SHARP NONZERO LOWER BOUNDS FOR THE SCHUR PRODUCT THEOREM

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Abstract. By a result of Schur [J. Reine Angew. Math. 1911], the entrywise product $M \circ N$ of two positive semidefinite matrices $M, N$ is again positive. Vybiral (2019) improved on this by showing the uniform lower bound $M \circ M \geq E_n/n$ for all $n \times n$ real or complex correlation matrices $M$, where $E_n$ is the all-ones matrix. This was applied to settle a conjecture of Novak [J. Complexity 1999] and to positive definite functions on groups. Vybiral then asked if one can obtain similar uniform lower bounds for higher entrywise powers of $M$, or for $M \circ N$ when $N \neq 0, 0$. A natural third question is to obtain a tighter lower bound that need not vanish, over infinite-dimensional Hilbert spaces as well.

In this note, we affirmatively answer all three questions by extending and refining Vybiral’s result to lower bound $M \circ N$ for arbitrary positive $M, N$. As a special case, the above bound of $E_n/n$ can be improved to $E_n/rk(M)$. In addition, our lower bounds – which we show are tracial Cauchy–Schwarz inequalities – are sharp. This is followed by a variant for Hilbert–Schmidt operators. We end with consequences for positive definite functions on groups, metric spaces, and Hilbert spaces.

Notation:

1. A positive semidefinite matrix is a complex Hermitian matrix with non-negative eigenvalues. Denote the space of such $n \times n$ matrices by $\mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$.
2. The Loewner ordering on $\mathbb{C}^{n \times n}$ is the partial order where $M \geq N$ if and only if $M - N \in \mathbb{P}_n$.
3. We say that a matrix in $\mathbb{P}_n$ is a real/complex correlation matrix if it has all diagonal entries 1, and all entries real/complex respectively.
4. The Schur product of two (possibly rectangular) $m \times n$ complex matrices $A = (a_{ij}), B = (b_{ij})$ equals the $m \times n$ matrix $A \circ B$ with $(i, j)$ entry $a_{ij} b_{ij}$.
5. Given a fixed integer $n \geq 1$, let $e = e(n) := (1, \ldots, 1)^T \in \mathbb{C}^n$, and $E_n := ee^T \in \mathbb{P}_n$.
6. Given a matrix $M_{n \times n}$ and a subset $J \subset \{1, \ldots, n\}$, let $M_J$ denote the principal submatrix of $M$ corresponding to the rows and columns indexed by $J$; and let $d_M := (m_{11}, \ldots, m_{nn})^T$.

1. Introduction and main result

1.1. The Schur product theorem and nonzero lower bounds. A seminal result by Schur [16] asserts that if $M, N$ are positive semidefinite matrices, then so is their entrywise product $M \circ N$. This fundamental observation has had numerous follow-ups and applications; perhaps the most relevant to the present short note is the development of the entrywise calculus in matrix analysis, with connections to numerous classical and modern works, both theoretical and applied. (See e.g. the survey [3].) It also extends immediately to positive self-adjoint operators on Hilbert spaces.

Note that the Schur product theorem is ‘qualitative’, in that it provides a lower bound of $0_{n \times n}$ (in the Loewner ordering) for $M \circ N$ for all $M, N \in \mathbb{P}_n$. It is natural to seek ‘quantitative’ results, i.e., nonzero lower bounds. Here are some known bounds: Fiedler’s inequality [5] says $A \circ A^{-1} \succeq \text{Id}_n$ whenever $A \in \mathbb{P}_n$ is invertible. Two more examples from the literature, including [6, 11], are:

$$M \circ N \succeq \lambda_{\text{min}}(N)(M \circ \text{Id}_n),$$

$$M \circ N \succeq \frac{1}{e^T N^{-1} e} M, \text{ if } \det(N) > 0. \tag{1.1}$$

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This note concerns the recent paper [17], in which Vybíral showed a new lower bound for all \( M \circ \overline{M} \), where \( M \) is a correlation matrix:

**Theorem 1.2** ([17]). If \( n \geq 1 \) and \( M_{n \times n} \) is a real or complex correlation matrix (so \( \overline{M} = M^T \)), then \( M \circ \overline{M} \geq \frac{1}{n} E_n \).

Theorem 1.2 is striking in its simplicity (and in that it seems to have been undiscovered for more than a century after the Schur product theorem [16]). Vybíral provided a direct proof in [17]; by repeating this proof, he then extended Theorem 1.2 to all matrices:

**Theorem 1.3** ([17]). Given a matrix \( M \in \mathbb{C}^{n \times n} \), let \( d_M := (m_{11}, \ldots, m_{nn})^T \) be the vector consisting of its diagonal entries. Now if \( M \in \mathbb{P}_n \), then \( M \circ \overline{M} \geq \frac{1}{n} d_M d_M^T \).

Vybíral used these results to prove a conjecture of Novak [9] in numerical integration (see Theorem 3.5), with applications to positive definite functions and in other areas. See [17] for details.

### 1.2. First improvement.

In this short note, we answer three questions (two of them posed by Vybíral at the end of [17]), via sharp matrix inequalities that extend his main result, Theorem 1.2. In doing so, (a) we then further extend the main result to Hilbert–Schmidt operators; and (b) we explain why all of these results are tracial Cauchy–Schwarz inequalities.

Vybíral asked in [17] if Theorem 1.2 admits variants (1) for \( M \circ N \) for \( N \neq M, \overline{M} \); and (2) for higher powers of \( M \). An additional question is (3) if Theorem 1.3 extends to infinite-dimensional Hilbert spaces. Note here that the naive extension \( d_M d_M^T / n \) would vanish and reduce back to the Schur product theorem, so it is natural to seek a nonzero lower bound.

