Moments in positivity:

metric geometry, covariance estimation, novel graph invariant

Apoorva Khare

IISc and APRG (Bangalore, India)

(Partly based on joint works with Alexander Belton, Dominique Guillot, Mihai Putinar, Bala Rajaratnam, and Terence Tao)

Working example

Definition. A real symmetric matrix $A_{N \times N}$ is *positive semidefinite* if all eigenvalues of A are $\geqslant 0$. (Equivalently, $u^T A u \geqslant 0$ for all $u \in \mathbb{R}^N$.)

Working example

Definition. A real symmetric matrix $A_{N\times N}$ is *positive semidefinite* if all eigenvalues of A are $\geqslant 0$. (Equivalently, $u^T Au \geqslant 0$ for all $u \in \mathbb{R}^N$.)

Example: Consider the following 5×5 correlation matrices:

$$A = \begin{pmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0.6 & 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 0.4 & 0 \\ 0 & 0 & 0.4 & 1 & 0.3 \\ 0 & 0 & 0 & 0.3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}.$$

(Pattern of zeros according to graphs: tree, banded graph.)

Question: Raise each entry to the αth power for some $\alpha > 0$.
For which α are the resulting matrices positive?

Positivity and Analysis

Introduction

Positivity (and preserving it) studied in many settings in the literature.

Different flavors of positivity:

- Positive semidefinite matrices (correlation and covariance matrices)
- Positive definite sequences/Toeplitz matrices (measures on S^1)
- Moment sequences/Hankel matrices (measures on $\mathbb{R}$)
- Totally positive matrices and kernels (Pólya frequency functions/sequences)
- Hilbert space kernels
- Positive definite functions on metric spaces, topological (semi)groups

Introduction

Positivity (and preserving it) studied in many settings in the literature.

Different flavors of positivity:

- Positive semidefinite matrices (correlation and covariance matrices)
- Positive definite sequences/Toeplitz matrices (measures on S^1)
- Moment sequences/Hankel matrices (measures on $\mathbb{R}$)
- Totally positive matrices and kernels (Pólya frequency functions/sequences)
- Hilbert space kernels
- Positive definite functions on metric spaces, topological (semi)groups

Question: Classify the positivity preservers in these settings.

Studied for the better part of a century.

Entrywise functions preserving positivity

Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive semidefinite matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

Problem: Given a function $f : I \to \mathbb{R}$, when is it true that

$$f[A] := (f(a_{ij})) \in \mathbb{P}_N \text{ for all } A \in \mathbb{P}_N(I)?$$

(Long history!)

Entrywise functions preserving positivity

Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive semidefinite matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

Problem: Given a function $f : I \to \mathbb{R}$, when is it true that

$$f[A] := (f(a_{ij})) \in \mathbb{P}_N \text{ for all } A \in \mathbb{P}_N(I)?$$

(Long history!) The *Hadamard product* (or Schur, or entrywise product) of two matrices is given by: $A \circ B = (a_{ij} b_{ij})$.

Schur Product Theorem (Schur, *J. Reine Angew. Math.* 1911)

If $A, B \in \mathbb{P}_N$, *then* $A \circ B \in \mathbb{P}_N$.

Entrywise functions preserving positivity

Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive semidefinite matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

Problem: Given a function $f : I \to \mathbb{R}$, when is it true that

$$f[A] := (f(a_{ij})) \in \mathbb{P}_N \text{ for all } A \in \mathbb{P}_N(I)?$$

(Long history!) The *Hadamard product* (or Schur, or entrywise product) of two matrices is given by: $A \circ B = (a_{ij} b_{ij})$.

Schur Product Theorem (Schur, *J. Reine Angew. Math.* 1911)

If $A, B \in \mathbb{P}_N$, then $A \circ B \in \mathbb{P}_N$.

Pólya–Szegö: As a consequence,

- $f(x) = x^2, x^3, \ldots, x^k$ preserves positivity on $\mathbb{P}_N$ for all N, k.

Entrywise functions preserving positivity

Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive semidefinite matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

Problem: Given a function $f : I \to \mathbb{R}$, when is it true that

$$f[A] := (f(a_{ij})) \in \mathbb{P}_N \text{ for all } A \in \mathbb{P}_N(I)?$$

(Long history!) The *Hadamard product* (or Schur, or entrywise product) of two matrices is given by: $A \circ B = (a_{ij}b_{ij})$.

Schur Product Theorem (Schur, *J. Reine Angew. Math.* 1911)

If $A, B \in \mathbb{P}_N$, then $A \circ B \in \mathbb{P}_N$.

Pólya–Szegö: As a consequence,

- $f(x) = x^2, x^3, \dots, x^k$ preserves positivity on $\mathbb{P}_N$ for all N, k.
- $f(x) = \sum_{k=0}^{l} c_k x^k$ preserves positivity if $c_k \geqslant 0$.

Entrywise functions preserving positivity

Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive semidefinite matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

Problem: Given a function $f : I \to \mathbb{R}$, when is it true that

$$f[A] := (f(a_{ij})) \in \mathbb{P}_N \text{ for all } A \in \mathbb{P}_N(I)?$$

(Long history!) The *Hadamard product* (or Schur, or entrywise product) of two matrices is given by: $A \circ B = (a_{ij} b_{ij})$.

Schur Product Theorem (Schur, *J. Reine Angew. Math.* 1911)

If $A, B \in \mathbb{P}_N$, *then* $A \circ B \in \mathbb{P}_N$.

Pólya–Szegö: As a consequence,

- $f(x) = x^2, x^3, \dots, x^k$ preserves positivity on $\mathbb{P}_N$ for all N, k.
- $f(x) = \sum_{k=0}^{l} c_k x^k$ preserves positivity if $c_k \geqslant 0$.
- Taking limits: if $f(x) = \sum_{k=0}^{\infty} c_k x^k$ is convergent and $c_k \geqslant 0$, then $f[-]$ preserves positivity.

Entrywise functions preserving positivity

Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive semidefinite matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

Problem: Given a function $f : I \to \mathbb{R}$, when is it true that

$$f[A] := (f(a_{ij})) \in \mathbb{P}_N \text{ for all } A \in \mathbb{P}_N(I)?$$

(Long history!) The *Hadamard product* (or Schur, or entrywise product) of two matrices is given by: $A \circ B = (a_{ij} b_{ij})$.

Schur Product Theorem (Schur, *J. Reine Angew. Math.* 1911)

If $A, B \in \mathbb{P}_N$, then $A \circ B \in \mathbb{P}_N$.

Pólya–Szegö: As a consequence,

- $f(x) = x^2, x^3, \dots, x^k$ preserves positivity on $\mathbb{P}_N$ for all N, k.
- $f(x) = \sum_{k=0}^{l} c_k x^k$ preserves positivity if $c_k \geqslant 0$.
- Taking limits: if $f(x) = \sum_{k=0}^{\infty} c_k x^k$ is convergent and $c_k \geqslant 0$, then $f[-]$ preserves positivity.
- Anything else?

Schoenberg's theorem

Question (Pólya–Szegö, 1925): Anything else?

Schoenberg's theorem

Question (Pólya–Szegö, 1925): Anything else? Remarkably, the answer is **no**, if we want to preserve positivity in *all* dimensions.

Theorem (Schoenberg, *Duke Math. J.* 1942; Rudin, *Duke Math. J.* 1959)

Suppose $I = (-1, 1)$ *and* $f : I \to \mathbb{R}$. *The following are equivalent:*

1. $f[A] \in \mathbb{P}_N$ *for all* $A \in \mathbb{P}_N(I)$ *and all* N.
2. f *is analytic on* I *and has nonnegative Maclaurin coefficients. In other words,* $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *on* $(-1, 1)$ *with all* $c_k \geqslant 0$.

Schoenberg's theorem

Question (Pólya–Szegö, 1925): Anything else? Remarkably, the answer is **no**, if we want to preserve positivity in *all* dimensions.

Theorem (Schoenberg, *Duke Math. J.* 1942; Rudin, *Duke Math. J.* 1959)

Suppose $I = (-1, 1)$ and $f : I \to \mathbb{R}$. The following are equivalent:

1. $f[A] \in \mathbb{P}_N$ *for all* $A \in \mathbb{P}_N(I)$ *and all* N.
2. f *is analytic on* I *and has nonnegative Maclaurin coefficients. In other words,* $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *on* $(-1, 1)$ *with all* $c_k \geqslant 0$.

Such functions f are said to be **absolutely monotonic** on $(0, 1)$.

