W_J -invariant – or as we now understand in the language of holes, invariant under the minimal hole reflections $s_j, j \in J$. We now explain a sense in which this phenomenon generalizes to all holes – when applied to the higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$.

We begin by first explaining not the invariance, but the *partial action* of the Weyl group on the weights of a highest weight module over Kac–Moody \mathfrak{g} . Consider once again the basic example $V_{00} = M(0, 0)/M(-2, -2)$ over $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. This is a length 3 highest weight module, and the unique hole here is $H = \{1, 2\}$. Now $s_1 s_2$ fixes the weight 0, but takes every other weight of

V_{00} to a non-weight. What does hold is that W stabilizes the weights of $L(0, 0)$; $W_{\{1\}} = \{1, s_1\}$ stabilizes the weights of $L(s_2 \bullet 0) = M(s_2 \bullet 0, \{1\})$; and similarly for $L(s_1 \bullet 0)$. Viewed differently, $\text{wt } V_{00} = \text{wt } M(\lambda, \{1\}) \cup \text{wt } M(\lambda, \{2\})$, and these are stable under the action of $W_{\{1\}}$ and $W_{\{2\}}$, respectively. (Hence their intersection $\text{wt } L(0, 0)$ is W -stable.)

The situation is similar in general, via Theorem A. Let \mathfrak{g} be a Kac–Moody algebra, and M an object in \mathcal{O} . Then M has a finite filtration by highest weight modules, say $M(\lambda_i) \twoheadrightarrow V_i$, and so

$$\text{wt } M = \bigcup_{i \geq 0} \text{wt } V_i = \bigcup_{i \geq 0} \bigcup_{J \subseteq J_{\lambda_i} : J \cap H \neq \emptyset \forall H \in \mathcal{H}_{V_i}} \text{wt } M(\lambda_i, J)$$

by Theorem A. The partial action now says that part of W acts on part of $\text{wt } M$. Namely, given $\mu \in \text{wt } M$, the orbit $W_J(\mu) \subseteq M$ still, whenever $\mu \in \text{wt } M(\lambda_i, J)$ for some (i, J) .

As a special case, if $M \in \mathcal{O}^{p, J'}$ for some $J' \subseteq I$, then in the above union every J satisfies: $J \cap \{j'\} \neq \emptyset$ for $j' \in J'$, and so $J \supseteq J'$ – which implies that $\text{wt } M$ is $W_{J'}$ -stable. In contrast, there need not be any global symmetries in the higher order case. E.g. for $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ one has $M := L(s_1 \bullet 0) \oplus L(s_2 \bullet 0) \in \mathcal{O}^{\{1, 2\}}$ – but $\text{wt } M$ is stable only under $W_{\{1\}} \cap W_{\{2\}}$, i.e. the identity.

Having discussed weights, we turn to characters. We return to the opening paragraph of this subsection, and attempt to generalize it to higher order holes. Begin once again with the above example V_{00} over $\mathfrak{sl}_2^{\oplus 2}$. In this case, the unique hole is $\{1, 2\}$, and one seeks to understand if (and how) $s_1 s_2$ preserves the character of V_{00} . The immediate approach would be to evaluate $s_1 s_2 (e^{(0,0)} + \sum_{n > 0} (e^{-n\alpha_1} + e^{-n\alpha_2}))$, and it is easy to check this does not leave the character unchanged. Instead, one needs to rewrite the character as

$$\text{ch } V_{00} = \frac{e^{(0,0)}}{(1 - e^{-\alpha_1})(1 - e^{-\alpha_2})} - \frac{e^{(-2,-2)}}{(1 - e^{-\alpha_1})(1 - e^{-\alpha_2})}.$$

Now acting on both numerators and both denominators by $s_1 s_2$ leaves this expression unchanged. This happens due to the “correct” way of expanding both ratios (after applying $s_1 s_2$) – via their “highest weight expansions”. As an illustration, if α is a positive root in a Kac–Moody algebra, then $(1 - e^{-\alpha})^{-1}$ equals $1 + e^{-\alpha} + e^{-2\alpha} + \dots$, whereas $(1 - e^{\alpha})^{-1}$ is expanded differently:

$$\frac{1}{1 - e^{\alpha}} = \frac{e^{-\alpha}}{e^{-\alpha} - 1} = -e^{-\alpha}(1 + e^{-\alpha} + e^{-2\alpha} + \dots).$$

This is originally due to Brion [7] (for rational polytopes) and Khovanskii–Pukhlikov [25], Lawrence [28], and Varchenko [38] – see Postnikov [31] for deformation arguments for generic nodes.

The invariance of the character of the higher (2nd) order Verma module V_{00} under the subgroup $\{1, s_1 s_2\}$ is a “higher order” version of the W_J -invariance of the character of the module $M(\lambda, J)$. We now extend the former phenomenon and parallel the latter, in the situation discussed above:

Proposition 7.5. *Let \mathfrak{g} , $\lambda \in \mathfrak{h}^*$, and $\mathcal{H}^{\min} = \{H_1, \dots, H_k\} \subseteq \text{Indep}(J_{\lambda})$ be as in Theorem 7.1 – as also the subgroup $W_{\mathcal{H}} \simeq (\mathbb{Z}/2\mathbb{Z})^k$ of W . Then,*

$$w(\text{ch } \mathbb{M}(\lambda, \mathcal{H})) = (-1)^{\ell(w) - \ell_{\mathcal{H}}(w)} \text{ch } \mathbb{M}(\lambda, \mathcal{H}), \quad \forall w \in W_{\mathcal{H}}. \quad (7.12)$$

