Pólya frequency sequences: analysis meets algebra

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Totally positive/nonnegative matrices

Definition. A rectangular matrix is *totally positive (TP)* if all minors are positive. (Similarly, totally non-negative (TN).)

Thus all entries > 0, all 2×2 minors > 0, ...

These matrices occur widely in mathematics:

Totally positive matrices in mathematics

TP and TN matrices occur in

- analysis and differential equations (Aissen, Edrei, Schoenberg, Pólya, Loewner, Whitney)
- probability and statistics (Efron, Karlin, Pitman, Proschan, Rinott)
- interpolation theory and splines (Curry, Schoenberg)
- Gabor analysis (Gröchenig, Stöckler)
- interacting particle systems (Gantmacher, Krein)
- matrix theory (Ando, Cryer, Fallat, Garloff, Johnson, Pinkus, Sokal)

- representation theory (Lusztig, Postnikov)
- cluster algebras (Berenstein, Fomin, Zelevinsky)
- integrable systems (Kodama, Williams)
- quadratic algebras (Borger, Davydov, Grinberg, Hô Hai)
- combinatorics (Brenti, Lindström–Gessel–Viennot, Skandera, Sturmfels)

$\vdots$

Examples of TP/TN matrices

1. The lower-triangular matrix $A = (\mathbf{1}_{j \geq k})_{j,k=1}^{n}$ is TN.
2. Generalized Vandermonde matrices are TP: if $0 < x_1 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ are real, then

$$\det(x_j^{y_k})_{j,k=1}^{n} > 0.$$

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3. (Pólya:) The *Gaussian kernel* is TP: given $\sigma > 0$ and scalars

$$x_1 < x_2 < \cdots < x_n, \qquad y_1 < y_2 < \cdots < y_n,$$

the matrix $G[\mathbf{x}; \mathbf{y}] := (e^{-\sigma(x_j - y_k)^2})_{j,k=1}^n$ is TP.

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Proof: It suffices to show $\det G[\mathbf{x}; \mathbf{y}] > 0$. Now factorize:
$$G[\mathbf{x}; \mathbf{y}] = \operatorname{diag}(e^{-\sigma x_j^2})_{j=1}^n \cdot ((e^{2\sigma x_j})^{y_k})_{j,k=1}^n \cdot \operatorname{diag}(e^{-\sigma y_k^2})_{k=1}^n.$$
The middle matrix is a generalized Vandermonde matrix, so all three factors have positive determinants. □

Pólya frequency sequences

The above notions of 'finite' matrices can be generalized to (bi-)infinite ones.
A real sequence $(a_n)_{n\in\mathbb{Z}}$ is a *Pólya frequency sequence* if for any integers

$$l_1 < l_2 < \cdots < l_n, \qquad m_1 < m_2 < \cdots < m_n,$$

the determinant $\det(a_{l_j - m_k})_{j,k=1}^n \geq 0$.

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In other words, these are bi-infinite Toeplitz matrices

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which are totally non-negative.

- Example: *Gaussians.* For $q \in (0,1)$, the sequence $(q^{n^2})_{n \in \mathbb{Z}}$ is not just a TN sequence, but TP. (Why? Set $q = e^{-\sigma}$.)

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Focus on two kinds of examples: *finite* and *one-sided infinite*.

Generating functions of Pólya frequency sequences

- Two remarkable results (1950s) say that finite and one-sided Pólya frequency sequences are simply products of 'atoms'!
- The 'atoms' are explained next. For now: why products?

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Suppose $\mathbf{a} = (\ldots, 0, 0, a_0, a_1, a_2, a_3, \ldots)$ is one-sided. Its *generating function* is

$$\Psi_{\mathbf{a}}(x) := a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \qquad a_0 \neq 0.$$

Now if $\mathbf{a}, \mathbf{b}$ are one-sided PF sequences, then their Toeplitz 'matrices' are TN:

$$T_{\mathbf{a}} := \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad T_{\mathbf{b}} := \begin{pmatrix} b_0 & 0 & 0 & \cdots \\ b_1 & b_0 & 0 & \cdots \\ b_2 & b_1 & b_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By the Cauchy–Binet formula, so also is $T_{\mathbf{a}} T_{\mathbf{b}} \rightsquigarrow$ Toeplitz matrix.