Toeplitz and Hankel matrices

Motivations: Rudin was motivated by harmonic analysis and Fourier analysis on locally compact groups. On $G = S^1$, he studied preservers of *positive definite sequences* $(a_n)_{n \in \mathbb{Z}}$. This means the Toeplitz kernel $(a_{i-j})_{i,j \geqslant 0}$ is positive semidefinite.

Toeplitz and Hankel matrices

Motivations: Rudin was motivated by harmonic analysis and Fourier analysis on locally compact groups. On $G = S^1$, he studied preservers of *positive definite sequences* $(a_n)_{n \in \mathbb{Z}}$. This means the Toeplitz kernel $(a_{i-j})_{i,j \geqslant 0}$ is positive semidefinite.

- In [*Duke Math. J.* 1959] Rudin showed: f preserves positive definite sequences (Toeplitz matrices) if and only if f is absolutely monotonic. Suffices to work with measures with 3-point supports.

Toeplitz and Hankel matrices

Motivations: Rudin was motivated by harmonic analysis and Fourier analysis on locally compact groups. On $G = S^1$, he studied preservers of *positive definite sequences* $(a_n)_{n \in \mathbb{Z}}$. This means the Toeplitz kernel $(a_{i-j})_{i,j \geqslant 0}$ is positive semidefinite.

- In [*Duke Math. J.* 1959] Rudin showed: f preserves positive definite sequences (Toeplitz matrices) if and only if f is absolutely monotonic. Suffices to work with measures with 3-point supports.
- Important parallel notion: **moment sequences**.
  Given positive measures μ on $[-1,1]$, with moment sequences
  $$\mathbf{s}(\mu) := (s_k(\mu))_{k \geqslant 0}, \qquad \text{where } s_k(\mu) := \int_{\mathbb{R}} x^k \, d\mu,$$
  classify the moment-sequence transformers: $f(s_k(\mu)) = s_k(\sigma_\mu), \ \forall k \geqslant 0.$

Toeplitz and Hankel matrices

Motivations: Rudin was motivated by harmonic analysis and Fourier analysis on locally compact groups. On $G = S^1$, he studied preservers of *positive definite sequences* $(a_n)_{n \in \mathbb{Z}}$. This means the Toeplitz kernel $(a_{i-j})_{i,j \geqslant 0}$ is positive semidefinite.

- In [*Duke Math. J.* 1959] Rudin showed: f preserves positive definite sequences (Toeplitz matrices) if and only if f is absolutely monotonic. Suffices to work with measures with 3-point supports.

- Important parallel notion: **moment sequences**.
  Given positive measures μ on $[-1, 1]$, with moment sequences

$$\mathbf{s}(\mu) := (s_k(\mu))_{k \geqslant 0}, \qquad \text{where } s_k(\mu) := \int_{\mathbb{R}} x^k \, d\mu,$$

  classify the moment-sequence transformers: $f(s_k(\mu)) = s_k(\sigma_\mu), \ \forall k \geqslant 0$.

- With Belton–Guillot–Putinar $\rightsquigarrow$ a parallel result to Rudin:

Toeplitz and Hankel matrices (cont.)

Let $0 < \rho \leqslant \infty$ be a scalar, and set $I = (-\rho, \rho)$.

Theorem (Rudin, *Duke Math. J.* 1959)

Given a function $f : I \to \mathbb{R}$, the following are equivalent:

1. $f[-]$ *preserves the set of* **positive definite sequences** *with entries in* I*.*
2. $f[-]$ *preserves positivity on* *Toeplitz* *matrices of all sizes and rank* $\leqslant 3$*.*

Toeplitz and Hankel matrices (cont.)

Let $0 < \rho \leqslant \infty$ be a scalar, and set $I = (-\rho, \rho)$.

Theorem (Rudin, *Duke Math. J.* 1959)

Given a function $f : I \to \mathbb{R}$, the following are equivalent:

1. $f[-]$ *preserves the set of* **positive definite sequences** *with entries in* I.
2. $f[-]$ *preserves positivity on Toeplitz matrices of all sizes and rank* $\leqslant 3$.
3. f *is analytic on* I *and has nonnegative Maclaurin coefficients.*
   In other words, $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *on* I *with all* $c_k \geqslant 0$.

Toeplitz and Hankel matrices (cont.)

Let $0 < \rho \leqslant \infty$ be a scalar, and set $I = (-\rho, \rho)$.

Theorem (Rudin, *Duke Math. J.* 1959)

Given a function $f : I \to \mathbb{R}$, the following are equivalent:

1. *$f[-]$ preserves the set of* **positive definite sequences** *with entries in I.*
2. *$f[-]$ preserves positivity on Toeplitz matrices of all sizes and rank $\leqslant 3$.*
3. *f is analytic on I and has nonnegative Maclaurin coefficients.*
   In other words, $f(x) = \sum_{k=0}^{\infty} c_k x^k$ on I with all $c_k \geqslant 0$.

Theorem (Belton–Guillot–K.–Putinar, *J. Eur. Math. Soc.*, accepted)

Given a function $f : I \to \mathbb{R}$, the following are equivalent:

1. *$f[-]$ preserves the set of* **moment sequences** *with entries in I.*
2. *$f[-]$ preserves positivity on Hankel matrices of all sizes and rank $\leqslant 3$.*
3. *f is analytic on I and has nonnegative Maclaurin coefficients.*

Positive semidefinite kernels

- These two results greatly weaken the hypotheses of Schoenberg's theorem – only need to consider positive semidefinite matrices of rank $\leqslant 3$.
- Note, such matrices are precisely the Gram matrices of vectors in a 3-dimensional Hilbert space. Hence Rudin (essentially) showed:

  Let $\mathcal{H}$ be a real Hilbert space of dimension $\geqslant 3$. If $f[-]$ preserves positivity on all Gram matrices in $\mathcal{H}$, then f is a power series on $\mathbb{R}$ with non-negative Maclaurin coefficients.

Positive semidefinite kernels

- These two results greatly weaken the hypotheses of Schoenberg's theorem – only need to consider positive semidefinite matrices of rank $\leqslant 3$.
- Note, such matrices are precisely the Gram matrices of vectors in a 3-dimensional Hilbert space. Hence Rudin (essentially) showed:

  Let $\mathcal{H}$ be a real Hilbert space of dimension $\geqslant 3$. If $f[-]$ preserves positivity on all Gram matrices in $\mathcal{H}$, then f is a power series on $\mathbb{R}$ with non-negative Maclaurin coefficients.

- But such functions are precisely the *positive semidefinite kernels on $\mathcal{H}$*! (Results of Pinkus et al.) Such kernels are important in modern day machine learning, via RKHS.

Positive semidefinite kernels

- These two results greatly weaken the hypotheses of Schoenberg's theorem – only need to consider positive semidefinite matrices of rank $\leqslant 3$.
- Note, such matrices are precisely the Gram matrices of vectors in a 3-dimensional Hilbert space. Hence Rudin (essentially) showed:

  Let $\mathcal{H}$ be a real Hilbert space of dimension $\geqslant 3$. If $f[-]$ preserves positivity on all Gram matrices in $\mathcal{H}$, then f is a power series on $\mathbb{R}$ with non-negative Maclaurin coefficients.
- But such functions are precisely the *positive semidefinite kernels on $\mathcal{H}$*! (Results of Pinkus et al.) Such kernels are important in modern day machine learning, via RKHS.
- Thus, Rudin (1959) classified positive semidefinite kernels on $\mathbb{R}^3$, which is relevant in machine learning. (Now also via our parallel 'Hankel' result.)

Positivity and Metric geometry

Distance geometry

How did the study of positivity and its preservers begin?

Distance geometry

How did the study of positivity and its preservers begin?

In the 1900s, the notion of a *metric space* emerged from the works of Fréchet and Hausdorff. . .

- Now ubiquitous in science (mathematics, physics, economics, statistics, computer science. . .).

Distance geometry

How did the study of positivity and its preservers begin?

In the 1900s, the notion of a *metric space* emerged from the works of Fréchet and Hausdorff...

- Now ubiquitous in science (mathematics, physics, economics, statistics, computer science...).
- Fréchet [*Math. Ann.* 1910]. If (X, d) is a metric space with $|X| = n + 1$, then (X, d) isometrically embeds into $(\mathbb{R}^n, \ell_\infty)$.
- This avenue of work led to the exploration of metric space embeddings.

  Natural question: *Which metric spaces isometrically embed into* ***Euclidean space****?*

Euclidean metric spaces and positive matrices

Which metric spaces isometrically embed into a Euclidean space?

Euclidean metric spaces and positive matrices

Which metric spaces isometrically embed into a Euclidean space?