Proof. With notation as in (7.5), write $W_{\mathcal{H}} = \{w_J = \prod_{j \in J} \mathbf{s}_{H_j} \mid J \subseteq \{1, \dots, k\}\}$. Now compute, starting from the Weyl–Kac type character formula (7.6):

$$\begin{aligned} w_K(\text{ch } \mathbb{M}(\lambda, \mathcal{H})) &= \sum_{J \subseteq \{1, \dots, k\}} w_K \left(\frac{(-1)^{|J|} e^{w_J \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_{\alpha}}} \right) \\ &= (-1)^{\ell(w_K)} \sum_{J \subseteq \{1, \dots, k\}} \frac{(-1)^{|J|} e^{(w_K w_J) \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_{\alpha}}} = (-1)^{\ell(w_K)} \cdot (-1)^{|K|} \text{ch } \mathbb{M}(\lambda, \mathcal{H}), \end{aligned}$$

since $(-1)^{|K \Delta J|} = (-1)^{|K|} (-1)^{|J|}$. This concludes the proof. \square

Remark 7.6. Suppose \mathfrak{g} is semisimple, and $\mathcal{H} \subseteq \text{Indep}(I)$ is such that the minimal sets in \mathcal{H}^{\min} are pairwise orthogonal. We explain a sense in which Proposition 7.5 extends to $\mathcal{O}^{\mathcal{H}}$ the W_J -invariance of $\text{ch } M$ for all objects M in the parabolic category $\mathcal{O}^{\mathfrak{p}_J}$. Indeed, via a triangular change of bases, the W_J -invariance of all characters in $\mathcal{O}^{\mathfrak{p}_J}$, is K_0 -equivalent to that of $\text{ch } M(\lambda, J)$ for all λ with $L(\lambda) \in \mathcal{O}^{\mathfrak{p}_J}$. The higher order analogue of this is given by Proposition 7.5 (via Proposition 6.4(2)):

If $L(\lambda) \in \mathcal{O}^{\mathcal{H}}$, then the character of $\mathbb{M}(\lambda, \mathcal{H})$ satisfies (7.12) with $W_{\mathcal{H}}$ replaced by $W_{\mathcal{H}'_{\lambda}}$, since if $\mathcal{H} = \mathcal{H}_J$ and $L(\lambda) \in \mathcal{O}^{\mathcal{H}} = \mathcal{O}^{\mathfrak{p}_J}$, then $\mathcal{H}'_{\lambda} = \mathcal{H}_J$ and $\ell_{\mathcal{H}_J} = \ell$.

7.3. Setting 2: Pairwise orthogonal integrable roots, and the parabolic Weyl semigroup. We next prove BGG resolutions and character formulas for the modules $\mathbb{M}(\lambda, \mathcal{H})$ in another setting: for $\mathfrak{g}_{J_{\lambda}} = \mathfrak{sl}_2^{\oplus n}$ for some $n \geq 1$. In this case, the BGG resolution turns out to involve a semigroup action of $W_{\mathcal{H}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus k}$ (as sets, with $k = |\mathcal{H}^{\min}|$) on λ , as we first explain for $k = 2$:

Theorem 7.7. Suppose $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$ for some $n \geq 1$, $\lambda \in \mathfrak{h}^*$, and $\mathcal{H}^{\min} = \{H_1, H_2\} \subseteq \text{Indep}(J_{\lambda}) = 2^{J_{\lambda}}$. Using the notation of $\lambda_H, \mathbf{f}_H^{(\lambda)}$ as in (7.1), the module $\mathbb{M}(\lambda, \mathcal{H}^{\min})$ has a BGG resolution

$$0 \longrightarrow M(\lambda_{H_1 \cup H_2}) \xrightarrow{d_2} M(\lambda_{H_1}) \oplus M(\lambda_{H_2}) \xrightarrow{d_1} M(\lambda) \xrightarrow{d_0} \mathbb{M}(\lambda, \mathcal{H}^{\min}) \rightarrow 0, \quad (7.13)$$

where the “Koszul-type” differentials are given by

$$\begin{aligned} d_1(X_1 m_{\lambda_{H_1}}, X_2 m_{\lambda_{H_2}}) &:= \left(X_1 \mathbf{f}_{H_1}^{(\lambda)} + X_2 \mathbf{f}_{H_2}^{(\lambda)} \right) m_{\lambda}, \\ d_2(X m_{\lambda_{H_1 \cup H_2}}) &:= (-X \mathbf{f}_{H_2 \setminus H_1}^{(\lambda)} m_{\lambda_{H_1}}, X \mathbf{f}_{H_1 \setminus H_2}^{(\lambda)} m_{\lambda_{H_2}}), \quad X, X_1, X_2 \in U\mathfrak{n}^-. \end{aligned}$$

This result can be verified by hand, and gives $\text{ch } \mathbb{M}(\lambda, \mathcal{H}^{\min})$ as the Euler characteristic. It also yields the same Weyl character formula as previously, involving $w \bullet \lambda$ for w in the set

$$W_{\mathcal{H}} = W_{\mathcal{H}^{\min}} := \{1, w_{\{1\}} = s_{H_1}, w_{\{2\}} = s_{H_2}, w_{\circ} = w_{\{1,2\}} = s_{H_1 \cup H_2}\}.$$

Define the associated length function $\ell_{\mathcal{H}} = \ell_{\mathcal{H}^{\min}}$, which sends $1 \mapsto 0$, $w_{\{j\}} \mapsto 1$, and $w_{\{1,2\}} \mapsto 2$.

Corollary 7.8. Notation as above. Then:

$$\text{ch } \mathbb{M}(\lambda, \mathcal{H}^{\min}) = \sum_{w \in W_{\mathcal{H}}} \frac{(-1)^{\ell_{\mathcal{H}}(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_{\alpha}}}. \quad (7.14)$$

Remark 7.9. While the Weyl character formula is unchanged from (7.6), the action of $W_{\mathcal{H}}$ on the orbit of λ is now different than in the previous case (7.2). (This action is useful in understanding the differential maps, which differ even for $\mathcal{H}^{\min} = \{H_1, H_2\}$ over $\mathfrak{g} = \mathfrak{sl}_2^{\oplus n}$.) Indeed, now we use

$$w_J \cdot' w_K := w_{J \cup K}, \quad w_J \bullet' \lambda_{H_K} := \lambda_{H_{J \cup K}} = \lambda_{H_J \cup H_K}, \quad \forall J, K \subseteq \{1, 2\}.$$

Thus, $(W_{\mathcal{H}}, \cdot')$ is what we term the *parabolic Weyl semigroup* in this situation. Moreover, the map \bullet' is a semigroup action of $(W_{\mathcal{H}}, \cdot')$ on the orbit $\{\lambda, \lambda_{H_1}, \lambda_{H_2}, \lambda_{H_1 \cup H_2}\}$. Of course, in the character formula (7.14), the only element on which $W_{\mathcal{H}}$ acts is λ itself, which is dominant integral for $\Delta_{H_1 \cup H_2}$, and so in those equations $\bullet' = \bullet$.