The main result of this note, Theorem 1.9, will answer Vybíral’s questions (1) and (2) affirmatively in \( \mathbb{P}_n \) – and we will extend it in Section 2 to arbitrary Hilbert spaces, resolving (3). Theorem 1.9 will also refine Vybíral’s results: for instance, Theorems 1.2 and 1.3 are improved by the following special case of it:

**Proposition 1.4.** If \( M \) is a correlation matrix, then \( M \circ \overline{M} \geq E_n / \text{rk}(M) \). More generally,

\[
M \circ \overline{M} \geq \frac{1}{\text{rk}(M)} d_M d_M^T, \quad \forall 0 \neq M \in \mathbb{P}_n.
\]

Note, the lower bound is purely intrinsic in \( M \), and does not (overtly) depend on the dimension of \( M \), but just on its rank and nonzero diagonal entries.

We now lead up to Theorem 1.9. Begin by noting that the coefficient \( 1/n \) in Vybíral’s Theorem 1.3 is sharp, in that \( v^T (M \circ \overline{M} - n^{-1} E_n) v = 0 \) for \( M = \text{Id}_n, v = e \). On the other hand, if we restrict the test set of matrices, then the coefficient \( 1/n \) can be improved – here are two possible ways:

- If the matrix \( M \) has nonzero entries only in the \( J \times J \) coordinates (for a nonempty subset \( J \subset \{1, \ldots, n\} \)), then the coefficient can be improved to \( 1/|J| \).
- Even with this improvement, if the matrix \( M_{J \times J} \) is rank-one, then the coefficient \( 1/|J| \) can in fact be improved to 1, since if \( M = uu^* \) (for \( u \in \mathbb{C}^n \) with \( u_i = 0 \forall i \not\in J \)), then

\[
M \circ \overline{M} = (u \circ \overline{u})(u^* \circ \overline{u}^*) = d_M d_M^T.
\]

It is thus natural to ask how the bound (sharp if possible) depends on the rank of \( M \).

With this motivation we present our ‘first improvement’ of the above results, which answers Vybíral’s second question for \( n \times n \) matrices (and partially his first question), but moreover incorporates both of these improvements:

**Theorem 1.5.** Given a vector \( u = (u_1, \ldots, u_n)^T \in \mathbb{C}^n \), let \( D_u \) denote the diagonal matrix whose diagonal entries are the coordinates \( u_1, \ldots, u_n \) of \( u \); and let \( J(u) \subset \{1, \ldots, n\} \) denote the nonzero coordinates of \( u \), i.e. \( \{j : 1 \leq j \leq n, \ u_j \neq 0\} \).
Now let \( k \geq 1 \), and fix vectors \( u_1, y_1, \ldots, u_k, y_k \in \mathbb{C}^n \) such that \( w := (u_1 \circ y_1) \circ \cdots \circ (u_k \circ y_k) \) is nonzero. Then we have the (rank \( \leq 1 \)) lower bound:

\[
\frac{1}{\text{rk}(M_J(w) \times J(w))} (w \circ d_{M_1} \circ \cdots \circ d_{M_k})(w \circ d_{M_1} \circ \cdots \circ d_{M_k}^*)^*, \quad \forall M_1, \ldots, M_k \in \mathbb{P}_n, \tag{1.6}
\]

where \( M := M_1 \circ \cdots \circ M_k \). Note, if the principal submatrix \( M_J(w) \times J(w) = 0 \) then \( w \circ d_{M_1} \circ \cdots \circ d_{M_k} \)

is also zero, so the coefficient is irrelevant.

Moreover, the coefficient \( \frac{1}{\text{rk}(M_J(w) \times J(w))} \) is best possible for all \( u_j, y_j \in \mathbb{C}^n \) for which \( w \neq 0 \), and all \( M_1, \ldots, M_k \) for which \( M_J(w) \times J(w) \neq 0 \).

Theorem 1.5 is a tighter refinement of the Schur product theorem than Theorem 1.3, which is the special case with \( k = 1 \) and \( u_1 = y_1 = e \). Moreover, Theorem 1.5 can (and does) extend to provide nonzero lower bounds in infinite-dimensional Hilbert spaces, unlike Theorems 1.2 and 1.3.

**Remark 1.7.** Define for a nonzero vector \( d \in \mathbb{C}^n \), the ‘level set’

\[
S_d := \{(M_1, \ldots, M_k) : M_j \in \mathbb{P}_n \ \forall j, \ d_{M_1} \circ \cdots \circ d_{M_n} = d\}.
\]

Then a consequence of Theorem 1.5 for \( u_j = y_j = e \ \forall j \), is that (1.6) provides a uniform lower bound on each set \( S_d \) (i.e., which depends only on \( d \)). In fact the case of \( M \) a correlation matrix in [17], is a special case of this consequence for \( d = e \) (and \( k = 1 \)).