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This product matrix corresponds to the coefficients of the power series $\Psi_{\mathbf{a}}(x)\Psi_{\mathbf{b}}(x)$. This gives new examples of PF sequences from old ones.

Finite Pólya frequency sequences – and real-rootedness

'Atomic' **finite** PF sequences:

- The sequence $(\ldots, 0, 0, a_0, 0, 0, \ldots)$ and $(\ldots, 0, 0, 1, \alpha, 0, 0, \ldots)$ are PF sequences if $a_0, \alpha > 0$.

  Indeed, every 'square submatrix' drawn from these sequences either has a zero row/column, or is triangular with positive diagonal entries.

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- The 'atom' $(\dots, 0, 0, 1, \alpha, 0, 0, \dots)$ corresponds to $\Psi_{\mathbf{a}}(x) = 1 + \alpha x$.

- By previous slide, $a_0(1 + \alpha_1 x)(1 + \alpha_2 x) \cdots (1 + \alpha_m x)$ generates a PF sequence $\mathbf{a}_m$, when all $\alpha_j > 0$.

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Theorem (Aissen–Schoenberg–Whitney and Edrei, 1950s)

Suppose $a_0, \ldots, a_m > 0$. *The following are equivalent.*

1. $\mathbf{a} = (\ldots, 0, 0, a_0, \ldots, a_m, 0, 0, \ldots)$ *is a PF sequence.*
2. *The generating function* $\Psi_{\mathbf{a}}(x)$ *has* m *negative real roots (i.e., the above form).*
3. *The generating function* $\Psi_{\mathbf{a}}(x)$ *has* m *real roots.*

Connection to combinatorics

'Finite-order' PF sequences: A real sequence $(a_n)_{n\in\mathbb{Z}}$ is PF$_r$ for $r \geq 1$ if for any size $1 \leq n \leq r$ and integers

$$l_1 < l_2 < \cdots < l_n, \qquad m_1 < m_2 < \cdots < m_n,$$

the determinant $\det(a_{l_j - m_k})_{j,k=1}^n \geq 0$.

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PF and related sequences are well-known to combinatorialists:

- A PF$_1$ sequence $(a_0, \dots, a_m)$ is simply a non-negative sequence. (Brenti: *the only ones in combinatorics that are "meaningful"*.)
- A positive tuple $(a_0, \dots, a_m)$ is a PF$_2$ sequence if and only if it is *log-concave*: $a_j^2 \geq a_{j-1} a_{j+1}$ for $0 < j < m$.

Connection to combinatorics (cont.)

Proposition

Fix a positive tuple (padded by zeros) $\mathbf{a} = (\dots, 0, 0, a_0, \dots, a_m, 0, 0, \dots)$. *Then each of the following parts implies the next.*

1. $\mathbf{a}$ *is a PF sequence – i.e.,, the polynomial* $\Psi_{\mathbf{a}}(x)$ *is real-rooted.*
2. $(a_0, \dots, a_m)$ *is strongly log-concave:* $(a_j / \binom{m}{j})_{j=0}^{m}$ *is log-concave.*
3. *The tuple* $(a_0, \dots, a_m)$ *is log-concave.*
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Well-studied in combinatorics. E.g. Stirling numbers of second kind:

$$E_n(x) = \sum_{k=1}^n k! S(n,k) x^k, \qquad \sum_{k=1}^n S(n,k) x^k$$

are real-rooted polynomials. For more on these connections to combinatorics:

- R.P. Stanley, *Graph theory and its applications*, 1989.
- F. Brenti, *Mem. Amer. Math. Soc.*, 1989.
- P. Brändén, *Handbook of Enumerative Combinatorics*, 2014.