Example 7.10. For a concrete working example, the reader can consider e.g. $\mathfrak{g} = \mathfrak{sl}_2^{\oplus 3}$, $I = \{1, 2, 3\}$, $\lambda \in P^+$, and $\mathcal{H}^{\min} = \{\{1, 2\}, \{2, 3\}\}$. The overlap between elements of \mathcal{H}^{\min} is what leads to the parabolic Weyl semigroup $W_{\mathcal{H}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ here (as sets), and its orbit is $W_{\mathcal{H}} \bullet' \lambda$.

We now write down the analogous picture – over \mathfrak{g} with orthogonal integrable roots for $\lambda \in \mathfrak{h}^*$, i.e., $\mathfrak{g}_{J_{\lambda}} = \mathfrak{sl}_2^{\oplus n}$ – with $\mathcal{H}^{\min} = \{H_1, \dots, H_k\} \subseteq \text{Indep}(J_{\lambda})$. Our BGG-type resolution again turns out to yield the same character formula (7.14); however, one now requires alternate notation from the earlier one, as is revealed by the simple case

$$\mathfrak{g} = \mathfrak{sl}_2^{\oplus 3}, \quad \mathcal{H}^{\min} = \{H_1 = \{1, 2\}, H_2 = \{1, 3\}, H_3 = \{2, 3\}\}. \quad (7.15)$$

In this case, the terms $M_1 \rightarrow M_0 = M(\lambda) \rightarrow \mathbb{M}(\lambda, \mathcal{H}^{\min})$ are as above, while in the above notation,

$$\lambda_{H_{\{1,2\}}} = \lambda_{H_{\{1,3\}}} = \lambda_{H_{\{2,3\}}} = \lambda_{H_{\{1,2,3\}}} = s_1 s_2 s_3 \bullet \lambda.$$

To distinguish between the corresponding four Verma modules occurring in M_2 and M_3 , define

$$\lambda(J) := \lambda_{H_J} = \lambda - \sum_{h \in \cup_{j \in J} H_j} (\langle \lambda, \alpha_h^\vee \rangle + 1) \alpha_h \in \lambda - \mathbb{Z}_{\geq 0} \Pi_{J_\lambda}, \quad J \subseteq \{1, \dots, k\}. \quad (7.16)$$

Choosing a maximal vector $m_\lambda \in M(\lambda)_\lambda$, the modules $M(\lambda(J))$ then again embed into one another via the maps $\iota(J', J)$ for $J \subseteq J' \subseteq \{1, \dots, k\}$ as in (7.7), and via this embedding into $M(\lambda)$, have intersections as in (7.8). Thus, also fix maximal vectors $m_{\lambda(J)} \in M(\lambda(J))_{\lambda(J)}$ such that $\iota(J', J)$ sends $m_{\lambda(J')}$ to $\mathbf{f}_{H_{J'} \setminus H_J}^{(\lambda)} m_{\lambda(J)}$ for all $J \subseteq J' \subseteq \{1, \dots, k\}$. Now define the modules

$$M_t := \bigoplus_{J \subseteq \{1, \dots, k\}, |J|=t} M(\lambda(J)), \quad 0 \leq t \leq k.$$

Also define the differential $d_1 : M_1 \rightarrow M_0$ via

$$d_1 \left(\sum_{j=1}^k X_j m_{\lambda(\{j\})} \right) := \sum_{j=1}^k X_j \mathbf{f}_{H_j}^{(\lambda)} m_\lambda, \quad X_j \in U\mathfrak{n}^-$$

and the differential d_t , $t > 1$ via its coordinates. Namely, $d_t(J', J) : M(\lambda(J')) \rightarrow M(\lambda(J))$ is zero unless $J \subseteq J'$ with $t = |J'| = |J| + 1$, in which case

$$\begin{aligned} J' &= \{i_1, \dots, i_t : 1 \leq i_1 < \dots < i_t \leq k\}, \quad X \in U\mathfrak{n}^- \\ \implies d_t(X m_{\lambda(J')}) &:= X \sum_{j=1}^t (-1)^{j-1} \mathbf{f}_{H_{i_j} \setminus H_{J' \setminus \{i_j\}}}^{(\lambda)} m_{\lambda(J' \setminus \{i_j\})}. \end{aligned} \quad (7.17)$$

Remark 7.11. The modules M_t and differentials d_t indeed specialize to their counterparts in Theorem 7.1 as well as in Theorem 7.7, earlier in this section.

As above, we now show this yields a resolution; notice that in conjunction with Theorem 7.1 and Proposition 7.5, this resolution would also imply our main theorem E.