**Remark 1.8.** Given the above results, a natural question is if even the original identity \( M \circ \overline{M} \geq \frac{1}{n} d_M d_M^T \) of Vybíral holds more widely. A natural extension to explore is from matrices \( M \circ \overline{M} \) to the larger class of **doubly non-negative matrices**: namely, matrices in \( \mathbb{P}_n \) with non-negative entries. In other words, given a doubly non-negative matrix \( A \in \mathbb{P}_n \), is it true that

\[
A \geq \frac{1}{n} d_A^{1/2} (d_A^{1/2})^T, \quad \text{where } d_A^{1/2} := (a_1^{1/2}, \ldots, a_n^{1/2})^T?
\]

While this question was not addressed in [17], it is easy to verify that it is indeed true for \( 2 \times 2 \) matrices. However, here is a family of counterexamples for \( n = 3 \) and \( k = 1 \); we leave the case of higher values of \( k \) to the interested reader. Consider the real matrix

\[
A = \begin{pmatrix} a & c & d \\ c & b & a \\ d & c & a \end{pmatrix}, \quad \text{where } a, b > 0, \ c \in [\sqrt{ab}/2, \sqrt{ab}], \ d = \frac{2c^2}{b} - a < a.
\]

These bounds imply \( A \) is doubly non-negative. Now we compute:

\[
A - d_A^{3/2} (d_A^{3/2})^T = \frac{1}{3} \begin{pmatrix} 2a & 3c - \sqrt{ab} & 3d - a \\ 3c - \sqrt{ab} & 2b & 3c - \sqrt{ab} \\ 3d - a & 3c - \sqrt{ab} & 2a \end{pmatrix}.
\]

Straightforward computations show that all entries and \( 2 \times 2 \) principal minors of this matrix are non-negative; but its determinant equals

\[
\frac{2}{3} (a - d)(2\sqrt{abc} + bd - 3c^2) = -\frac{2}{3} (a - d)(\sqrt{abc} - c)^2 < 0.
\]

This shows that one cannot hope to go much beyond the above test-set of matrices \( M \circ \overline{M} \), along the lines of the lower bound in (1.6).
1.3. The main result and its proof. We now present our main result, which provides a lower bound for $M \circ N$ for arbitrary positive matrices $M = AA^*, N = BB^*$:

**Theorem 1.9.** Given integers $n, k \geq 1$ and a complex matrix $C_{k \times n}$, let $J \subset \{1, \ldots, n\}$ index the nonzero columns of $C$. Then for all integers $a, b \geq 1$ and matrices $A \in \mathbb{C}^{a \times a}$, $B \in \mathbb{C}^{b \times b}$ such that $(AA^*)_J \cdot (BB^*)_J$ are nonzero, we have the (rank $\leq 1$) lower bound:

$$C(AA^* \circ BB^*)C^* \geq \gamma(A, B, J) \cdot C_{ABT}d_{ABT}^*C^*, \quad \text{(1.10)}$$

where $\gamma$ is a scalar, and $P$ the restriction of a projection operator:

$$\gamma(A, B, J) := \frac{1}{\min(\text{rk}(AA^*_J), \text{rk}(BB^*_J))} \quad \text{and} \quad P := \text{proj}_{\ker(A)} \cdot \text{lim}(B^T). \quad \text{(1.11)}$$

Here the coefficient $\gamma(A, B, J)$ is best possible. Also, if $a = b$, one can use $P = \text{Id}_n$ instead.

**Remark 1.12.** In the special case $n = a = b$ and $C = \text{Id}_n$ (so $J = \{1, \ldots, n\}$), the result yields

$$M \circ N \geq \frac{1}{\min(\text{rk}(M), \text{rk}(N))} \cdot C_{ABT}d_{ABT}^*, \quad \forall M, N \in \mathbb{P}_n(\mathbb{C}), \quad \text{(1.13)}$$

whenever we have the factorizations $M = AA^*, N = BB^*$. Even in this special case – albeit involving any two positive operators $M, N$ – there are no obvious upper bounds for the left-hand side, while it is even more intriguing that there is a lower bound.

**Remark 1.14.** Using $A, B$ to be the positive square roots of $M, N$ respectively, (1.10) implies:

$$M \circ N \geq \frac{1}{\min(\text{rk}(M), \text{rk}(N))} \cdot \sqrt{M} \cdot \sqrt{N} \cdot \sqrt{M} \cdot \sqrt{N}, \quad \forall M, N \in \mathbb{P}_n(\mathbb{C}), \quad n \geq 1. \quad \text{(1.15)}$$

This provides a connection between the entrywise and functional calculus.

Before proving Theorem 1.9, we explore some consequences. The first is a special case that Vybíral has communicated to us [18] for square matrix decompositions:

$$AA^* \circ BB^* \geq \frac{1}{\max(\text{rk}(AA^*), \text{rk}(BB^*))} \cdot d_{ABT}^*d_{ABT}^*, \quad \forall A, B \in \mathbb{C}^{n \times n}. \quad \text{(1.16)}$$

More precisely, Vybíral mentioned that given any two positive matrices $M, N \in \mathbb{P}_n$, one has the lower bound (1.16) for every pair of decompositions $M = AA^*, N = BB^*$ for square matrices $A, B \in \mathbb{C}^{n \times n}$. Clearly, (1.16) follows from the special case of (1.10) with $k = a = b = n$, $J = \{1, \ldots, n\}$, $C = \text{Id}_n$.

It is also not tight, as the coefficient of $1/ \text{max}$ can be improved by our bound $\gamma(A, B, J)$ (see the proof of Theorem 1.9).