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$$\Psi_{\mathbf{a}_1}(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

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Claim: The function $\mathbf{a}_c := (\dots, 0, 0, 1, c, c^2, \dots)$ is a PF sequence for $c > 0$.

Proof: Given increasing tuples of integers $(l_j), (m_k)$ for $1 \leq j, k \leq n$,

$$((\mathbf{a}_c)_{l_j - m_k}) = \operatorname{diag}(c^{l_j})_{j=1}^n \cdot (\mathbf{1}_{l_j \geq m_k})_{j,k=1}^n \cdot \operatorname{diag}(c^{-m_k})_{k=1}^n,$$

and this has a non-negative determinant since $\mathbf{a}_1$ is PF. □

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- Therefore $(1 - \beta x)^{-1}$ is a PF sequence for $\beta > 0$.
- **Limits:** If $\mathbf{a}_m$ are PF sequences, converging 'pointwise' to $\mathbf{a}$, then $\mathbf{a}$ is a PF sequence.

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- **Limits:** If $\mathbf{a}_m$ are PF sequences, converging 'pointwise' to $\mathbf{a}$, then $\mathbf{a}$ is a PF sequence.
- *Example:* Since $(1 + \delta x/m)^m$ generates a PF sequence for $\delta \geq 0$ and all $m \geq 1$, so does $e^{\delta x}$. (E.g., $(\dots, 0, 0, 1, \frac{1}{1!}, \frac{1}{2!}, \dots)$ is a PF sequence.)

Infinite one-sided Pólya frequency sequences (cont.)

- More examples: if $\alpha_j, \beta_j \geq 0$ for all $j \geq 0$ are summable, then

$$\prod_{j=1}^{\infty}(1+\alpha_j x), \qquad \prod_{j=1}^{\infty}(1-\beta_j x)^{-1}$$

both generate PF sequences.

- Hence so does their product:

$$e^{\delta x}\frac{\prod_{j=1}^{\infty}(1+\alpha_j x)}{\prod_{j=1}^{\infty}(1-\beta_j x)}.$$

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Remarkably, these are *all* of the PF sequences!

Theorem (Aissen–Schoenberg–Whitney and Edrei, 1950s)

A one-sided sequence $\mathbf{a} = (\dots, 0, 0, a_0 = 1, a_1, \dots)$ *is a PF sequence if and only if it is of the above form.*

(Uses Hadamard's thesis (1892) and Nevanlinna's refinement (1929) of Picard's theorem.)

From Pólya–Schur multipliers to Ramanujan graphs

What if $\Psi_{\mathbf{a}}(x)$ is an entire function? It must be $e^{\delta x} \prod_{j \geq 1}(1 + \alpha_j x)$.

Theorem (Pólya–Schur, *Crelle*, 1914)

An entire function $\Psi(x) = \sum_{n \geq 0} a_n x^n$ with $\Psi(0) = 1$ generates a one-sided PF sequence, if and only if the sequence $n! a_n$ is a multiplier sequence of the first kind.

In other words, if $\sum_{j \geq 0} c_j x^j$ is a real-rooted *polynomial*, so is $\sum_{j \geq 0} j! a_j c_j x^j$.

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- Taken forward by Marcus–Spielman–Srivastava (2010s):
  - Kadison–Singer conjecture.
  - Existence of bipartite Ramanujan (expander) graphs of every degree and every order.

The Riemann Hypothesis

Pólya frequency sequences also connect to number theory:

Theorem (Katkova, *Comput. Meth. Funct. Th.*, 2000)

Let $\xi(s) = \binom{s}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$ *be the Riemann xi-function. If*

$$\xi_1(s) := \xi(1/2 + \sqrt{s})$$

generates a PF sequence, then the Riemann Hypothesis is true.