- Menger [*Amer. J. Math.* 1931] and Fréchet [*Ann. of Math.* 1935] provided characterizations.

Euclidean metric spaces and positive matrices

Which metric spaces isometrically embed into a Euclidean space?

- Menger [*Amer. J. Math.* 1931] and Fréchet [*Ann. of Math.* 1935] provided characterizations.
- Reformulated by Schoenberg, using... matrix positivity!

Theorem (Schoenberg, *Ann. of Math.* 1935)

Fix a finite metric space (X, d), where $X = \{x_0, \dots, x_n\}$. Then (X, d) isometrically embeds into some $\mathbb{R}^m$ (with the Euclidean distance/norm) if and only if the $n \times n$ matrix

$$A := (d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2)_{i,j=1}^n$$

is positive semidefinite.

Euclidean metric spaces and positive matrices

Which metric spaces isometrically embed into a Euclidean space?

- Menger [*Amer. J. Math.* 1931] and Fréchet [*Ann. of Math.* 1935] provided characterizations.
- Reformulated by Schoenberg, using. . . matrix positivity!

Theorem (Schoenberg, *Ann. of Math.* 1935)

Fix a finite metric space (X, d)*, where* $X = \{x_0, \dots, x_n\}$*. Then* (X, d) *isometrically embeds into some* $\mathbb{R}^m$ *(with the Euclidean distance/norm) if and only if the* $n \times n$ *matrix*

$$A := (d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2)_{i,j=1}^n$$

is positive semidefinite. Moreover, the smallest such m *is the rank of* A*.*

This is how Schoenberg connected metric geometry and matrix positivity.

Positive definite functions on spheres

Schoenberg was interested in embedding metric spaces into Euclidean spheres.

- Notice that every sphere S^{r-1} – whence the Hilbert sphere S^∞ – has a rotation-invariant distance. Namely, the *arc-length* along a great circle:

$$d(x,y) := \sphericalangle(x,y) = \arccos\langle x,y\rangle, \qquad x,y \in S^\infty.$$

Positive definite functions on spheres

Schoenberg was interested in embedding metric spaces into Euclidean spheres.

- Notice that every sphere S^{r-1} – whence the Hilbert sphere S^∞ – has a rotation-invariant distance. Namely, the *arc-length* along a great circle:

$$d(x,y) := \sphericalangle(x,y) = \arccos\langle x,y\rangle, \qquad x,y \in S^\infty.$$

- Applying $\cos[-]$ entrywise to any distance matrix on S^∞ yields:

$$\cos[(d(x_i,x_j))_{i,j\geqslant 0}] = (\langle x_i,x_j\rangle)_{i,j\geqslant 0},$$

  and this is a Gram matrix, so $\cos(\cdot)$ is positive definite on S^∞.

Positive definite functions on spheres

Schoenberg was interested in embedding metric spaces into Euclidean spheres.

- Notice that every sphere S^{r-1} – whence the Hilbert sphere S^∞ – has a rotation-invariant distance. Namely, the *arc-length* along a great circle:

$$d(x,y) := \sphericalangle(x,y) = \arccos\langle x,y\rangle, \qquad x,y \in S^\infty.$$

- Applying $\cos[-]$ entrywise to any distance matrix on S^∞ yields:

$$\cos[(d(x_i,x_j))_{i,j\geqslant 0}] = (\langle x_i,x_j\rangle)_{i,j\geqslant 0},$$

  and this is a Gram matrix, so $\cos(\cdot)$ is positive definite on S^∞.

Schoenberg then classified *all* continuous f such that $f\circ\cos(\cdot)$ is p.d.:

Theorem (Schoenberg, *Duke Math. J.* 1942)

Suppose $f:[-1,1]\to\mathbb{R}$ is continuous, and $r\geqslant 2$. Then $f(\cos\cdot)$ is positive definite on the unit sphere $S^{r-1}\subset\mathbb{R}^r$ if and only if

$$f(\cdot) = \sum_{k\geqslant 0} a_k C_k^{(\frac{r-2}{2})}(\cdot) \qquad \text{for some } a_k \geqslant 0,$$

where $C_k^{(\lambda)}(\cdot)$ are the ultraspherical / Gegenbauer / Chebyshev polynomials.

From spheres to correlation matrices

- Any Gram matrix of vectors $x_j \in S^{r-1}$ is the same as a rank $\leqslant r$ correlation matrix $A = (a_{ij})_{i,j=1}^n$, i.e.,

$$A = \begin{pmatrix} 1 & & * & \\ & 1 & & \\ * & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} — & x_1^T & — \\ — & x_2^T & — \\ & \vdots & \\ — & x_n^T & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{pmatrix} = (\langle x_i, x_j \rangle)_{i,j=1}^n.$$

From spheres to correlation matrices

- Any Gram matrix of vectors $x_j \in S^{r-1}$ is the same as a rank $\leqslant r$ correlation matrix $A = (a_{ij})_{i,j=1}^n$, i.e.,

$$A = \begin{pmatrix} 1 & & * & \\ & 1 & & \\ * & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} — & x_1^T & — \\ — & x_2^T & — \\ & \vdots & \\ — & x_n^T & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{pmatrix} = (\langle x_i, x_j \rangle)_{i,j=1}^n.$$

- So,

$$\begin{aligned} f(\cos \cdot) \text{ positive definite on } S^{r-1} \quad & \iff \quad (f(\cos d(x_i, x_j)))_{i,j=1}^n \in \mathbb{P}_n \\ & \iff \quad (f(\langle x_i, x_j \rangle))_{i,j=1}^n \in \mathbb{P}_n \\ & \iff \quad (f(a_{ij}))_{i,j=1}^n \in \mathbb{P}_n \ \forall n \geqslant 1, \end{aligned}$$

From spheres to correlation matrices

- Any Gram matrix of vectors $x_j \in S^{r-1}$ is the same as a rank $\leqslant r$ correlation matrix $A = (a_{ij})_{i,j=1}^n$, i.e.,

$$A = \begin{pmatrix} 1 & & * & \\ & 1 & & \\ & * & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} - & x_1^T & - \\ - & x_2^T & - \\ & \vdots & \\ - & x_n^T & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{pmatrix} = (\langle x_i, x_j \rangle)_{i,j=1}^n.$$

- So,

$$\begin{aligned} f(\cos\cdot) \text{ positive definite on } S^{r-1} \quad & \Longleftrightarrow \quad (f(\cos d(x_i,x_j)))_{i,j=1}^n \in \mathbb{P}_n \\ & \Longleftrightarrow \quad (f(\langle x_i, x_j\rangle))_{i,j=1}^n \in \mathbb{P}_n \\ & \Longleftrightarrow \quad (f(a_{ij}))_{i,j=1}^n \in \mathbb{P}_n \ \forall n \geqslant 1, \end{aligned}$$

i.e., f preserves positivity on correlation matrices of rank $\leqslant r$.

From spheres to correlation matrices

- Any Gram matrix of vectors $x_j \in S^{r-1}$ is the same as a rank $\leqslant r$ correlation matrix $A = (a_{ij})_{i,j=1}^n$, i.e.,

$$A = \begin{pmatrix} 1 & & * & \\ & 1 & & \\ * & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} - & x_1^T & - \\ - & x_2^T & - \\ & \vdots & \\ - & x_n^T & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{pmatrix} = (\langle x_i, x_j \rangle)_{i,j=1}^n.$$

- So,

$$\begin{aligned} f(\cos \cdot) \text{ positive definite on } S^{r-1} \quad & \iff \quad (f(\cos d(x_i, x_j)))_{i,j=1}^n \in \mathbb{P}_n \\ & \iff \quad (f(\langle x_i, x_j \rangle))_{i,j=1}^n \in \mathbb{P}_n \\ & \iff \quad (f(a_{ij}))_{i,j=1}^n \in \mathbb{P}_n \ \forall n \geqslant 1, \end{aligned}$$

  i.e., f preserves positivity on correlation matrices of rank $\leqslant r$.

- If instead $r = \infty$, such a result would classify the entrywise positivity preservers on all correlation matrices.