Theorem 7.12. *Fix Kac-Moody \mathfrak{g} and $\lambda \in \mathfrak{h}^*$ such that the nodes J_λ have no edges, and let $\mathcal{H}^{\min} = \{H_1, \dots, H_k\} \subseteq \text{Indep}(J_\lambda)$. With the modules M_t and differentials d_t as above, the complex*

$$0 \longrightarrow M_k \xrightarrow{d_k} M_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \mathbb{M}(\lambda, \mathcal{H}) \rightarrow 0 \quad (7.18)$$

is a free $U\mathfrak{n}^-$ -resolution of $\mathbb{M}(\lambda, \mathcal{H})$. As a consequence, with $\lambda(J)$ as in (7.16),

$$\text{ch } \mathbb{M}(\lambda, \mathcal{H}) = \sum_{J \subseteq \{1, \dots, k\}} (-1)^{|J|} \text{ch } M(\lambda(J)). \quad (7.19)$$

Proof. First consider the variant of this complex over $\mathfrak{g}_{J_\lambda} \simeq \mathfrak{sl}_2^{\oplus n}$. Writing $\mathbf{f}_{H_{i_j} \setminus H_{J' \setminus \{i_j\}}}^{(\lambda)}$ as $\mathbf{f}_{H_{i_j} \setminus H_{J' \setminus \{i_j\}}}^{(\lambda)} = \frac{\text{lcm}\{\mathbf{f}_{H_i}^{(\lambda)} : i \in J'\}}{\text{lcm}\{\mathbf{f}_{H_i}^{(\lambda)} : i \in J', i \neq i_j\}}$, (7.18) is precisely the Taylor resolution [36] (see also [13]) for $\mathbb{M}(\lambda, \mathcal{H})$ over the commutative ring – in fact UFD – $R := \mathbb{C}[\{f_i : i \in J_\lambda\}]$. Now the result follows over $U\mathfrak{g}$ by tensoring this Taylor resolution with the R -module $U\mathfrak{n}^-$, which is R -free by the PBW theorem. \square

In particular, the Weyl character formula again follows from this resolution, as for $k = 2$. Let

$$W_{\mathcal{H}} = \{(w_J, J) \mid J \subseteq \{1, \dots, k\}\}, \quad w_J := \mathbf{s}_{\cup_{j \in J} H_j}, \quad (w_J, J) \cdot' (w_K, K) := (w_{J \cup K}, J \cup K)$$

for $J, K \subseteq \{1, \dots, k\}$ denote the **parabolic Weyl semigroup** in this situation. Define the length of (w_J, J) to be $\ell_{\mathcal{H}}((w_J, J)) := |J|$. Then $(W_{\mathcal{H}}, \cdot')$ acts on the orbit $\{\lambda_{H_J} : J \subseteq \{1, \dots, k\}\}$ (with $\lambda_{H_\emptyset} = \lambda$) via: $(w_J, J) \bullet' \lambda_{H_K} := \lambda_{H_{J \cup K}}$. On λ , this is the dot-action: $(w_J, J) \bullet' \lambda = w_J \bullet \lambda \ \forall J$.

Corollary 7.13. *The resolution (7.18) once again implies, for arbitrary \mathcal{H}^{\min} of size $k \geq 1$:*

$$\text{ch } \mathbb{M}(\lambda, \mathcal{H}) = \sum_{w=(w_J, J) \in W_{\mathcal{H}}} \frac{(-1)^{\ell_{\mathcal{H}}(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \quad (7.20)$$

7.4. Resolutions over dihedral groups. Having obtained BGG resolutions and Weyl character formulas in the above two settings using $W_{\mathcal{H}}$, we briefly discuss another potentially simple setting in which such formulas and resolutions can be explored. We begin with a small lemma on Coxeter generators in finite Weyl groups. As used in Corollary 7.4, if two sets $H_1, H_2 \subseteq J_\lambda$ of independent nodes are orthogonal, then $(\mathbf{s}_{H_1} \mathbf{s}_{H_2})^2 = 1$ in W . The next lemma computes this order in the more general case when H_1, H_2 are merely pairwise disjoint. Even more generally:

Lemma 7.14. *Suppose \mathfrak{g} is of finite type, and $H_1, \dots, H_k \in \text{Indep}(I)$ are pairwise disjoint. Let $H_1 \sqcup \dots \sqcup H_k$ have connected Dynkin components J_1, \dots, J_l . Then the product $\mathbf{s}_{H_1} \cdots \mathbf{s}_{H_k}$ has order precisely $\text{lcm}(c_1, \dots, c_l)$, where c_t is the Coxeter number of the parabolic Weyl subgroup W_{J_t} .*

Proof. For $1 \leq i \leq k$ and $1 \leq t \leq l$, define $J_{it} := H_i \cap J_t$. It is clear that $\prod_{i=1}^k \mathbf{s}_{H_i} = \prod_{t=1}^l \mathbf{s}_t$, where $\mathbf{s}_t := \mathbf{s}_{J_{1t}} \cdots \mathbf{s}_{J_{kt}}$ are pairwise commuting. Hence the order of $\mathbf{s}_{H_1} \cdots \mathbf{s}_{H_k}$ is the lcm of the orders of the \mathbf{s}_t . But each \mathbf{s}_t is a Coxeter element for the Weyl group on $\sqcup_i J_{it} = J_t$, hence has order c_t . \square

Remark 7.15. The assumption of finite type is needed in Lemma 7.14, because if W is an irreducible, infinite Coxeter group then its Coxeter elements have infinite order [19, 34].

As an “application”, let \mathfrak{g} be of finite type and $\mathcal{H}^{\min} = \{H_1, H_2\}$, with $H_1, H_2 \in \text{Indep}(J_\lambda)$ disjoint subsets of nodes. The corresponding subgroup $W_{\mathcal{H}} = \langle \mathbf{s}_{H_1}, \mathbf{s}_{H_2} \rangle$ is then dihedral in these two Coxeter generators, with the longest word in them given by \mathbf{w}_\circ , say. If $\mathbf{s}_{H_1} \mathbf{s}_{H_2}$ has order $m \geq 2$ (computed via Lemma 7.14), then we would expect the BGG resolution to be of the form

$$\begin{aligned} 0 \longrightarrow M(\mathbf{w}_\circ \bullet \lambda) &\xrightarrow{d_m} M(\mathbf{s}_{H_1} \mathbf{w}_\circ \bullet \lambda) \oplus M(\mathbf{s}_{H_2} \mathbf{w}_\circ \bullet \lambda) \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_2} \\ &\xrightarrow{d_2} M(\mathbf{s}_{H_1} \bullet \lambda) \oplus M(\mathbf{s}_{H_2} \bullet \lambda) \xrightarrow{d_1} M(\lambda) \xrightarrow{d_0} \mathbb{M}(\lambda, \mathcal{H}) \longrightarrow 0. \end{aligned} \quad (7.21)$$

Moreover, such a resolution would again lead to a Weyl character formula akin to (7.20).