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Katkova proved that ξ_1 is PF of order at least 43, and is 'asymptotically PF' of all orders.

Hilbert series and PF sequences

PF sequences also show up in *algebra*. Given a $\mathbb{Z}^{\geq 0}$-graded vector space $V = \oplus_{n \geq 0} V[n]$ over a field $\mathbb{F}$, its *Hilbert series* is

$$H(V, x) = \sum_{n \geq 0} x^n \dim V[n].$$

If $V \cong \mathbb{F}^m$ for $m \geq 1$, then $H(\wedge^\bullet V, x) = (1+x)^m, \ H(\mathbb{S}^\bullet V, x) = \dfrac{1}{(1-x)^m}$, and from above, these Koszul-dual algebras both generate PF sequences.

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More generally, say $R : V \otimes V \to V \otimes V$ satisfies

- the Yang–Baxter equation $R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}$, and
- the Iwahori–Hecke relation $(R+1)(R-q) = 0$, $q \in \mathbb{F}^\times$.

Define two graded algebas – the R-exterior algebra and R-symmetric algebra:

$$\wedge^\bullet_R(V) := T^\bullet(V)/(\mathrm{im}(R + \mathrm{Id})), \qquad \mathbb{S}^\bullet_{q,R}(V) := T^\bullet(V)/(\mathrm{im}(R - q\,\mathrm{Id})).$$

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Theorem (Hô Hai 1999, Davydov 2000)

Suppose $\mathbb{F}$ has characteristic zero, $\dim V < \infty$, and either $q = 1$ or q is not a root of unity. Then the Hilbert series $H(\wedge^\bullet_R(V), x)$, $H(\mathbb{S}^\bullet_{q,R}(V), x)$ both generate PF sequences. (Skryabin, 2019)

Elementary symmetric polynomials

Return to the case $q = 1$ and $R = \tau =$ flip, but now with V having a countable $\mathbb{R}^{\geq 0}$*-graded basis* v_j *of degree* $\alpha_j > 0$. Then the Hilbert series of $\wedge^\bullet(V)$ is:

$$H(\wedge^\bullet(V), x) = \prod_{j \geq 1}(1 + \alpha_j x),$$

and this generates a PF sequence if $\alpha_j \geq 0$ are summable.

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$$1, \quad \sum_j \alpha_j, \quad \sum_{j<k} \alpha_j \alpha_k, \quad \sum_{j<k<l} \alpha_j \alpha_k \alpha_l, \quad \dots$$
which are precisely the *elementary symmetric polynomials* in the roots α_j.
- Thus, the corresponding infinite Toeplitz TN matrix is
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Complete homogeneous symmetric polynomials

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Monomial-positivity and the (dual) Jacobi–Trudi identity

Theorem

All minors of the matrices $(e_{j-k}(\mathbf{u})\mathbf{1}_{j\geq k})_{j,k\geq 0}$ *and* $(h_{j-k}(\mathbf{u})\mathbf{1}_{j\geq k})_{j,k\geq 0}$ *are* ***monomial-positive****.*

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  'upgrading' to Schur-positivity.
- They are called the *(dual) Jacobi–Trudi identities* (1800s).

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This brings us to Schur polynomials.

Schur polynomials

Given a decreasing N-tuple $n_{N-1} > n_{N-2} > \cdots > n_0 \geqslant 0$, the corresponding **Schur polynomial** over a field $\mathbb{F}$ is the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{(n_{N-1},\ldots,n_0)}(u_1,\ldots,u_N) := \frac{\det(u_j^{n_{k-1}})}{\det(u_j^{k-1})}$$

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Example: If $N = 2$ and $\mathbf{n} = (m < n)$, then

$$s_{\mathbf{n}}(u_1, u_2) = \frac{u_1^n u_2^m - u_1^m u_2^n}{u_1 - u_2} = (u_1 u_2)^m (u_1^{n-m-1} + u_1^{n-m-2} u_2 + \cdots + u_2^{n-m-1}).$$

- Basis of homogeneous symmetric polynomials in $u_1, \dots, u_N$.
- Characters of irreducible polynomial representations of $GL_N(\mathbb{C})$.