From spheres to correlation matrices

- Any Gram matrix of vectors $x_j \in S^{r-1}$ is the same as a rank $\leqslant r$ correlation matrix $A = (a_{ij})_{i,j=1}^n$, i.e.,

$$A = \begin{pmatrix} 1 & & * & \\ & 1 & & \\ * & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} — & x_1^T & — \\ — & x_2^T & — \\ & \vdots & \\ — & x_n^T & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{pmatrix} = (\langle x_i, x_j \rangle)_{i,j=1}^n.$$

- So,

$$\begin{aligned} f(\cos\cdot) \text{ positive definite on } S^{r-1} \quad & \Longleftrightarrow \quad (f(\cos d(x_i, x_j)))_{i,j=1}^n \in \mathbb{P}_n \\ & \Longleftrightarrow \quad (f(\langle x_i, x_j \rangle))_{i,j=1}^n \in \mathbb{P}_n \\ & \Longleftrightarrow \quad (f(a_{ij}))_{i,j=1}^n \in \mathbb{P}_n \ \forall n \geqslant 1, \end{aligned}$$

  i.e., f preserves positivity on correlation matrices of rank $\leqslant r$.

- If instead $r = \infty$, such a result would classify the entrywise positivity preservers on all correlation matrices. Interestingly, 70 years later the subject has acquired renewed interest because of its immediate impact in high-dimensional covariance estimation, in several applied fields.

Schoenberg's theorem on positivity preservers

And indeed, Schoenberg did make the leap from S^{r-1} to S^∞:

Theorem (Schoenberg, *Duke Math. J.* 1942)

Suppose $f : [-1,1] \to \mathbb{R}$ is continuous. Then $f(\cos \cdot)$ is positive definite on the Hilbert sphere $S^\infty \subset \mathbb{R}^\infty = \ell^2$ if and only if

$$f(\cos\theta) = \sum_{k \geqslant 0} c_k \cos^k \theta,$$

where $c_k \geqslant 0 \ \forall k$ are such that $\sum_k c_k < \infty$.

Schoenberg's theorem on positivity preservers

And indeed, Schoenberg did make the leap from S^{r-1} to S^∞:

Theorem (Schoenberg, *Duke Math. J.* 1942)

Suppose $f : [-1,1] \to \mathbb{R}$ is continuous. Then $f(\cos \cdot)$ is positive definite on the Hilbert sphere $S^\infty \subset \mathbb{R}^\infty = \ell^2$ if and only if

$$f(\cos\theta) = \sum_{k \geqslant 0} c_k \cos^k \theta,$$

where $c_k \geqslant 0 \ \forall k$ are such that $\sum_k c_k < \infty$.

(By the Schur product theorem, $\cos^k \theta$ is positive definite on S^∞.)

Freeing this result from the sphere context, one obtains Schoenberg's theorem on entrywise positivity preservers.

Schoenberg's theorem on positivity preservers

And indeed, Schoenberg did make the leap from S^{r-1} to S^∞:

Theorem (Schoenberg, *Duke Math. J.* 1942)

Suppose $f : [-1,1] \to \mathbb{R}$ is continuous. Then $f(\cos \cdot)$ is positive definite on the Hilbert sphere $S^\infty \subset \mathbb{R}^\infty = \ell^2$ if and only if

$$f(\cos\theta) = \sum_{k \geqslant 0} c_k \cos^k \theta,$$

where $c_k \geqslant 0 \ \forall k$ are such that $\sum_k c_k < \infty$.

(By the Schur product theorem, $\cos^k \theta$ is positive definite on S^∞.)

Freeing this result from the sphere context, one obtains Schoenberg's theorem on entrywise positivity preservers.

For more information: *A panorama of positivity* – arXiv, Dec. 2018.
(Survey, 80+ pp., by A. Belton, D. Guillot, A.K., and M. Putinar.)

Positivity and Statistics

Modern motivation: covariance estimation

Schoenberg's result has recently attracted renewed attention, owing to the statistics of big data.

Modern motivation: covariance estimation

Schoenberg's result has recently attracted renewed attention, owing to the statistics of big data.

- Major challenge in science: detect structure in vast amount of data.
- Covariance/correlation is a fundamental measure of dependence between random variables:

$$\Sigma = (\sigma_{ij})_{i,j=1}^{p}, \qquad \sigma_{ij} = \mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j].$$

Modern motivation: covariance estimation

Schoenberg's result has recently attracted renewed attention,
owing to the statistics of big data.

- Major challenge in science: detect structure in vast amount of data.
- Covariance/correlation is a fundamental measure of dependence between random variables:
$$\Sigma = (\sigma_{ij})_{i,j=1}^p, \qquad \sigma_{ij} = \mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j].$$
- **Important question:** Estimate Σ from data $x_1, \dots, x_n \in \mathbb{R}^p$.

Modern motivation: covariance estimation

Schoenberg's result has recently attracted renewed attention, owing to the statistics of big data.

- Major challenge in science: detect structure in vast amount of data.
- Covariance/correlation is a fundamental measure of dependence between random variables:
$$\Sigma = (\sigma_{ij})_{i,j=1}^p, \qquad \sigma_{ij} = \mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j].$$
- **Important question:** Estimate Σ from data $x_1, \dots, x_n \in \mathbb{R}^p$.
- In modern-day settings (small samples, ultra-high dimension), covariance estimation can be very challenging.
- Classical estimators (e.g. sample covariance matrix (MLE)):
$$S = \frac{1}{n} \sum_{j=1}^{n} (x_j - \overline{x})(x_j - \overline{x})^T$$
perform poorly, are singular/ill-conditioned, etc.

Modern motivation: covariance estimation

Schoenberg's result has recently attracted renewed attention, owing to the statistics of big data.

- Major challenge in science: detect structure in vast amount of data.
- Covariance/correlation is a fundamental measure of dependence between random variables:
$$\Sigma = (\sigma_{ij})_{i,j=1}^p, \qquad \sigma_{ij} = \mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j].$$
- **Important question:** Estimate Σ from data $x_1, \dots, x_n \in \mathbb{R}^p$.
- In modern-day settings (small samples, ultra-high dimension), covariance estimation can be very challenging.
- Classical estimators (e.g. sample covariance matrix (MLE)):
$$S = \frac{1}{n} \sum_{j=1}^n (x_j - \overline{x})(x_j - \overline{x})^T$$
perform poorly, are singular/ill-conditioned, etc.
- Require some form of *regularization* – and resulting matrix has to be positive semidefinite (in the parameter space) for applications.

Motivation from high-dimensional statistics

Graphical models: Connections between statistics and combinatorics.

Let $X_1, \dots, X_p$ be a collection of random variables.

- Very large vectors: rare that all X_j depend strongly on each other.
- Many variables are (conditionally) independent; not used in prediction.

Motivation from high-dimensional statistics

Graphical models: Connections between statistics and combinatorics.

Let $X_1, \dots, X_p$ be a collection of random variables.

- Very large vectors: rare that all X_j depend strongly on each other.
- Many variables are (conditionally) independent; not used in prediction.
- Leverage the independence/conditional independence structure to reduce dimension – translates to zeros in covariance/inverse covariance matrix.

Motivation from high-dimensional statistics

Graphical models: Connections between statistics and combinatorics.

Let $X_1, \dots, X_p$ be a collection of random variables.

- Very large vectors: rare that all X_j depend strongly on each other.
- Many variables are (conditionally) independent; not used in prediction.
- Leverage the independence/conditional independence structure to reduce dimension – translates to zeros in covariance/inverse covariance matrix.
- **Modern approach:** Compressed sensing methods (Daubechies, Donoho, Candes, Tao, ...) use convex optimization to obtain a sparse estimate of Σ (e.g., ℓ^1-penalized likelihood methods).
- State-of-the-art for ~ 20 years.

  Works well for dimensions of a few thousands.

Motivation from high-dimensional statistics

Graphical models: Connections between statistics and combinatorics.

Let $X_1, \dots, X_p$ be a collection of random variables.

- Very large vectors: rare that all X_j depend strongly on each other.
- Many variables are (conditionally) independent; not used in prediction.
- Leverage the independence/conditional independence structure to reduce dimension – translates to zeros in covariance/inverse covariance matrix.
- **Modern approach:** Compressed sensing methods (Daubechies, Donoho, Candes, Tao, ...) use convex optimization to obtain a sparse estimate of Σ (e.g., ℓ^1-penalized likelihood methods).
- State-of-the-art for ~ 20 years.

  Works well for dimensions of a few thousands.
- Not scalable to modern-day problems with $100,000+$ variables (disease detection, climate sciences, finance...).