Example 7.16. For additional intuition about such a resolution, we discuss the simplest “nontrivial” case. Let $\mathfrak{g} = \mathfrak{sl}_4$, $\lambda = 0$, and $\mathcal{H} = \{H_1 = \{1, 3\}, H_2 = \{2\}\}$, so that $\mathbf{s}_{H_1} \mathbf{s}_{H_2}$ has order 4 by Lemma 7.14. (Here, $\mathbf{s}_{H_1} = s_1 s_3$ and $\mathbf{s}_{H_2} = s_2$.) Then does $\mathbb{M}(0, \mathcal{H})$ have a resolution of the form

$$\begin{aligned} 0 \longrightarrow M(\mathbf{w}_\circ \bullet 0) &\xrightarrow{d_4} M(\mathbf{s}_{H_1} \mathbf{w}_\circ \bullet 0) \oplus M(\mathbf{s}_{H_2} \mathbf{w}_\circ \bullet 0) \xrightarrow{d_3} M(\mathbf{s}_{H_1} \mathbf{s}_{H_2} \bullet 0) \oplus M(\mathbf{s}_{H_2} \mathbf{s}_{H_1} \bullet 0) \\ &\xrightarrow{d_2} M(\mathbf{s}_{H_1} \bullet 0) \oplus M(\mathbf{s}_{H_2} \bullet 0) \xrightarrow{d_1} M(0) \xrightarrow{d_0} \mathbb{M}(0, \mathcal{H}) \longrightarrow 0, \end{aligned}$$

where $\mathbf{w}_\circ = \mathbf{s}_{H_1} \mathbf{s}_{H_2} \mathbf{s}_{H_1} \mathbf{s}_{H_2}$, and the differentials need to be worked out using the relations in $U(\mathfrak{sl}_4)$. For instance, $d_1 : (X_1 m_{\mathbf{s}_{H_1} \bullet 0}, X_2 m_{\mathbf{s}_{H_2} \bullet 0}) \mapsto (X_1 f_1 f_3 + X_2 f_2) m_0$ for $X_1, X_2 \in U\mathfrak{n}^-$ with some fixed choice of highest weight vectors m , and similarly, one coordinate of d_2 is given by

$$d_2(X m_{\mathbf{s}_{H_1} \mathbf{s}_{H_2} \bullet 0}) := (X(f_2 f_1 f_3 + 2f_{12} f_3 - 2f_{23} f_1 + 2f_{123}) m_{\mathbf{s}_{H_1} \bullet 0}, -X f_1^2 f_3^2 m_{\mathbf{s}_{H_2} \bullet 0}),$$

where $f_{12} = [f_1, f_2]$, $f_{23} = [f_2, f_3]$, f_{123} are the non-simple negative root vectors, and $X \in U\mathfrak{n}^-$. Note, for all other subsets $\mathcal{H}' \subseteq \text{Indep}(I)$ for $\mathfrak{g} = \mathfrak{sl}_4, \lambda = 0$, a BGG resolution of $\mathbb{M}(0, \mathcal{H}')$ is known.

Remark 7.17. Given the expected resolution (7.21) for two minimal holes, one can ask what happens when \mathcal{H} has ≥ 3 minimal holes. This is not always “finite Coxeter”, as we explain.

Given (7.21), a natural question is if all such resolutions involve a finite Weyl/Coxeter group in *every* situation over \mathfrak{g} of finite type. In order to avoid Weyl semigroups arising due to overlap of nodes in holes (see Remark 7.9), one can restrict to the case of say disjoint holes in \mathcal{H} . Even with these constraints, we now explain why the question above cannot have a uniformly positive answer – even with three holes and \mathfrak{g} of type A . Indeed, suppose

$$\mathfrak{g} = \mathfrak{sl}_5, \quad \lambda \in P^+, \quad \mathcal{H} = \{\{1\}, \{3\}, \{2, 4\}\}.$$

Then the subgroup $W_{\mathcal{H}}$ of $W \simeq S_5$ generated by $s_1, s_3, t_2 := s_2s_4$ has at least the relations

$$(s_1t_2)^6 = (s_1s_3)^2 = (t_2s_3)^4 = 1$$

by Lemma 7.14, and one would like to know if these relations give a Coxeter presentation of $W_{\mathcal{H}}$. But this is necessarily false, since the only finite Coxeter group with connected underlying Coxeter graph containing an edge labeled 6, is a dihedral group [11, 12] – with 2 Coxeter generators.

7.5. Concluding remarks and questions. Looking back – in the course of this paper, we have uncovered a host of objects and properties: (a) The higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$ – whose weight-sets have closed-form expressions and Minkowski sum decompositions, and comprise the weights of all highest weight \mathfrak{g} -modules – and moreover, these modules themselves comprise all highest weight $\mathfrak{sl}_2^{\oplus n}$ -modules. (b) BGG resolutions and $W_{\mathcal{H}}$ -invariant characters of these modules. (c) k th order upper- and lower-approximations of all highest weight modules. (d) The k th order integrability of highest weight modules, leading to iteratively stratifying the quotients of each $M(\lambda)$, each time into intervals. (e) Category $\mathcal{O}^{\mathcal{H}}$, with enough projectives and BGG-type reciprocity.

All of these are natural extensions of their zeroth and first order versions in the literature – $M(\lambda), M(\lambda, J)$; their characters and resolutions; $\mathcal{O}, \mathcal{O}^{p_J}$ – i.e., they “occur in nature”, and are not “artificial” constructs. Yet, surprisingly, to our knowledge they were unexplored to date. The present paper provides information on the above objects/properties to varying degrees of completeness – from Theorem A to Theorem E. We end this paper with a few questions that are natural to explore going ahead, given the above results.