Schur polynomials via semi-standard Young tableaux

Schur polynomials are also defined using semi-standard Young tableaux:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (0, 2, 4)$. The tableaux are:

3	3	3	3	3	2	3	2	3	1	3	1	2	2	2	1
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Example 2: Suppose $N = 3$ and $\mathbf{n} = (0, 2, 3)$:

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Then $s_{(0,2,3)}(u_1, u_2, u_3) = u_1 u_2 + u_2 u_3 + u_3 u_1$.

Schur Monotonicity Lemma

Example: Continuing from the previous slide,

$$f(u_1, u_2, u_3) = \frac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1 u_2 + u_2 u_3 + u_3 u_1}, \qquad u_1, u_2, u_3 > 0.$$

Note: both numerator and denominator are monomial-positive (in fact Schur-positive, obviously) – hence non-decreasing in each coordinate.

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Theorem (K.–Tao, *Amer. J. Math.*, in press)

For integer tuples $0 \leq n_0 < \cdots < n_{N-1}$ and $0 \leq m_0 < \cdots < m_{N-1}$ such that $n_j \leq m_j \ \forall j$, the function

$$f : (0, \infty)^N \to \mathbb{R}, \qquad f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$$

is non-decreasing in each coordinate. *(In fact we show Schur-positivity.)*

(Recent example of numerical positivity ⇝ monomial-pos. ⇝ Schur-positivity.)

Schur Monotonicity Lemma (cont.)

Claim: *The ratio* $f(u_1, u_2, u_3) = \dfrac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1 u_2 + u_2 u_3 + u_3 u_1}$,

treated as a ***function*** *on the orthant* $(0, \infty)^3$, *is coordinatewise non-decreasing.*

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(Why?) Applying the quotient rule of differentiation to f,

$$s_{\mathbf{n}}(\mathbf{u})\partial_{u_3} s_{\mathbf{m}}(\mathbf{u}) - s_{\mathbf{m}}(\mathbf{u})\partial_{u_3} s_{\mathbf{n}}(\mathbf{u}) = (u_1 + u_2)(u_1u_3 + 2u_1u_2 + u_2u_3)u_3,$$

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and this is monomial-positive.

Now if we write this as $\sum_{j \geqslant 0} p_j(u_1,u_2)u_3^j$, then each p_j is Schur-positive, i.e. a sum of Schur polynomials:

$$p_0(u_1,u_2) = 0,$$

$$p_1(u_1,u_2) = 2u_1u_2^2 + 2u_1^2u_2 = 2\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} + 2\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} = 2s_{(3,1)}(u_1,u_2),$$

$$p_2(u_1,u_2) = (u_1+u_2)^2 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$$

$$= s_{(3,0)}(u_1,u_2) + s_{(2,1)}(u_1,u_2).$$

Proof-sketch of Schur Monotonicity Lemma

The proof for general $\mathbf{m} \geq \mathbf{n}$ is similar:

By symmetry, and the quotient rule of differentiation, it suffices to show that

$$s_{\mathbf{n}} \cdot \partial_{u_N}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_N}(s_{\mathbf{n}})$$

is numerically positive on $(0,\infty)^N$. (Note, the coefficients in $s_{\mathbf{n}}(\mathbf{u})$ of each u_N^j are skew-Schur polynomials in $u_1, \dots, u_{N-1}$.)