The next observation is that (1.10) – say its special case with $C = \text{Id}_n$ – can be extended to Schur products of any number of positive matrices. Here is a sample consequence:

**Corollary 17.** Let $m, n, l \geq 1$ and matrices $M_1, \ldots, M_m \in \mathbb{P}_n$. Given a partition of $\{1, \ldots, m\}$ into subsets $J_1 \sqcup \cdots \sqcup J_{2l}$, let

$$M'_j := \circ_{i \in J_j} M_i, \quad 1 \leq j \leq 2l.$$  

Now if $M'_j = A_jA_j^*$ for all $j \geq 1$, with each $A_j$ square, then we have the (rank $\leq 1$) lower bound:

$$M'_1 \circ \cdots \circ M'_{2l} \geq \prod_{j=1}^{k} \frac{1}{\min(\text{rk}(M'_j), \text{rk}(M'_{j+l}))} \cdot \text{ww}^*, \quad \text{where} \; \text{ww} := \circ_{j=1}^l d_{A_jA_j^*}. \quad \text{(1.17)}$$

While this result implies (1.10) for $l = 1$, $n = k$, and $C = \text{Id}_n$, it is also implied by it, via the ‘monotonicity’ of the Schur product: if $A \geq B$ and $A' \geq B'$, then $A \circ A' \geq B \circ B'$.

Our third and final consequence of Theorem 1.9 is Theorem 1.5 (and hence, the results discussed prior to it):
Proof of Theorem 1.3. Note that (1.6) follows from (1.16) (which follows from (1.10)). Indeed, in (1.16) we set \( M := M_1 \circ \cdots \circ M_k \) and choose any square matrix \( A \) such that \( M = AA^* \). Now set \( B := A \). Then pre- and post-multiplying (1.16) by \( D_w \) and \( D_w^* \) respectively, followed by straightforward computations, yields (1.6), since
\[
D_w(M \circ M)D_w^* = D_w((M_{J \times J} \oplus 0_{J \times J}) \circ (M_{J \times J} \oplus 0_{J \times J}))D_w^*.
\]
where \( J^c := \{1, \ldots, n\} \setminus J \).

We next show the sharpness of the coefficient \( 1/\text{rk}(M_j(w) \times J(w)) \) in (1.6). In other words, we are given \( u_j, y_j \in \mathbb{C}^n \) with \( w \neq 0 \), and a rank \( 0 < r \leq |J(w)| \); and we seek matrices \( M_j \) such that \( M_j(w) \times J(w) \) has rank \( r \) and equality is attained in (1.6) by pre- and post-multiplying by \( v^* \) and \( v \) respectively, for some nonzero vector \( v \in \mathbb{C}^n \).

Below, for clarity we decouple the complex phase from the modulus. More precisely, given a vector \( v = (v_1, \ldots, v_n)^T \in \mathbb{C}^n \), define \( |v| \in [0, \infty)^n \) and \( (v)_S \in (S^1 \cup \{0\})^n \) via:
\[
|v| := (|v_1|, \ldots, |v_n|)^T, \quad (v)_S := \left( \frac{v_1}{|v_1|}, \ldots, \frac{v_n}{|v_n|} \right)^T,
\]
where \( 0/0 := 0 \) by convention.

For a diagonal matrix \( D \) with diagonal entries \( d_{jj} \in [0, \infty) \), recall that \( D^{1/2} \) denotes the positive semidefinite square root of the Moore–Penrose inverse of \( D \) – i.e., \( (D^{1/2})_{jj} \) equals \( d_{jj}^{1/2} \) if \( d_{jj} > 0 \), and 0 otherwise. Now given \( u_j, u_j \) such that \( w \neq 0 \), fix a subset \( J_0 \subset J(w) \) of size \( r \), and define
\[
M_j := (\text{Id}_{J_0} \oplus 0_{J_0^c \times J_0^c}) \circ (D_{u_j} D_{y_j} D_{u_j}^* \oplus 0_{J_0^c \times J_0^c})^{1/2} = M_j^c, \quad \forall 1 \leq j \leq k.
\]

Let \( L \) denote the left-hand side of (1.6); then straightforward computations yield, with a mild abuse of notation:
\[
L = \text{Id}_{J_0} \oplus 0_{J_0^c \times J_0^c}, \quad w = D_{M_1} \circ \cdots \circ D_{M_k} = (w \circ e(J_0))_{S^1},
\]
where \( e(J_0) \) is the vector with \( i \)th coordinate \( 1 \in J_0 \). Setting \( v := (w \circ e(J_0))_{S^1} \), we compute:
\[
v^* \left( L - \frac{1}{\text{rk}(M_j(w) \times J(w))} (w \circ e(J_0))_{S^1} (w \circ e(J_0))_{S^1}^* \right) v = |J_0| - \frac{1}{r} |J_0|^2,
\]
and this vanishes, with sharp threshold \( 1/r \).

Next, we prove our main result.

Proof of Theorem 1.4. A preliminary observation is that if \( (AA^*)_{J \times J} = 0 \) then \( \text{Id}_J(AA^*) \text{Id}_J = 0 \), where \( \text{Id}_J \in \mathbb{P}_n \) has diagonal entries \( 1_{i \in J} \). But then the submatrix \( A_{J \times \{1, \ldots, a\}} = 0 \), whence
\[
Cd_{APB^*} = C \text{Id}_J d_{APB^*} = 0.
\]

Thus the matrices on both sides of (1.10) are zero, and so the coefficient is irrelevant. The same conclusion is obtained by a similar argument if \( (BB^*)_{J \times J} = 0 \).

Before proceeding, we next show the tightness of the bound \( \gamma(A, B, J) \) (e.g. over 1/ max). Choose integers \( 1 \leq r \leq a, 1 \leq s \leq b \) with both \( r, s \leq n \), and complex block diagonal matrices
\[
A_{n \times a} := \begin{pmatrix} D_{r \times r} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{n \times b} := \begin{pmatrix} D_{s \times s}^* & 0 \\ 0 & 0 \end{pmatrix},
\]
with both \( D, D' \) nonsingular. Let \( J \subset \{1, \ldots, n\} \) be such that \( J' := J \cap \{1, \ldots, \min(r, s)\} \) is nonempty, and let \( C_{n \times n} := \text{Id}_J \). In this situation, \( P := \begin{pmatrix} \text{Id}_{\min(r, s)} & 0 \\ 0 & 0 \end{pmatrix} \), and the bound of \( \gamma(A, B, J) = 1/|J'| \) is indeed tight, as can be verified using the Cauchy–Schwarz identity.