- (1) Much is understood about the first order approximations of highest weight modules, but less is known about the higher order analogues. Thus, a natural question is to better understand the higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$ for all $\mathfrak{g}, \lambda, \mathcal{H}$. Can one “upgrade” the weight-sets of the modules $\mathbb{M}(\lambda, \mathcal{H})$ to their characters, at once extending the classical formulas (1.1), (1.6) for parabolic Verma modules as well as (7.6) and (7.20)?
- (2) Second, do the modules $\mathbb{M}(\lambda, \mathcal{H})$ occur at the end of a BGG-type resolution that uses Verma modules? More concretely, in what generality in $(\mathfrak{g}, \lambda, \mathcal{H})$ can one show a resolution of $\mathbb{M}(\lambda, \mathcal{H})$, which at once extends the usual BGG resolution for $M(\lambda, J)$ as well as (7.2) and (7.18)? (Informally, can one extend these Koszul–Taylor resolutions from $(\mathbb{Z}/2\mathbb{Z})^{\oplus n}$ to other Weyl groups.) And is (7.21) true?
- (3) Is there a general version of the parabolic Weyl (semi)group W_J or $W_{\mathcal{H}}$, and of its action \bullet' on the orbit of λ , that works in the cases in (2) above? Does this (semi)group have a Coxeter-type enumeration as words in the generators $\{\mathbf{s}_H : H \in \mathcal{H}^{\min}\}$, which act on λ ? And can Proposition 7.5 – or a refinement thereof – be proved for more general $\mathfrak{g}, \lambda, \mathcal{H}$? (Note, it also holds in all parabolic categories \mathcal{O}^{p_J} .)
- (4) Another follow-up is to interpret the modules $\mathbb{M}(\lambda, \mathcal{H})$ and their resolutions, geometrically on the flag variety – then extend this to more general V , and to the categories $\mathcal{O}^{\mathcal{H}}$.
- (5) Extend the treatment of \mathcal{O}^{p_J} in the literature (see e.g. [20, 21, 32]) to the higher order parabolic categories $\mathcal{O}^{\mathcal{H}}$, for all $\mathcal{H} \subseteq \text{Indep}(I)$. Natural questions include:
 - (a) *BGG-type reciprocity*: Extending Theorem 6.17 to arbitrary $\mathcal{O}^{\mathcal{H}}$ over all \mathfrak{g} .
 - (b) *Parabolic Verma modules*: Finding “higher order Jantzen filtrations” for $\mathbb{M}(\lambda, \mathcal{H})$ (see [42, 43]). (Also note, if $\mathbb{M}(\lambda, \mathcal{H})$ is simple then it is a parabolic Verma module.)

(c) *Blocks:* Endomorphism rings of block-progenerators and other projectives in $\mathcal{O}^{\mathcal{H}}$, and their properties, including Koszulity and Koszul duals [2, 4, 33, 35]. Homomorphisms between higher order Vermas. Determining the block structure of $\mathcal{O}^{\mathcal{H},[\lambda]}$ (see [41]).

(6) This question is more speculative. It involves Theorem 5.5, which characterizes the first order integrability of a highest weight module V , i.e. the set J_V , in terms of the convex hull of the set of weights – or in other words, the extremal rays and edges of the W_{J_V} -invariant polyhedral shape $\text{conv}(\text{wt } V)$. Similarly, the zeroth order integrability of V is the vertex λ . It could be interesting to examine what “higher order geometric combinatorics” emerges out of the higher order integrabilities $(\mathcal{H}_V^{\min})_{\leq k}$ (see Section 5.2, including Definition 5.12).

In parallel, one can ask if Proposition 6.21 can be extended to all blocks $\mathcal{O}^{\mathcal{H},[\lambda]}$ over arbitrary \mathfrak{g} . Namely, if the categories $\mathcal{O}^{\mathcal{H}}$ lead to “refined” Kazhdan–Lusztig combinatorics and (quotient) spaces associated to Hecke modules – working internally in $\mathcal{O}^{\mathcal{H}}$ rather than first converting to Verma characters via BGG resolutions – which specialize to relative/parabolic versions (e.g. [9, 14]) at first order.

APPENDIX A. PROOF OF ENOUGH PROJECTIVES – DETAILS

Here we include details of the proof of a part of Theorem D: $\mathcal{O}^{\mathcal{H}}$ has enough projectives. As said above, we do so because there are multiple standard objects for the simple objects (which do get used in BGG reciprocity, see Section 6.3). We thus list and verify the five conditions in [4, Theorem 3.2.1]. The setting in [4] does hold: by Lemma 6.2, \mathcal{A} is an abelian subcategory of \mathbb{C} -vector spaces.

- (1) Every object of $\mathcal{A} \subseteq \mathcal{O}$ has finite length.
- (2) There are only finitely many simple isoclasses in $\mathcal{A} \subseteq \mathcal{O}^{W \bullet \lambda}$. Index these by S .
- (3) The endomorphisms of all simple objects in $\mathcal{A} \subseteq \mathcal{O}$ are scalars.

Moreover, $S \subseteq \mathfrak{h}^*$ has the partial order \leqslant , via: $\nu \leqslant \mu$ if $\mu - \nu \in \mathbb{Z}_{\geq 0} \Pi$. For every lower-closed subset $T \subseteq S$ (if $s \in T, s' \leqslant s$ then $s' \in T$), let \mathcal{A}_T be the full subcategory of objects in \mathcal{A} , all of whose simple subquotients are $L(\mu)$ with $\mu \in T$. E.g., $(\leqslant_\mu) := \{s \in S : s \leqslant \mu\}$ for $\mu \in S$, and similarly $(<_\mu)$. Finally, define the (co)standard objects in $\mathcal{A}_{(\leqslant_\mu)}$ for $\mu \in S$ via Proposition 6.4:

$$\mathcal{H}'_\mu := \{J_\mu \cap H : H \in \mathcal{H}^{\min}\}, \quad \Delta(\mu) := \mathbb{M}(\mu, \mathcal{H}'_\mu), \quad \nabla(\mu) := \mathbb{M}(\mu, \mathcal{H}'_\mu)^\vee. \quad (\text{A.1})$$

Moreover, since $L(\mu)^\vee \cong L(\mu)$, one has canonical maps (up to scaling)

$$\pi : \Delta(\mu) \rightarrow L(\mu), \quad \pi^\vee : L(\mu) \hookrightarrow \nabla(\mu).$$

- (5) The objects $K := \ker(\pi)$, $\text{coker}(\pi^\vee) = K^\vee$ lie in $\mathcal{A}_{(<_\mu)}$, in that their simple subquotients are all of the form $L(\nu)$ with $\nu < \mu$. (The assertion for K^\vee follows from that for K , since restricted duality is an exact contravariant functor on \mathcal{O} , hence on \mathcal{A} via Lemma 6.2.)