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Key ingredient: Schur-positivity result by Lam–Postnikov–Pylyavskyy (2007). In turn, this emerged out of Skandera's results (2004) on determinant inequalities for *totally non-negative matrices*. □

Weak majorization through Schur polynomials

- Our Schur Monotonicity Lemma implies in particular:

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geq \frac{s_{\mathbf{m}}(1,\dots,1)}{s_{\mathbf{n}}(1,\dots,1)}, \qquad \forall \mathbf{u} \in [1,\infty)^N.$$

  if $\mathbf{m}$ dominates $\mathbf{n}$ coordinatewise.
- Natural to ask: for which other tuples $\mathbf{m}, \mathbf{n}$ does this inequality hold?

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Theorem (K.–Tao, *Amer. J. Math.*, in press)

Given integers $0 \leq n_0 < \cdots < n_{N-1}$ and $0 \leq m_0 < \cdots < m_{N-1}$, the above inequality holds for all $\mathbf{u} \in [1,\infty)^N$, if and only if $\mathbf{m}$ weakly majorizes $\mathbf{n}$.

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This problem was studied originally by Skandera and others in 2011, *on the entire positive orthant* $(0,\infty)^N$:

Cuttler–Greene–Skandera conjecture

Theorem (Cuttler–Greene–Skandera, *Eur. J. Comb.*, 2011)

Given integers $0 \leq n_0 < \cdots < n_{N-1}$ *and* $0 \leq m_0 < \cdots < m_{N-1}$ *such that*

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Theorem (Sra, *Eur. J. Comb.*, 2016)

The Cuttler–Greene–Skandera conjecture is true.

These results provide novel characterizations of (weak) majorization, through Schur polynomials and through proof-techniques originating in total positivity.

Open question: Optimizing over $[-1,1]^N$?

- Our work with Tao (2017) concerned entrywise operations preserving positive semidefiniteness in a fixed dimension.
- The maximization of $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ over $(0,1]^N$ reveals tight bounds on certain classes of polynomial preservers, acting on correlation matrices with non-negative entries. (By homogeneity and continuity, maximize only over the cube-boundary $(0,1]^N \cap \partial(0,1]^N$.)

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- What about on *all* correlation matrices? Need to bound $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ over all of $[-1,1]^N \setminus \{0\}$.
- For this, need to ensure $s_{\mathbf{n}}(\mathbf{u})$ does not vanish except at 0. **Facts:**
  (1) The only such $\mathbf{n} = (0,1,\dots,N-2,N-1+2r)$ for $r \in \mathbb{Z}^{\geq 0}$.
  (2) All such $s_{\mathbf{n}}(\mathbf{u})$ are *complete symmetric homogeneous polynomials* $h_{2r}(\mathbf{u})$, and they are positive on $\mathbb{R}^N \setminus \{0\}$.

Open question: Optimizing over $[-1,1]^N$?

- Our work with Tao (2017) concerned entrywise operations preserving positive semidefiniteness in a fixed dimension.
- The maximization of $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ over $(0,1]^N$ reveals tight bounds on certain classes of polynomial preservers, acting on correlation matrices with non-negative entries. (By homogeneity and continuity, maximize only over the cube-boundary $(0,1]^N \cap \partial(0,1]^N$.)
- What about on *all* correlation matrices? Need to bound $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ over all of $[-1,1]^N \setminus \{0\}$.
- For this, need to ensure $s_{\mathbf{n}}(\mathbf{u})$ does not vanish except at 0. **Facts:**
  (1) The only such $\mathbf{n} = (0,1,\dots,N-2,N-1+2r)$ for $r \in \mathbb{Z}^{\geq 0}$.
  (2) All such $s_{\mathbf{n}}(\mathbf{u})$ are *complete symmetric homogeneous polynomials* $h_{2r}(\mathbf{u})$, and they are positive on $\mathbb{R}^N \setminus \{0\}$.

Question: *Say $m_j \geq j$ for $j = 0,1,\dots,N-2$, and $m_{N-1} \geq N-1+2r$. Maximize $\dfrac{s_{\mathbf{m}}(\mathbf{u})}{h_{2r}(\mathbf{u})}$ on $[-1,1]^N \setminus \{0\}$ – or just on its cube-boundary.*

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