Finally, we prove (1.10). The first reduction is via observing that \( C = C \text{Id}_J \), so that (1.10) for \((C, A, B)\) follows from (1.10) for \((C = \text{Id}_J, A, B)\). But this is precisely (1.10) for \((\text{Id}_J, A, B, B)\).
In other words, by restricting to the $J \times J$ principal submatrices on both sides, we may assume without loss of generality that $J = \{1, \ldots, n\}$ and $C = \text{Id}_n$; the hypotheses imply $A, B$ are nonzero.

Now the key identity needed to prove (1.10) is algebraic: given any square $n \times n$ matrices $A, B$ and vectors $u, v$ with $n$ coordinates (over a unital commutative ring),
\[ u^T (A \circ B)v = \text{tr}(B^T D_u A D_v). \]  
Thus, pre- and post-multiplying the left-hand side of (1.10) by $u^*, u$ respectively, we compute:
\[ u^*(AA^* \circ BB^*)u = \text{tr}(BB^T D_{u^*} AA^* D_u) = \text{tr}(N^* N), \quad \text{where} \quad N := A^* D_u B. \]

Consider the inner product on $\mathbb{C}^{a \times b}$, given by $\langle X, Y \rangle := \text{tr}(X^* Y)$, and let $P$ be as in (1.11). Then,
\[ \langle P, P \rangle \leq \min(\text{dim}(\ker A), \text{dim}(B)) = \min(\text{rk}(A^*), \text{rk}(B^*)) = \min(\text{rk}(AA^*), \text{rk}(BB^*)). \]

Hence by the Cauchy–Schwarz inequality (for this tracial inner product),
\[ u^*(AA^* \circ BB^*)u = \langle N, N \rangle \geq \frac{|\langle N, P \rangle|^2}{\langle P, P \rangle} = \frac{|\text{tr}(APB^T D_{u^*})|^2}{\langle P, P \rangle} \geq \frac{1}{\text{min}(\text{rk}(AA^*), \text{rk}(BB^*))} u^* d_{APB^T} d_{APB^T}^* u. \]

But this holds for all vectors $u$, showing (1.10).

Finally, if $a = b$, then it is straightforward to verify (by the choice of $P$) that $APB^T = AB^T$, so one can make this substitution in the final expression above.

We end this section with some remarks, beginning by attaining equality in (1.10).

**Example 1.19.** Suppose $A = u, B = v$ are nonzero vectors in $\mathbb{C}^n$. Then (1.10) says (without using $C = C \text{Id}_J$, as these are ‘less general’ – see the proof below):
\[ uu^* \circ vv^* \geq d_{uv^*} d_{uv^*} = (u \circ v)(u \circ v)^*. \]

Thus, the inequality (1.10) reduces to an equality for rank-one matrices $AA^*, BB^*$.

**Remark 1.20.** Another way to consider Theorem 1.9 is to start with matrices $M, N \in \mathbb{P}_n(\mathbb{C})$ and then obtain the bound (1.10) for every decomposition $M = AA^*, N = BB^*$. In this case, it is clear that the constant $\gamma = \min(\text{rk}(M_{J \times J}), \text{rk}(N_{J \times J}))^{-1}$ does not change; but the rank-one lower bound can indeed change. Even if one runs over decompositions in terms of square matrices $A, B$ (to dispense with the role of $P$), and assumes $C = \text{Id}_J$, it would be interesting to obtain some understanding of the possible rank-one matrices obtained as lower bounds.

This is also linked to the possibility of obtaining higher-rank lower bounds for $AA^* \circ BB^*$. One way to do so is to realize that the left-hand side of (1.10) is bi-additive in $(AA^*, BB^*)$, so one can decompose both $AA^*$ and $BB^*$ as sums of lower-rank matrices and obtain rank-one lower bounds for each pair of lower-rank matrices. Example (1.19) is relevant here: it shows that if one writes $AA^*, BB^*$ as sums of rank-one matrices, then each corresponding inequality is an equality, and adding these yields the unique best lower bound of $AA^* \circ BB^*$.

### 2. Extension to Hilbert spaces

As mentioned in the discussion preceding Theorem 1.5, we now extend our results to Hilbert spaces. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a real or complex Hilbert space with a fixed (complete) orthonormal basis $\{e_x : x \in X\}$ – so its span is dense in $\mathcal{H}$. We recall a few standard notions for linear operators $A, B : \mathcal{H} \to \mathcal{H}'$, with $\mathcal{H}'$ another Hilbert space with complete orthonormal basis $\{f_y : y \in Y\}$:

1. The adjoint $A^* : \mathcal{H}' \to \mathcal{H}$ of $A$ is given by: $\langle A^* f_y, e_x \rangle := \langle f_y, A e_x \rangle$ for all $x \in X, y \in Y$. We will also freely use $u^*$ for a vector $u \in \mathcal{H}$ to denote the linear functional $\langle u, \cdot \rangle$.
2. The transpose of $A$ is $A^T : \mathcal{H}' \to \mathcal{H}$, given by: $\langle A^T f_y, e_x \rangle := \langle A e_x, f_y \rangle$ for $x \in X, y \in Y$. 