It remains to verify the fourth condition:

- (4) Fix a lower-closed subset $T \subseteq S$ and a maximal element $\mu \in (T, \leqslant)$. Then in \mathcal{A}_T , $\Delta(\mu) \rightarrow L(\mu)$ is a projective cover and $L(\mu) \hookrightarrow \nabla(\mu)$ is an injective hull.

The exactness and contravariance of restricted duality on \mathcal{A}_T shows that the assertion for $\nabla(\mu)$ follows from that for $\Delta(\mu)$. Thus, it suffices to show that $\Delta(\mu)$ is a projective cover of $L(\mu)$.

Begin by noting that $\Delta(\mu) = \mathbb{M}(\mu, \mathcal{H}'_\mu)$ is indecomposable. Moreover, $N := \ker(\pi : \Delta(\mu) \rightarrow L(\mu))$ is indeed a superfluous submodule of $\Delta(\mu)$, since it does not intersect the highest weight line $\Delta(\mu)_\mu$ which generates $\Delta(\mu)$. Thus, it remains to show that $\Delta(\mu)$ is a projective object in \mathcal{A}_T if μ is maximal in $(T, \leqslant) \subseteq (S, \leqslant)$. We do so by showing that the functor $\text{Hom}_{\mathcal{A}_T}(\Delta(\mu), -)$ is exact. More strongly, denote by M_μ the μ -weight space of M , and assert the functorial isomorphism

$$\text{Hom}_{\mathcal{A}_T}(\Delta(\mu), M) \cong M_\mu, \quad M \in \mathcal{A}_T.$$

Indeed, fix a highest weight vector $v_\mu \in \Delta(\mu)_\mu$, and send $\varphi \in \text{Hom}_{\mathcal{A}_T}(\Delta(\mu), M)$ to $\varphi(v_\mu) \in M_\mu$, for $M \in \mathcal{A}_T$. Clearly this assignment is linear and injective, and we now prove the surjectivity.

Given $M \in \mathcal{A}_T \subseteq \mathcal{O}^{\mathcal{H}} = \mathcal{O}^{\mathcal{H}^{\min}}$, choose $0 \neq m_\mu \in M_\mu$. We claim that the corresponding map

$$\varphi : \Delta(\mu) = \mathbb{M}(\mu, \mathcal{H}'_\mu) \rightarrow M, \quad F \cdot v_\mu \mapsto F \cdot m_\mu, \quad F \in U(\mathfrak{g})$$

is indeed a \mathfrak{g} -module map. Since μ is maximal in T , it suffices to verify that for all $H \in \mathcal{H}^{\min}$,

$$\mathbf{f}_{J_\mu \cap H}^{(\mu)} \cdot M_\mu = 0, \quad \text{where we recall,} \quad \mathbf{f}_{J_\mu \cap H}^{(\mu)} := \prod_{h \in J_\mu \cap H} f_h^{\langle \mu, \alpha_h^\vee \rangle + 1}. \quad (\text{A.2})$$

By the definition of $\mathcal{O}^{\mathcal{H}}$, $\mathbf{f}_H = \mathbf{f}_H^{(0)}$ acts nilpotently on $m_\mu \in M_\mu$, so there exists n such that

$$0 = \mathbf{f}_H^n v_\mu = \mathbf{f}_{J_\mu \cap H}^n \mathbf{f}_{H \setminus J_\mu}^n v_\mu.$$

Acting by e_h^n for all $h \in H \setminus J_\mu$, it follows via (6.2) as above that $\mathbf{f}_{J_\mu \cap H}^n m_\mu = 0$. Now define

$$m_h = \langle \mu, \alpha_h^\vee \rangle + 1, \quad \forall h \in J_\mu, \quad H_1 := \{h \in J_\mu \cap H : n \geq m_h\}.$$

Repeating the proof of Lemma 6.6, one obtains: (i) H_1 is nonempty, and (ii) $\mathbf{f}_{H_1}^{(\mu)} m_\mu = 0$. This implies (A.2); hence, $\Delta(\mu)$ is projective in \mathcal{A}_T . Thus by [4, Theorem 3.2.1], \mathcal{A} and hence $\mathcal{O}^{\mathcal{H}}$ has enough projectives. Using the properties of restricted duality, $\mathcal{O}^{\mathcal{H}}$ also has enough injectives. \square