Thresholding and regularization

Thresholding covariance/correlation matrices

$$\text{True } \Sigma = \begin{pmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0.95 & 0.18 & 0.02 \\ 0.18 & 0.96 & 0.47 \\ 0.02 & 0.47 & 0.98 \end{pmatrix}$$

Thresholding and regularization

Thresholding covariance/correlation matrices

$$\text{True } \Sigma = \begin{pmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0.95 & 0.18 & 0.02 \\ 0.18 & 0.96 & 0.47 \\ 0.02 & 0.47 & 0.98 \end{pmatrix}$$

Natural to *threshold* small entries (thinking the variables are independent):

$$\tilde{S} = \begin{pmatrix} 0.95 & 0.18 & \mathbf{0} \\ 0.18 & 0.96 & 0.47 \\ \mathbf{0} & 0.47 & 0.98 \end{pmatrix}$$

Thresholding and regularization

Thresholding covariance/correlation matrices

$$\text{True } \Sigma = \begin{pmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0.95 & 0.18 & 0.02 \\ 0.18 & 0.96 & 0.47 \\ 0.02 & 0.47 & 0.98 \end{pmatrix}$$

Natural to *threshold* small entries (thinking the variables are independent):

$$\tilde{S} = \begin{pmatrix} 0.95 & 0.18 & \mathbf{0} \\ 0.18 & 0.96 & 0.47 \\ \mathbf{0} & 0.47 & 0.98 \end{pmatrix}$$

Can be significant if $p = 100,000$ and only, say, $\sim 1\%$ of the entries of the true Σ are nonzero.

Entrywise functions – regularization

More generally, we could apply a function $f : \mathbb{R} \to \mathbb{R}$ to the elements of the matrix S – *regularization*:

Entrywise functions – regularization

More generally, we could apply a function $f : \mathbb{R} \to \mathbb{R}$ to the elements of the matrix S – *regularization*:

$$\widehat{\Sigma} = f[S] := \begin{pmatrix} f(\sigma_{11}) & f(\sigma_{12}) & \dots & f(\sigma_{1N}) \\ f(\sigma_{21}) & f(\sigma_{22}) & \dots & f(\sigma_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ f(\sigma_{N1}) & f(\sigma_{N2}) & \dots & f(\sigma_{NN}) \end{pmatrix}$$

(Example on previous slide is $f_\epsilon(x) = x \cdot \mathbf{1}_{|x|>\epsilon}$ for some $\epsilon > 0$.)

Entrywise functions – regularization

More generally, we could apply a function $f : \mathbb{R} \to \mathbb{R}$ to the elements of the matrix S – *regularization*:

$$\widehat{\Sigma} = f[S] := \begin{pmatrix} f(\sigma_{11}) & f(\sigma_{12}) & \dots & f(\sigma_{1N}) \\ f(\sigma_{21}) & f(\sigma_{22}) & \dots & f(\sigma_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ f(\sigma_{N1}) & f(\sigma_{N2}) & \dots & f(\sigma_{NN}) \end{pmatrix}$$

(Example on previous slide is $f_\epsilon(x) = x \cdot \mathbf{1}_{|x|>\epsilon}$ for some $\epsilon > 0$.)

- Highly scalable. Analysis on the cone – no optimization.

Entrywise functions – regularization

More generally, we could apply a function $f : \mathbb{R} \to \mathbb{R}$ to the elements of the matrix S – *regularization*:

$$\widehat{\Sigma} = f[S] := \begin{pmatrix} f(\sigma_{11}) & f(\sigma_{12}) & \dots & f(\sigma_{1N}) \\ f(\sigma_{21}) & f(\sigma_{22}) & \dots & f(\sigma_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ f(\sigma_{N1}) & f(\sigma_{N2}) & \dots & f(\sigma_{NN}) \end{pmatrix}$$

(Example on previous slide is $f_\epsilon(x) = x \cdot \mathbf{1}_{|x|>\epsilon}$ for some $\epsilon > 0$.)

- Highly scalable. Analysis on the cone – no optimization.
- Regularized matrix $f[S]$ further used in applications, where (estimates of) Σ required in procedures such as PCA, CCA, MANOVA, etc.

Entrywise functions – regularization

More generally, we could apply a function $f : \mathbb{R} \to \mathbb{R}$ to the elements of the matrix S – *regularization*:

$$\widehat{\Sigma} = f[S] := \begin{pmatrix} f(\sigma_{11}) & f(\sigma_{12}) & \dots & f(\sigma_{1N}) \\ f(\sigma_{21}) & f(\sigma_{22}) & \dots & f(\sigma_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ f(\sigma_{N1}) & f(\sigma_{N2}) & \dots & f(\sigma_{NN}) \end{pmatrix}$$

(Example on previous slide is $f_\epsilon(x) = x \cdot \mathbf{1}_{|x|>\epsilon}$ for some $\epsilon > 0$.)

- Highly scalable. Analysis on the cone – no optimization.
- Regularized matrix $f[S]$ further used in applications, where (estimates of) Σ required in procedures such as PCA, CCA, MANOVA, etc.
- **Question:** When does this procedure preserve positive (semi)definiteness? Critical for applications since $\Sigma \in \mathbb{P}_N$.

Entrywise functions – regularization

More generally, we could apply a function $f : \mathbb{R} \to \mathbb{R}$ to the elements of the matrix S – *regularization*:

$$\widehat{\Sigma} = f[S] := \begin{pmatrix} f(\sigma_{11}) & f(\sigma_{12}) & \dots & f(\sigma_{1N}) \\ f(\sigma_{21}) & f(\sigma_{22}) & \dots & f(\sigma_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ f(\sigma_{N1}) & f(\sigma_{N2}) & \dots & f(\sigma_{NN}) \end{pmatrix}$$

(Example on previous slide is $f_\epsilon(x) = x \cdot \mathbf{1}_{|x|>\epsilon}$ for some $\epsilon > 0$.)

- Highly scalable. Analysis on the cone – no optimization.
- Regularized matrix $f[S]$ further used in applications, where (estimates of) Σ required in procedures such as PCA, CCA, MANOVA, etc.
- **Question:** When does this procedure preserve positive (semi)definiteness? Critical for applications since $\Sigma \in \mathbb{P}_N$.

Problem: For what functions $f : \mathbb{R} \to \mathbb{R}$, does $f[-]$ preserve $\mathbb{P}_N$?

Preserving positivity in fixed dimension

Schoenberg's result characterizes functions preserving positivity for matrices of *all* dimensions: $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N$ and **all** N.

Preserving positivity in fixed dimension

Schoenberg's result characterizes functions preserving positivity for matrices of *all* dimensions: $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N$ and **all** N.

Similar/related problems studied by many others, including:

- Bharali, Bhatia, Christensen, FitzGerald, Helson, Hiai, Holtz,
- Horn, Jain, Kahane, Karlin, Katznelson, Loewner, Menegatto,
- Micchelli, Pinkus, Pólya, Ressel, Vasudeva, Willoughby, ...

Preserving positivity in fixed dimension

Schoenberg's result characterizes functions preserving positivity for matrices of *all* dimensions: $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N$ and **all** N.

Similar/related problems studied by many others, including:

- Bharali, Bhatia, Christensen, FitzGerald, Helson, Hiai, Holtz,
- Horn, Jain, Kahane, Karlin, Katznelson, Loewner, Menegatto,
- Micchelli, Pinkus, Pólya, Ressel, Vasudeva, Willoughby, ...

Preserving positivity for fixed N:

- Natural refinement of original problem of Schoenberg.
- In applications: dimension of the problem is known.
  Unnecessarily restrictive to preserve positivity in all dimensions.

Preserving positivity in fixed dimension

Schoenberg's result characterizes functions preserving positivity for matrices of *all* dimensions: $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N$ and **all** N.

Similar/related problems studied by many others, including:

- Bharali, Bhatia, Christensen, FitzGerald, Helson, Hiai, Holtz,
- Horn, Jain, Kahane, Karlin, Katznelson, Loewner, Menegatto,
- Micchelli, Pinkus, Pólya, Ressel, Vasudeva, Willoughby, ...

Preserving positivity for fixed N:

- Natural refinement of original problem of Schoenberg.
- In applications: dimension of the problem is known.
  Unnecessarily restrictive to preserve positivity in all dimensions.
- Known for $N = 2$ (Vasudeva, *IJPAM* 1979):

$$f \textit{ is nondecreasing and } f(x)f(y) \geqslant f(\sqrt{xy})^2 \textit{ on } (0,\infty).$$

Preserving positivity in fixed dimension

Schoenberg's result characterizes functions preserving positivity for matrices of *all* dimensions: $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N$ and **all** N.

Similar/related problems studied by many others, including:

- Bharali, Bhatia, Christensen, FitzGerald, Helson, Hiai, Holtz,
- Horn, Jain, Kahane, Karlin, Katznelson, Loewner, Menegatto,
- Micchelli, Pinkus, Pólya, Ressel, Vasudeva, Willoughby, ...