(3) The Schur product of $A, B$ is the operator $A \circ B$ determined by: $\langle f_y, (A \circ B)e_x \rangle := \langle f_y, Ae_x \rangle \langle f_y, Be_x \rangle$ for all $x \in X, y \in Y$.

(4) We say $A$ is Hilbert–Schmidt if its Hilbert–Schmidt / Frobenius norm is finite:

$$\sum_{x \in X} \| Ae_x \|^2 < \infty.$$ 

Denote the set of Hilbert–Schmidt operators by $S_2(\mathcal{H}, \mathcal{H}')$ (the Schatten 2-class).

(5) For $\mathcal{H} = \mathcal{H}'$, a Hilbert–Schmidt operator $A : \mathcal{H} \to \mathcal{H}$ is trace class if the square root of the sum of the singular values of $\sqrt{A^*A}$ is convergent. We then define $\text{tr}(A)$ to be this sum – so $\text{tr}(A) = \sum_{x \in X} \langle e_x, Ae_x \rangle$. The vector space of trace class operators is denoted by $S_1(\mathcal{H})$.

(6) Given a vector $u \in \mathcal{H}$, the corresponding Hilbert–Schmidt multiplier $M_u : \mathcal{H} \to \mathcal{H}$ is given by: $\langle e_x, M_u e_y \rangle := \delta_{x,y} \langle e_x, u \rangle$ for all $x, y \in X$. In other words, $M_u$ is a diagonal operator with respect to the given basis $\{e_x\}$, with the corresponding coordinates of the vector $u$ as its diagonal entries.

Next, we collect together some well-known properties of these operators; see e.g. [7].

**Lemma 2.1.** Suppose $(\mathcal{H}, \langle \cdot, \cdot \rangle, \{e_x : x \in X\})$ is as above, and $u, v \in \mathcal{H}$. Also fix another Hilbert space $\mathcal{H}'$ with a fixed complete orthonormal basis $\{f_y : y \in Y\}$.

1. The space $S_2(\mathcal{H})$ is a two-sided *-ideal in $\mathcal{B}(\mathcal{H})$, which contains the multipliers $M_u$.
2. The subspace $S_2(\mathcal{H}, \mathcal{H}') \subset \mathcal{B}(\mathcal{H}, \mathcal{H}')$ contains all rank-one operators $\lambda uv^* := \lambda u \langle v, \cdot \rangle$ for $\lambda \in \mathbb{C}, v \in \mathcal{H}, u \in \mathcal{H}'$. Moreover, $*: S_2(\mathcal{H}, \mathcal{H}') \to S_2(\mathcal{H}', \mathcal{H})$.
3. If $A \in S_2(\mathcal{H}, \mathcal{H}'), B \in S_2(\mathcal{H}', \mathcal{H})$, then $AB, BA$ are trace class, and their traces coincide.
4. The assignment $(A, B) \mapsto \text{tr}(A^*B)$ is an inner product on $S_2(\mathcal{H}, \mathcal{H}')$.
5. $S_2(\mathcal{H})$ is closed under taking Schur products (with respect to $\{e_x : x \in X\}$).

(7) The multipliers $M_u, u \in \mathcal{H}$ pairwise commute and are Hilbert–Schmidt.

We require a few more notions:

**Definition 2.2.** Let $\mathcal{H}, X$ be as above.

1. Given an operator $A : \mathcal{H} \to \mathcal{H}$ and a subset $J \subset X$, define its ‘principal submatrix’ $A_{J \times J} : \mathcal{H} \to \mathcal{H}$ via:

$$\langle e_x, A_{J \times J} e_y \rangle := 1_{x \in J} 1_{y \in J} \langle e_x, Ae_y \rangle, \quad \forall x, y \in X.$$ 

2. For $A \in S_2(\mathcal{H})$, define its ‘diagonal vector’ $d_A \in \mathcal{H}$ via: $\langle e_x, d_A \rangle := \langle e_x, Ae_x \rangle$.

3. An operator $A \in \mathcal{B}(\mathcal{H})$ is positive if $A = A^*$ (self-adjoint) and $\langle u, Au \rangle \geq 0$ for all $u \in \mathcal{H}$.

Finally, we recall the spectral theorem for compact self-adjoint operators, by which every finite-rank self-adjoint operator $A$ can be written as a finite sum

$$A = \sum_{j=1}^{r} \lambda_j u_j u_j^* = \sum_{j=1}^{r} \lambda_j u_j \langle u_j, \cdot \rangle, \quad \lambda_j \in \mathbb{R} \forall j,$$

with the $u_j$ orthonormal. In particular, if $A$ is positive then one can define its square root $\sqrt{A} := \sum_{j=1}^{r} \sqrt{\lambda_j} u_j \langle u_j, \cdot \rangle$.