REFERENCES

- [1] Michael F. Atiyah and Raoul Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math.*, 88:451–491, 1968.
- [2] Eric Backelin. Koszul duality for parabolic and singular category \mathcal{O} . *Represent. Theory*, 3:139–152, 1999.
- [3] Alexander A. Beilinson and Joseph Bernstein. Localisation de \mathfrak{g} -modules. *C. R. Acad. Sci. Paris*, 292(1):15–18, 1981.
- [4] Alexander A. Beilinson, Victor Ginzburg, and Wolfgang Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996.
- [5] Joseph Bernstein, Israel M. Gelfand, and Sergei I. Gelfand. Differential operators on the base affine space and a study of \mathfrak{g} -modules. In: *Lie groups and their representations*, 1971 Summer School of the Bolyai János Math. Society (I.M. Gelfand, Ed.), Adam Hilger Ltd., London, 1975.
- [6] Joseph Bernstein, Israel M. Gelfand, and Sergei I. Gelfand. A category of \mathfrak{g} -modules. *Funct. Anal. Appl.*, 10(2):87–92, 1976.
- [7] Michel Brion. Points entiers dans les polyèdres convexes. *Ann. Sci. École Normale Sup.*, 21(4):653–663, 1988.
- [8] Jean-Luc Brylinski and Masaki Kashiwara. Kazhdan–Lusztig conjecture and holonomic systems. *Invent. Math.*, 64(3):387–410, 1981.
- [9] Luis G. Casian and David H. Collingwood. The Kazhdan–Lusztig conjecture for generalized Verma modules. *Math. Z.*, 195(4):581–600.
- [10] Edward Cline, Brian Parshall, and Leonard Scott. Finite dimensional algebras and highest weight categories. *J. reine angew. Math.*, 391:85–99, 1988.
- [11] Harold S.M. Coxeter. Discrete groups generated by reflections. *Ann. of Math.*, 35(3):588–621, 1934.
- [12] Harold S.M. Coxeter. The complete enumeration of finite groups of the form $R_i^2 = (R_i R_j)^{k_{ij}} = 1$. *J. London Math. Soc.*, s1-10(1):21–25, 1935.
- [13] Corrado de Concini, David Eisenbud, and Claudio Procesi. Hodge algebras. *Astérisque*, 91, 1982.
- [14] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan–Lusztig polynomials. *J. Algebra*, 111(2):483–506, 1987.
- [15] Gurbir Dhillon and Apoorva Khare. Faces of highest weight modules and the universal Weyl polyhedron. *Adv. Math.*, 319:111–152, 2017.
- [16] Gurbir Dhillon and Apoorva Khare. The weights of simple modules in Category \mathcal{O} for Kac–Moody algebras. *J. Algebra*, 603:164–200, 2022.
- [17] Edward Frenkel and Dennis Gaitsgory. Localization of \mathfrak{g} -modules on the affine Grassmannian. *Ann. of Math.*, 170(3):1339–1381, 2009.
- [18] Howard Garland and James Lepowsky. Lie algebra homology and the Macdonald–Kac formulas. *Invent. Math.*, 34:37–76, 1976.
- [19] Robert B. Howlett. Coxeter groups and M -matrices. *Bull. London Math. Soc.*, 14(2):137–141, 1982.
- [20] James E. Humphreys. *Representations of semisimple Lie algebras in the BGG Category \mathcal{O}* . Graduate Studies in Mathematics, vol. 94 (289 pp.), Amer. Math. Soc., Providence, RI, 2008.

- [21] Jens C. Jantzen. *Moduln mit einem höchsten Gewicht*. Lecture Notes in Mathematics, vol. 750, Springer, Berlin, 1979.
- [22] David A. Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.
- [23] Apoorva Khare. Faces and maximizer subsets of highest weight modules. *J. Algebra*, 455:32–76, 2016.
- [24] Apoorva Khare. Standard parabolic subsets of highest weight modules. *Trans. Amer. Math. Soc.*, 369(4):2363–2394, 2017.
- [25] Askold G. Khovanskii and Alexander V. Pukhlikov. Finitely additive measures of virtual polyhedra. In: *St. Petersburg Math. J.*, 4(2):337–356, 1993.
- [26] Bertram Kostant. A formula for the multiplicity of a weight. *Proc. Natl. Acad. Sci.*, 44(6):588–589, 1958; *Trans. Amer. Math. Soc.*, 93(1):53–73, 1959.
- [27] Shrawan Kumar. Bernstein–Gelfand–Gelfand resolution for arbitrary Kac–Moody algebras. *Math. Ann.*, 286(4):709–729, 1990.
- [28] Jim Lawrence. Polytope volume computation. *Math. Comp.*, 57(195):259–271, 1991.
- [29] James Lepowsky. Generalized Verma modules, the Cartan–Helgason theorem, and the Harish-Chandra homomorphism. *J. Algebra*, 49(2):470–495, 1977.
- [30] James Lepowsky. Personal communication. 2015.
- [31] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.
- [32] Alvany Rocha-Caridi. Splitting criteria for \mathfrak{g} -modules induced from a parabolic and the Bernstein–Gelfand–Gelfand resolution of a finite-dimensional, irreducible \mathfrak{g} -module. *Trans. Amer. Math. Soc.*, 262(2):335–366, 1980.
- [33] Wolfgang Soergel. Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3:421–445, 1990.
- [34] David E. Speyer. Powers of Coxeter elements in infinite groups are reduced. *Proc. Amer. Math. Soc.*, 137(4):1295–1302, 2009.
- [35] Catharina Stroppel. Category \mathcal{O} : Quivers and endomorphism rings of projectives. *Represent. Theory*, 7:322–345, 2003.
- [36] Diana K. Taylor. *Ideals generated by monomials in an R-sequence*. Ph.D. thesis, University of Chicago, 1966.
- [37] G. Krishna Teja. Moving between weights of weight modules. *Preprint*, arXiv:2012.07775, 2020.
- [38] Alexander N. Varchenko. Combinatorics and topology of the disposition of affine hyperplanes in real space. *Funct. Anal. Appl.*, 21(1):9–19, 1987.
- [39] Daya-Nand Verma. Structure of certain induced representations of complex semisimple Lie algebras. *Bull. Amer. Math. Soc.*, 74(1):160–166, 1968.
- [40] Hermann Weyl. Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen. I, II, III. *Math. Z.*, 23: pp. 271–309, 24: pp. 328–376, 24: pp. 377–395, 1925–1926.
- [41] Wei Xiao. Janzen coefficients and Blocks of category \mathcal{O}^p . *Preprint*, arXiv:2004.08756, 2020.
- [42] Wei Xiao. Janzen Conjecture for singular characters. *Preprint*, arXiv:2106.14216, 2021.
- [43] Wei Xiao and Ailin Zhang. Janzen coefficients and simplicity of generalized Verma modules. *Preprint*, arXiv:2004.08755, 2020.

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