Preserving positivity for fixed N:

- Natural refinement of original problem of Schoenberg.
- In applications: dimension of the problem is known.
  Unnecessarily restrictive to preserve positivity in all dimensions.
- Known for $N = 2$ (Vasudeva, *IJPAM* 1979):

  $$f \textit{ is nondecreasing and } f(x)f(y) \geqslant f(\sqrt{xy})^2 \textit{ on } (0,\infty).$$

- **Open** for $N \geqslant 3$.

Problems motivated by applications

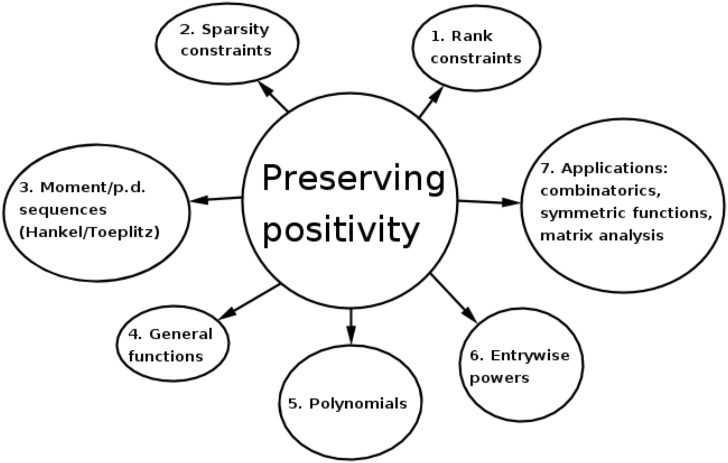
- We revisit this problem with modern applications in mind.

Problems motivated by applications

- We revisit this problem with modern applications in mind.
- Applications motivate many new exciting problems:

Problems motivated by applications

- We revisit this problem with modern applications in mind.
- Applications motivate many new exciting problems:



Further connections: total positivity, symmetric functions

Two more broad areas:

1. *Total positivity:* Pólya frequency functions and sequences.

   Rich history, from Laguerre and Fekete–Pólya, to Schoenberg, Gantmacher–Krein, Karlin...

Further connections: total positivity, symmetric functions

Two more broad areas:

1. *Total positivity:* Pólya frequency functions and sequences.

   Rich history, from Laguerre and Fekete–Pólya, to Schoenberg, Gantmacher–Krein, Karlin...

2. Connections of positivity preservers, as well as of total positivity, to
   $\longleftrightarrow$ algebraic combinatorics, Schur polynomials. (K.–Tao)

Preserving positivity in fixed dimension

Question: Find a power series with a negative coefficient, preserving positivity on $\mathbb{P}_N$ with $N \geqslant 3$.

Preserving positivity in fixed dimension

Question: Find a power series with a negative coefficient, preserving positivity on $\mathbb{P}_N$ with $N \geqslant 3$. (**Was not known** since Schoenberg's *Duke* 1942 paper.)

Preserving positivity in fixed dimension

Question: Find a power series with a negative coefficient, preserving positivity on $\mathbb{P}_N$ with $N \geqslant 3$. (**Was not known** since Schoenberg's *Duke* 1942 paper.)

Fixed $N \geqslant 3$ and general $f \rightsquigarrow$ only known necessary condition, by Loewner:

Theorem (Horn–Loewner, Guillot–K.–Rajaratnam, *Trans. AMS* 1969, 2017)

*Fix $I = (0, \infty)$ and $f : I \to \mathbb{R}$ of class C^{N-1}. Suppose $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N(I)$ Hankel of rank $\leqslant 2$, with N **fixed**.*

Preserving positivity in fixed dimension

Question: Find a power series with a negative coefficient, preserving positivity on $\mathbb{P}_N$ with $N \geqslant 3$. (**Was not known** since Schoenberg's *Duke* 1942 paper.)

Fixed $N \geqslant 3$ and general $f \rightsquigarrow$ only known necessary condition, by Loewner:

Theorem (Horn–Loewner, Guillot–K.–Rajaratnam, *Trans. AMS* 1969, 2017)

*Fix $I = (0, \infty)$ and $f : I \to \mathbb{R}$ of class C^{N-1}. Suppose $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N(I)$ Hankel of rank $\leqslant 2$, with N **fixed**. Then*

$$f, f', f'', \cdots, f^{(N-1)} \geqslant 0 \text{ on } I.$$

Preserving positivity in fixed dimension

Question: Find a power series with a negative coefficient, preserving positivity on $\mathbb{P}_N$ with $N \geqslant 3$. (**Was not known** since Schoenberg's *Duke* 1942 paper.)

Fixed $N \geqslant 3$ and general $f \rightsquigarrow$ only known necessary condition, by Loewner:

Theorem (Horn–Loewner, Guillot–K.–Rajaratnam, *Trans. AMS* 1969, 2017)

*Fix $I = (0, \infty)$ and $f : I \to \mathbb{R}$ of class C^{N-1}. Suppose $f[A] \in \mathbb{P}_N$ for all $A \in \mathbb{P}_N(I)$ Hankel of rank $\leqslant 2$, with N **fixed**. Then*

$$f, f', f'', \cdots, f^{(N-1)} \geqslant 0 \text{ on } I.$$

- Implies Schoenberg–Rudin result for matrices with positive entries.
- Loewner had initially summarized these computations in a letter to Josephine Mitchell (Penn. State University) on October 24, 1967:

Loewner's computations

when I got interested in the following question: Let $f(t)$ be a function defined in some interval (a,b), $a \geq 0$ and consider all real symmetric matrices $(a_{ij}) \geq 0^{*}$ of order n with elements $a_{ij} \in (a,b)$. What properties must ~~fo~~ f have in order that the matrices $(f(a_{ij})) \geq 0$.
I found ~~as necessary conditions: $f(t) \geq 0$, $f'(t)$~~ that if f is $(n-1)$ times differentiable the following conditions are necessary

(C) $f(t) \geq 0, f'(t) \geq 0, \ \dots \ f^{(n-1)}(t) \geq 0$

The functions t^{ρ} $(\rho > 1)$ do not satisfy these conditions for all $\rho > 1$ if $n > 3$.
The proof is obtained by considering matrices of the form $a_{ij} = \alpha + \omega \alpha_i \alpha_j$ with $\alpha \in (a,b)$, $\omega > 0$ and the α_i arbitrary. Then $(f(a_{ij})) \geq 0$ for sufficiently small ω and hence the determinant $\Delta(\omega) = \det (f(a_{ij})) \geq 0$.
The first ~~term~~ term in the Taylor expansion of $\Delta(\omega)$ at $\omega = 0$ is $f(\alpha) f'(\alpha) \cdots f^{(n-1)}(\alpha) \cdot \left(\prod (\alpha_i - \alpha_\beta)\right)^2$ and hence $f(\alpha) f'(\alpha) \cdots f^{(n-1)}(\alpha) \geq 0$, from which one easily derives that (C) must hold.

Entrywise polynomial preservers in fixed dimension

Consequence: Let $N \in \mathbb{N}$ and $c_0, \ldots, c_{2N} \neq 0$. Suppose

$$f(x) = \sum_{j=0}^{N-1} c_j x^j + c_N x^N + \sum_{j=N+1}^{2N} c_j x^j$$

preserves positivity on $\mathbb{P}_N$. Then:

- By considering $f(x)$, we obtain $c_0, \ldots, c_{N-1} > 0$.
- By considering $f(1/x)$, we obtain: $c_{N+1}, \ldots, c_{2N} > 0$.

Entrywise polynomial preservers in fixed dimension

Consequence: Let $N \in \mathbb{N}$ and $c_0, \dots, c_{2N} \neq 0$. Suppose

$$f(x) = \sum_{j=0}^{N-1} c_j x^j + c_N x^N + \sum_{j=N+1}^{2N} c_j x^j$$

preserves positivity on $\mathbb{P}_N$. Then:

- By considering $f(x)$, we obtain $c_0, \dots, c_{N-1} > 0$.
- By considering $f(1/x)$, we obtain: $c_{N+1}, \dots, c_{2N} > 0$.

Can c_N be negative?

(More generally, which coefficients in a polynomial preserver can be negative?)

Entrywise polynomial preservers in fixed dimension

Consequence: Let $N \in \mathbb{N}$ and $c_0, \dots, c_{2N} \neq 0$. Suppose

$$f(x) = \sum_{j=0}^{N-1} c_j x^j + c_N x^N + \sum_{j=N+1}^{2N} c_j x^j$$

preserves positivity on $\mathbb{P}_N$. Then:

- By considering $f(x)$, we obtain $c_0, \dots, c_{N-1} > 0$.
- By considering $f(1/x)$, we obtain: $c_{N+1}, \dots, c_{2N} > 0$.