With these preparations, we are ready to extend Theorem 1.9 to Hilbert–Schmidt operators:

**Theorem 2.3.** Fix $\mathcal{H}, X$ as above, and Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$. Suppose $C_j \in S_2(\mathcal{H}_j, \mathcal{H})$ for $1 \leq j \leq 3$, and define $J := \{x \in X : C_3^* e_x \neq 0\}$. If $(C_1 C_1^*)_{J \times J}, (C_2 C_2^*)_{J \times J}$ are nonzero, then

$$C_3^* (C_1 C_1^* \circ C_2 C_2^*) C_3 \geq \gamma(C_1, C_2, J) \cdot C_3^* d_{C_1}^* d_{C_2}^* d_{C_1}^* d_{C_2}^* C_3,$$ 

where $\gamma(C_1, C_2, J)$ and $P$ are as in (1.11). Moreover, the coefficient $\gamma(C_1, C_2, J)$ is best possible.
Sketch of proof. If both $(C_1^*C_1)_{J \times J}$ and $(C_2^*C_2)_{K \times K}$ have infinite rank, then (the denominator on) the right-hand side vanishes and the inequality reduces to the Schur product theorem. It is when at least one of these ranks is finite that the theorem provides a nonzero lower bound. In this case, one repeats the proof of Theorem 1.9 carefully observing at each step that the relevant operations are permitted because various operators are trace class or Hilbert–Schmidt, by the preliminaries collected in Lemma 2.1. For example, the key identity (1.18) holds as both sides are additive and continuous in $u, v$ and hence can be reduced to (the easily verifiable case of) $u = e_x, v = e_y$. We also use that $\text{rk}(A_{J \times J}) \leq \text{rk}(A)$ for all $J \subseteq X$ and $A$ of finite rank, as well as the standard identity $\langle u, v \rangle = \sum_{x \in X} \langle u, e_x \rangle \langle e_x, v \rangle$ for $u, v \in H$. □

Remark 2.5. It is natural to ask if Theorem 2.3 follows from Theorem 1.5 by restricting all operators in question to some common finite-dimensional space, e.g. the column space of the matrix on the left side. However, for infinite $X$ this is not clear, because such a subspace need not contain a subset of $\{e_x : x \in X\}$ as a basis, and our Schur product is with respect to this basis $\{e_x : x \in X\}$.

3. Further ramifications

3.1. Entrywise polynomial preservers in fixed dimension. The above results reinforce the subtlety of the entrywise calculus. As observed by Pólya–Szegő [10] Problem 37, the Schur product theorem implies that every convergent power series $f(x)$ with real non-negative Maclaurin coefficients, when applied entrywise to positive matrices of all sizes with all entries in the domain of $f$, preserves matrix positivity. A famous result by Schoenberg [15] and its strengthening by Rudin [12] provide the converse for $I = (-1, 1)$: there are no other such positivity preservers. These works have led to a vast amount of activity on entrywise preservers – see e.g. [3] for more on this.

If one restricts to matrices of a fixed dimension $n$, the situation is far more challenging and a complete characterization remains open even for $n = 3$. In this setting, partial results are available when one restricts the class of test functions, or the class of test matrices in $\mathbb{P}_n$ – see [3] for details.

We restrict here to a brief comparison of Vybíral’s Theorem 1.2 with basic results in [1, 8] about entrywise polynomial maps that preserve positivity on $\mathbb{P}_n$ for fixed $n$. These latter say that for real matrices in $\mathbb{P}_n$ with entries in $(0, \epsilon)$ (resp. $(\epsilon, \infty)$) for any $\epsilon > 0$, if an entrywise polynomial preserves positivity on such matrices of rank one, then its first (resp. last) $n$ nonzero Maclaurin coefficients must be positive. Contrast this with Theorem 1.5 (or Theorem 1.2 together with the Schur product theorem), which shows that for all real correlation matrices in $\mathbb{P}_n$, of all dimensions $n$, the polynomials $x^{2k} - 1/n, k \geq 1$ preserve matrix positivity when applied entrywise. One hopes that this contrast, together with Remark 1.7 and the work [17], will lead to further new bounds and refined results for the entrywise calculus on classes of positive matrices.

3.2. Positive definite functions and related kernels. As Vybíral remarks in [17], if $g$ is any positive definite function on $\mathbb{P}^d$, or on a locally compact abelian group $G$, then Theorem 1.2 immediately implies a sharpening of the ‘easy half of Bochner’s theorem’ for $|g|^2$. We elaborate on this and other applications through the following unifying notion:

Definition 3.1. Given a set $X$ and a sequence of positive matrices $\mathcal{M} = \{M_n \in \mathbb{P}_n : n \geq 1\}$, a complex positive kernel on $X$ with lower bound $\mathcal{M}$ is any function $K : X \times X \to \mathbb{C}$ such that for all integers $n \geq 1$ and points $x_1, \ldots, x_n \in X$, the matrix $(K(x_i, x_j))_{i,j=1}^n \geq M_n \geq 0_{n \times n}$.

Note, positive definite functions/kernels are special cases with $M_n = 0_{n \times n} \forall n$. Now by Theorem 1.5

Proposition 3.2. Suppose $k \geq 1$, and for each $1 \leq j \leq k$, the function $K_{j}$ is a complex positive kernel on a set $X_{j}$, with common lower bound $\{0_{n \times n} : n \geq 1\}$. Also suppose $K_{j}(x_{j}, x_{j}) = \ell_{j} > 0 \forall x_{j} \in X_{j}, 1 \leq j \leq k$. Then the kernel $K$ on $X_{1} \times \cdots \times X_{k}$ given by

$$K((x_{1}, \ldots, x_{k}), (x_{1}', \ldots, x_{k}')) := \prod_{j=1}^{k} K_{j}(x_{j}, x_{j}') K_{j}(x_{j}', x_{j}), \quad x_{j}, x_{j}' \in X_{j},$$
is complex positive on $X_1 \times \cdots \times X_k$ with lower bound $\{ \frac{1}{n} \prod_{j=1}^{k} \ell_j \cdot E_n : n \geq 1 \}$.