Can c_N be negative?
(More generally, which coefficients in a polynomial preserver can be negative?)

Theorem (K.–Tao, *Amer. J. Math.*, accepted)

There exists a polynomial preserver of positivity on $\mathbb{P}_N$, with a (sufficiently small) negative coefficient, if and only if there are N positive coefficients occurring 'before' it, and N positive coefficients occurring 'after' it.

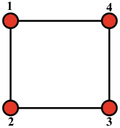
Positivity and Combinatorics

Matrices with zeros according to graphs

- In many applications, rare for all variables to depend strongly on each other – simplifies prediction.
- Many variables are (conditionally) independent – domain-specific knowledge in applications.

Matrices with zeros according to graphs

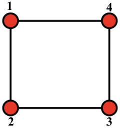
- In many applications, rare for all variables to depend strongly on each other – simplifies prediction.
- Many variables are (conditionally) independent – domain-specific knowledge in applications. Leverage the (conditional) independence structure to reduce dimension.



$$\begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{pmatrix}$$

Matrices with zeros according to graphs

- In many applications, rare for all variables to depend strongly on each other – simplifies prediction.
- Many variables are (conditionally) independent – domain-specific knowledge in applications. Leverage the (conditional) independence structure to reduce dimension.

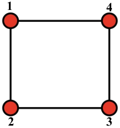


$$\begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{pmatrix}$$

- Natural to encode dependencies via a graph, where lack of an edge signifies conditional independence (given other variables).

Matrices with zeros according to graphs

- In many applications, rare for all variables to depend strongly on each other – simplifies prediction.
- Many variables are (conditionally) independent – domain-specific knowledge in applications. Leverage the (conditional) independence structure to reduce dimension.



$$\begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{pmatrix}$$

- Natural to encode dependencies via a graph, where lack of an edge signifies conditional independence (given other variables).

Study matrices with zeros according to graphs:

Given a graph $G = (V, E)$ on N vertices, and $I \subset \mathbb{R}$, define

$$\mathbb{P}_G(I) := \{A = (a_{ij}) \in \mathbb{P}_N(I) : a_{ij} = 0 \text{ if } i \neq j,\ (i,j) \notin E\}.$$

Note: a_{ij} can be zero if $(i,j) \in E$.

Powers preserving positivity: Working example

Distinguished family of functions: the power maps $x^\alpha, \alpha \in \mathbb{R},\ x \geqslant 0$.
(Here, $0^\alpha := 0$.)

Powers preserving positivity: Working example

Distinguished family of functions: the power maps $x^\alpha, \alpha \in \mathbb{R},\ x \geqslant 0$.
(Here, $0^\alpha := 0$.)

Example: Suppose $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$.

Powers preserving positivity: Working example

Distinguished family of functions: the power maps $x^\alpha, \alpha \in \mathbb{R},\ x \geqslant 0$.
(Here, $0^\alpha := 0$.)

Example: Suppose $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$.

Raise each entry to the αth power for some $\alpha > 0$.
When is the resulting matrix positive semidefinite?

Intriguing "phase transition" discovered by two students of Loewner:

Powers preserving positivity: Working example

Distinguished family of functions: the power maps $x^\alpha, \alpha \in \mathbb{R},\ x \geqslant 0$.
(Here, $0^\alpha := 0$.)

Example: Suppose $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$.

Raise each entry to the αth power for some $\alpha > 0$.
When is the resulting matrix positive semidefinite?

Intriguing "phase transition" discovered by two students of Loewner:

Theorem (FitzGerald–Horn, *J. Math. Anal. Appl.* 1977)

Let $N \geqslant 2$. Then $f(x) = x^\alpha$ preserves positivity on $\mathbb{P}_N([0,\infty))$ if and only if $\alpha \in \mathbb{N} \cup [N-2, \infty)$.

Powers preserving positivity: Working example

Distinguished family of functions: the power maps $x^\alpha, \alpha \in \mathbb{R},\ x \geqslant 0$.
(Here, $0^\alpha := 0$.)

Example: Suppose $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$.

Raise each entry to the αth power for some $\alpha > 0$.
When is the resulting matrix positive semidefinite?

Intriguing "phase transition" discovered by two students of Loewner:

Theorem (FitzGerald–Horn, *J. Math. Anal. Appl.* 1977)

*Let $N \geqslant 2$. Then $f(x) = x^\alpha$ preserves positivity on $\mathbb{P}_N([0,\infty))$ if and only if $\alpha \in \mathbb{N} \cup [N-2,\infty)$. The threshold $N-2$ is called the **critical exponent**.*

Powers preserving positivity: Working example

Distinguished family of functions: the power maps $x^\alpha, \alpha \in \mathbb{R},\ x \geqslant 0$.
(Here, $0^\alpha := 0$.)

Example: Suppose $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$.

Raise each entry to the αth power for some $\alpha > 0$.
When is the resulting matrix positive semidefinite?

Intriguing "phase transition" discovered by two students of Loewner:

Theorem (FitzGerald–Horn, *J. Math. Anal. Appl.* 1977)

*Let $N \geqslant 2$. Then $f(x) = x^\alpha$ preserves positivity on $\mathbb{P}_N([0,\infty))$ if and only if $\alpha \in \mathbb{N} \cup [N-2, \infty)$. The threshold $N-2$ is called the **critical exponent**.*

So for T_5 as above, all powers $\alpha \in \mathbb{N} \cup [3, \infty)$ work.

Powers preserving positivity: Working example

Distinguished family of functions: the power maps $x^\alpha, \alpha \in \mathbb{R},\ x \geqslant 0$. (Here, $0^\alpha := 0$.)

Example: Suppose $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$.

Raise each entry to the αth power for some $\alpha > 0$.
When is the resulting matrix positive semidefinite?

Intriguing "phase transition" discovered by two students of Loewner:

Theorem (FitzGerald–Horn, *J. Math. Anal. Appl.* 1977)

*Let $N \geqslant 2$. Then $f(x) = x^\alpha$ preserves positivity on $\mathbb{P}_N([0,\infty))$ if and only if $\alpha \in \mathbb{N} \cup [N-2, \infty)$. The threshold $N-2$ is called the **critical exponent**.*

So for T_5 as above, all powers $\alpha \in \mathbb{N} \cup [3, \infty)$ work. Can we do better?

Digression: the Pólya frequency function of Karlin

In fact when FitzGerald–Horn were students (at Stanford), in the next building S. Karlin had discovered this same 'Wallach set' of powers, via total positivity!

- Karlin studied powers of the Pólya frequency function $\Omega(x) := xe^{-x}\mathbf{1}_{x \geqslant 0}$, and showed that if $n \geqslant 0$ is an integer, $\Omega(x)^n$ has the following property:

Digression: the Pólya frequency function of Karlin

In fact when FitzGerald–Horn were students (at Stanford), in the next building S. Karlin had discovered this same 'Wallach set' of powers, via total positivity!

- Karlin studied powers of the Pólya frequency function $\Omega(x) := xe^{-x}\mathbf{1}_{x \geqslant 0}$, and showed that if $n \geqslant 0$ is an integer, $\Omega(x)^n$ has the following property:

  For all $N \geqslant 1$, the function $\Omega(x)^n$ is totally non-negative of order N.

  That is, for all scalars $x_1 < \cdots < x_N$, $y_1 < \cdots < y_N$, the matrix

$$\begin{pmatrix} \Omega(x_1-y_1)^n & \Omega(x_1-y_2)^n & \cdots & \Omega(x_1-y_N)^n \\ \Omega(x_2-y_1)^n & \Omega(x_2-y_2)^n & \cdots & \Omega(x_2-y_N)^n \\ \vdots & \vdots & \ddots & \vdots \\ \Omega(x_N-y_1)^n & \Omega(x_N-y_2)^n & \cdots & \Omega(x_N-y_N)^n \end{pmatrix}$$

  has all $1 \times 1, \dots, N \times N$ minors non-negative.

Digression: the Pólya frequency function of Karlin (cont.)

Karlin asked: What if we consider non-integer powers $\alpha > 0$? These are never TN, but are TN_N for various N:

Theorem (Karlin, *Trans. Amer. Math. Soc.* 1964)

Let $2 \leqslant N \in \mathbb{Z}$, *and* $\alpha \in \mathbb{N} \cup [N-2, \infty)$.
Then $\Omega(x)^\alpha = x^\alpha e^{-\alpha x} \mathbf{1}_{x \geqslant 0}$ *is* TN_N.

What about the remaining powers?

Digression: the Pólya frequency function of Karlin (cont.)

Karlin asked: What if we consider non-integer powers $\alpha > 0$? These are never TN, but are TN_N for various N:

Theorem (Karlin, *Trans. Amer. Math. Soc.* 1964)

Let $2 \leqslant N \in \mathbb{Z}$, *and* $\alpha \in \mathbb{N} \cup [N-2, \infty)$.
Then $\Omega(x)^\alpha = x^\alpha e^{-\alpha x} \mathbf{1}_{x \geqslant 0}$ *is* TN_N.

What about the remaining powers?

Theorem (K., 2020)

Let $\alpha \in (0, N-2) \setminus \mathbb{Z}$. *Then* $x^\alpha e^{-\alpha x} \mathbf{1}_{x \geqslant 0}$ *is not* TN_N.

(Key ingredient in proof: 2020 results of Tanvi Jain.)

Critical exponent of a graph

Back to entrywise powers preserving *positivity*. E.g., can we improve on the set of powers $\mathbb{N} \cup [3, \infty)$ for $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$?

Exploit the sparsity structure of $\mathbb{P}_G$.

Critical exponent of a graph

Back to entrywise powers preserving *positivity*. E.g., can we improve on the set of powers $\mathbb{N} \cup [3, \infty)$ for $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$?

Exploit the sparsity structure of $\mathbb{P}_G$.

Problem: Compute the set of powers preserving positivity on $\mathbb{P}_G$:

$$\mathcal{H}_G := \{\alpha \geqslant 0 : A^{\circ\alpha} \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G([0, \infty))\}$$

$$CE(G) := \text{ smallest } \alpha_0 \text{ s.t. } x^\alpha \text{ preserves positivity on } \mathbb{P}_G, \forall \alpha \geqslant \alpha_0.$$

How do $CE(G)$ and $\mathcal{H}_G$ depend on the geometry of G?

Critical exponent of a graph

Back to entrywise powers preserving *positivity*. E.g., can we improve on the set of powers $\mathbb{N} \cup [3, \infty)$ for $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$?

Exploit the sparsity structure of $\mathbb{P}_G$.

Problem: Compute the set of powers preserving positivity on $\mathbb{P}_G$:

$$\mathcal{H}_G := \{\alpha \geqslant 0 : A^{\circ\alpha} \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G([0,\infty))\}$$

$$CE(G) := \text{ smallest } \alpha_0 \text{ s.t. } x^\alpha \text{ preserves positivity on } \mathbb{P}_G, \forall \alpha \geqslant \alpha_0.$$

How do $CE(G)$ and $\mathcal{H}_G$ depend on the geometry of G?

- By FitzGerald–Horn, $CE(G)$ always exists and is $\leqslant |V(G)| - 2$. Call this the *critical exponent of the graph* G.

Critical exponent of a graph

Back to entrywise powers preserving *positivity*. E.g., can we improve on the set of powers $\mathbb{N} \cup [3, \infty)$ for $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$?

Exploit the sparsity structure of $\mathbb{P}_G$.

Problem: Compute the set of powers preserving positivity on $\mathbb{P}_G$:

$$\mathcal{H}_G := \{\alpha \geqslant 0 : A^{\circ\alpha} \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G([0,\infty))\}$$

$CE(G) :=$ smallest α_0 s.t. x^α preserves positivity on $\mathbb{P}_G, \forall \alpha \geqslant \alpha_0$.

How do $CE(G)$ and $\mathcal{H}_G$ depend on the geometry of G?

- By FitzGerald–Horn, $CE(G)$ always exists and is $\leqslant |V(G)| - 2$. Call this the *critical exponent of the graph* G.
- FitzGerald–Horn studied the case $G = K_r$: $CE(K_r) = r - 2$.
- Guillot–K.–Rajaratnam [*Trans. AMS* 2016] studied trees: $CE(T) = 1$.

Critical exponent of a graph

Back to entrywise powers preserving *positivity*. E.g., can we improve on the set of powers $\mathbb{N} \cup [3, \infty)$ for $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$?

Exploit the sparsity structure of $\mathbb{P}_G$.

Problem: Compute the set of powers preserving positivity on $\mathbb{P}_G$:

$$\mathcal{H}_G := \{\alpha \geqslant 0 : A^{\circ\alpha} \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G([0,\infty))\}$$

$$CE(G) := \text{ smallest } \alpha_0 \text{ s.t. } x^\alpha \text{ preserves positivity on } \mathbb{P}_G, \forall \alpha \geqslant \alpha_0.$$

How do $CE(G)$ and $\mathcal{H}_G$ depend on the geometry of G?

- By FitzGerald–Horn, $CE(G)$ always exists and is $\leqslant |V(G)| - 2$. Call this the *critical exponent of the graph* G.
- FitzGerald–Horn studied the case $G = K_r$: $CE(K_r) = r - 2$.
- Guillot–K.–Rajaratnam [*Trans. AMS* 2016] studied trees: $CE(T) = 1$.

- Compute $CE(G)$ for a family containing complete graphs and trees?

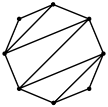
Chordal graphs – powers preserving positivity

Trees have no cycles of length $n \geqslant 3$.

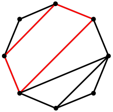
Chordal graphs – powers preserving positivity

Trees have no cycles of length $n \geqslant 3$.

Definition: G is *chordal* if it does not contain induced cycles of length $n \geqslant 4$.



Chordal

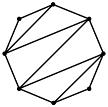


Not Chordal

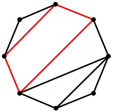
Chordal graphs – powers preserving positivity

Trees have no cycles of length $n \geqslant 3$.

Definition: G is *chordal* if it does not contain induced cycles of length $n \geqslant 4$.



Chordal



Not Chordal

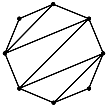
Theorem (Guillot–K.–Rajaratnam, *J. Combin. Theory Ser. A* 2016)

Let $K_r^{(1)}$ be the 'almost complete' graph on r nodes – missing one edge. Let $r = r(G)$ be the largest integer such that either K_r or $K_r^{(1)}$ is an induced subgraph of G.

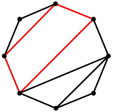
Chordal graphs – powers preserving positivity

Trees have no cycles of length $n \geqslant 3$.

Definition: G is *chordal* if it does not contain induced cycles of length $n \geqslant 4$.



Chordal



Not Chordal

Theorem (Guillot–K.–Rajaratnam, *J. Combin. Theory Ser. A* 2016)

Let $K_r^{(1)}$ be the 'almost complete' graph on r nodes – missing one edge. Let $r = r(G)$ be the largest integer such that either K_r or $K_r^{(1)}$ is an induced subgraph of G.

If G is chordal with $|V| \geqslant 2$, then $\mathcal{H}_G = \mathbb{N} \cup [r-2, \infty)$.

In particular, $CE(G) = r - 2$.

Unites complete graphs, trees, band graphs, split graphs. . .

Open to date: non-chordal graphs

Example: Band graphs with bandwidth d: $CE(G) = \min(d, n-2)$.

So for $T_5 = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$ as above, all powers $\geqslant 2 = d$ work.

Non-chordal graphs? $CE(G)$ in terms of 'known' graph invariants?
Not known to date.

American Institute of Mathematics
NSF
Department of Science & Technology, Government of India
सत्यमेव जयते
SERB INDIA
Indian Institute of Science
भारतीय विज्ञान संस्थान
DARPA
ICMS International Centre for Mathematical Sciences
ILAS International Linear Algebra Society
Taylor & Francis
TATA TRUSTS
Infosys Foundation

Selected publications

D. Guillot, A. Khare, and B. Rajaratnam:

[1] *Preserving positivity for rank-constrained matrices*, Trans. AMS, 2017.

[2] *Preserving positivity for matrices with sparsity constraints*, Tr. AMS, 2016.

[3] *Critical exponents of graphs*, J. Combin. Theory Ser. A, 2016.

A. Belton, D. Guillot, A. Khare, and M. Putinar:

[4] *Matrix positivity preservers in fixed dimension. I*, Advances in Math., 2016.

[5] *Moment-sequence transforms*, J. Eur. Math. Soc., accepted.

[6] *A panorama of positivity* (survey), Shimorin volume + Ransford-60 proc.

---

[7] *On the sign patterns of entrywise positivity preservers in fixed dimension*, (With T. Tao) Amer. J. Math., accepted.

[8] *Matrix analysis and entrywise positivity preservers*, Lecture notes (website); forthcoming book – Cambridge Univ. Press, 2021.