This setting and result unify several different notions in the literature, as we now explain:

1. **Positive definite functions on groups:** Here $X$ is a group with identity $e_X$, and $K$ is the composite of the map $(x, x') \mapsto x^{-1} x'$ and a function $g : X \to \mathbb{C}$ satisfying: $g(x^{-1}) = \overline{g(x)}$. Then the hypotheses of Proposition 3.2 apply in this case, with $\ell := g(e_X)$.

   For instance, in [17] the author uses the positive definiteness of the cosine function on $\mathbb{R}$ to apply Theorem 1.2 and prove a conjecture of Novak [9] – see Theorem 3.5 below. This now follows from Proposition 3.2 – we present here a more general version than in [17]:

   **Proposition 3.3.** Let $\mu_1, \ldots, \mu_k$ be finite non-negative Borel measures on $X$, and $g_l$ the Fourier transform of $\mu_l$ for all $l$. Then,

   $$
   \left( \prod_{i=1}^{k} |g_l(x_i^{-1} x_j)|^2 \right)_{i,j=1}^{n} \geq \frac{1}{n} \prod_{i=1}^{k} g_l(e_X)^2 \cdot E_n.
   $$

2. **Positive semidefinite kernels on Hilbert spaces:** Here $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, and $K$ is the composite of the map $(x, x') \mapsto \langle x, x' \rangle$ and a function $g : \mathbb{C} \to \mathbb{C}$ satisfying: $g(\overline{z}) = g(z)$. (See e.g. the early work by Rudin [12], which classified the positive semidefinite kernels on $\mathbb{R}^d$ for $d \geq 3$, and related this to harmonic analysis and to the entrywise calculus.)

   In this case Theorem 1.5 applies; if one restricts to kernels that are positive definite on the unit sphere in $X$, then Proposition 3.2 applies here as well, with $\ell := g(1)$ – and thus applies to covariance kernels, widely used in the (statistics) literature.

3. **Positive definite functions on metric spaces:** In this case, $(X, d)$ is a metric space, and $K$ is the composite of the map $(x, x') \mapsto d(x, x')$ and a function $g : [0, \infty) \to \mathbb{R}$. This was studied by several experts including Bochner, Weil, and Schoenberg. For instance, Schoenberg observed in [13] that $\cos(\cdot)$ is positive definite on unit spheres in Euclidean spaces, and went on to classify in [15] the positive definite functions $f \circ \cos$ on spheres of each fixed dimension $d$. The $d = \infty$ case is the aforementioned ‘converse’ to the Schur product theorem (i.e., it shows that the Pólya–Szegö observation above is ‘sharp’).

   We conclude with a specific example, which leads to another result similar to Novak’s conjecture (shown by Vybíral). A well-known result of Schoenberg [14] says that the Gaussian kernel $\exp(-\lambda x^2)$ is positive definite on Euclidean space for all $\lambda > 0$ (In fact Schoenberg shows this characterizes Hilbert space $\ell^2(\mathbb{N})$, i.e. the closure of $\bigcup_{d \geq 1} (\mathbb{R}^d, \| \cdot \|_2)$.) Thus:

   **Proposition 3.4.** Given $x_{11}, \ldots, x_{ln} \in \ell^2(\mathbb{N})$ for $l = 1, \ldots, k$, the $n \times n$ real matrix with $(i,j)$ entry $\prod_{l=1}^{k} \exp(-\|x_{li} - x_{lj}\|^2) - \frac{1}{n}$ is positive semidefinite.

   This is similar to Novak’s conjecture, now shown by Vybíral:

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1. On a related note: Vybíral mentions in [17] that $\cos(\cdot)$ is positive definite on $\mathbb{R}^1$ using Bochner’s theorem. A simpler way to see this uses trigonometry: given reals $x_1, \ldots, x_n$, the matrix $(\cos(x_i - x_j))_{i,j=1}^{n} = uu^T + vv^T$, where $u = (\cos x_i)_{i=1}^{n}$ and $v = (\sin x_i)_{i=1}^{n}$. 

2. On a related note: Schoenberg shows the positive definiteness of the Gaussian kernel using Fourier analysis. In the spirit of the preceding footnote, we provide a purely matrix-theoretic proof in three steps – we also include this in the recent survey [3]:

   1. A result of Gantmakher–Krein says square generalized Vandermonde matrices $(x_{ij}^{nk})$ have positive determinant if $0 < x_1 < x_2 < \cdots$ and $\alpha_1 < \alpha_2 < \cdots$ are real.

   2. This implies an observation of Pólya: the Gaussian kernel is positive definite on $\mathbb{R}^1$. Indeed, given $x_1 < x_2 < \cdots$, the matrix $(\exp(-(x_j - x_k)^2))$ equals $DV D$, where $D$ is the diagonal matrix with diagonal entries $\exp(-x_j^2)$, and $V = (\exp(2x_j) e^k)$ is a generalized Vandermonde matrix.

   3. The positivity of the Gaussian kernel on every Euclidean space $\mathbb{R}^d$, whence on Hilbert space $\ell^2(\mathbb{N})$, now follows from Pólya’s observation via the Schur product theorem.
Theorem 3.5 ([9] [17]). Given \( x_1, \ldots, x_n \in \mathbb{R} \) for \( l = 1, \ldots, k \), the \( n \times n \) real matrix with entry \( \prod_{i=1}^{k} \cos^2(x_{li} - x_{lj}) - \frac{1}{n} \) is positive semidefinite.

The two results are similar in that Novák’s conjecture uses \( \cos(\cdot) \) and \( \mathbb{R}^1 \) in place of \( \exp(-\cdot)^2 \) and \( \ell^2(\mathbb{N}) \) respectively. Both results follow from Proposition 3.2 (up to rescaling the variables).

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