Entrywise positivity preservers in fixed dimension: II

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(Joint with Alexander Belton, Dominique Guillot, and Mihai Putinar; and with Terence Tao)

The entrywise calculus

Definitions.

1. A real symmetric matrix $A_{N \times N}$ is *positive semidefinite* if all eigenvalues of A are $\geqslant 0$. (Equivalently, $u^T A u \geqslant 0$ for all $u \in \mathbb{R}^N$.)
2. Given $N \geqslant 1$ and $I \subset \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive semidefinite matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)
3. A function $f : I \to \mathbb{R}$ acts *entrywise* on a matrix A via: $f[A] := (f(a_{ij}))$.

Schoenberg and Rudin's theorems

Problem: Given a function $f : I \to \mathbb{R}$, when is it true that

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- Pólya–Szegö (1925 book) via the Schur product theorem (*Crelle* 1911):

  If $f(x) = \sum_{k=0}^{\infty} c_k x^k$ is convergent and $c_k \geqslant 0$, then $f[-]$ preserves positivity on $\mathbb{P}_N$ in all dimensions.

- Schoenberg (*Duke* 1942):

  The converse also holds, *if f is continuous.*

- Rudin (*Duke* 1959); resp. Belton–Guillot–K.–Putinar (*JEMS*, accepted):

  The converse holds for any f, and we only need to assume $f[-]$ preserves positivity on all Toeplitz (resp. Hankel) matrices of rank $\leqslant 3$.

Positivity preservers in fixed dimension

Preserving positivity for fixed N:

- Natural refinement of original problem of Schoenberg.
- In applications: dimension of the problem is known.
  Unnecessarily restrictive to preserve positivity in all dimensions.

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In this talk, we focus on *sums of powers* $\sum_{\alpha \geqslant 0} c_\alpha x^\alpha$ – with $\alpha \in (0, \infty)$ – acting on $\mathbb{P}_N((0, \rho))$.

Question: Find such a function with a negative coefficient, preserving positivity on $\mathbb{P}_N$ for a fixed $N \geqslant 3$.

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- **Previous talk:** Belton–Guillot–K.–Putinar, *Adv. Math.* 2016:
  Yes, if the first N degrees are consecutive.

Outstanding questions: 1. More general polynomials

Analogue of Loewner's necessary condition implies:

Suppose $c_0, c_2, c_3 \neq 0$ are real, $M \geqslant 4$, and $c_0 + c_2 x^2 + c_3 x^3 + c_M x^M$ entrywise preserves positivity on 3×3 correlation matrices.
Then $c_0, c_2, c_3 > 0$. **Can c_M be negative? (Not known.)**

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General case: Fix integers $N \geqslant 3$ and $0 \leqslant n_0 < \cdots < n_{N-1} < M$, not all n_j consecutive. Also fix real scalars $\rho > 0$ and $c_{n_0}, \ldots, c_{n_{N-1}} \neq 0$. Suppose

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Reformulation: Multiplying by $t = |c_M|^{-1}$, does

$$p_t(x) := t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M$$

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entrywise preserve positivity on $\mathbb{P}_N((0, \rho))$ **for any $t > 0$? No example known.**

Main result (for integer powers)

Theorem (K.–Tao, *Amer. J. Math.*, in press)

Fix integers $N \geqslant 1$ *and* $0 \leqslant n_0 < \cdots < n_{N-1} < M$, *and real scalars* $\rho > 0$ *and* $c_{n_0}, \dots, c_{n_{N-1}}$. *For* $t > 0$, *define* $p_t(x) := t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M$.

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$$t \geqslant \mathcal{K}_{\rho, \mathbf{n}, M} := \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}},$$

where $\mathbf{n} := (n_0, \ldots, n_{N-1})$*, the tuples*

$$\mathbf{n}_j := (n_0, \ldots, n_{j-1}, \widehat{n_j}, n_{j+1}, \ldots, n_{N-1}, M), \quad 0 \leqslant j \leqslant N-1,$$

and given a vector $\mathbf{u} = (u_1, \ldots, u_N)$*, its 'Vandermonde determinant' is*

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3. $p_t[-]$ *preserves positivity on Hankel rank-one matrices in* $\mathbb{P}_N((0, \rho))$.

Consequences

1. For the 'initial', consecutive powers $n_j = j$ as in previous talk,

$$\mathcal{K}_{\rho,\mathbf{n},M} = \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{c_j} = \mathcal{K}_{\rho,M}.$$

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4. Surprisingly, the sharp bound on the negative threshold is obtained on rank 1 matrices.

Sketch of the proof

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(3) $\implies$ (2): How does the constant $\mathcal{K}_{\rho,\mathbf{n},M}$ appear from rank-one matrices?

Study the determinants of linear pencils

$$\det p_t[A] = \det \left(t(c_{n_0} A^{\circ n_0} + \cdots + c_{n_{N-1}} A^{\circ n_{N-1}}) - A^{\circ M} \right)$$

for rank-one matrices $A = \mathbf{u}\mathbf{v}^T$.

Schur polynomials

Given an increasing N-tuple of integers $0 \leqslant n_0 < \cdots < n_{N-1}$, the corresponding **Schur polynomial** over a field $\mathbb{F}$ is the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{\mathbf{n}}(u_1, \ldots, u_N) := \frac{\det(u_i^{n_{j-1}})_{i,j=1}^N}{\det(u_i^{j-1})} = \frac{\det(u_i^{n_{j-1}})_{i,j=1}^N}{V(\mathbf{u})}$$

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- Characters of irreducible polynomial representations of $GL_N(\mathbb{C})$, usually defined in terms of semi-standard Young tableaux.
- Weyl Character (Dimension) Formula in Type A:

$$s_{\mathbf{n}}(1, \dots, 1) = \prod_{1 \leqslant i < j \leqslant N} \frac{n_j - n_i}{j - i} = \frac{V(\mathbf{n})}{V((0, 1, \dots, N-1))}.$$

Schur polynomials via semi-standard Young tableaux

Schur polynomials are also defined using semi-standard Young tableaux:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (0, 2, 4)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

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$$\begin{aligned} & s_{(0,2,4)}(u_1, u_2, u_3) \\ = {} & u_3^2 u_2 + u_3^2 u_1 + u_3 u_2^2 + 2 u_3 u_2 u_1 + u_3 u_1^2 + u_2^2 u_1 + u_2 u_1^2 \\ = {} & (u_1 + u_2)(u_2 + u_3)(u_3 + u_1). \end{aligned}$$

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Example 2: Suppose $N = 3$ and $\mathbf{n} = (0,2,3)$:

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1

2
1

Then $s_{(0,2,3)}(u_1,u_2,u_3) = u_1u_2 + u_2u_3 + u_3u_1$.

Sketch of the proof of the main result (cont.)

Technical result used in the proof: Jacobi–Trudi type identity for p_t:

Theorem (K.–Tao, in press)

Let $N \geqslant 1$ and $0 \leqslant n_0 < \cdots < n_{N-1} < M$ be integers. Suppose $c_0, \ldots, c_{N-1} \in \mathbb{F}^\times$ are non-zero scalars in a field $\mathbb{F}$. Define the polynomial

$$p_t(x) := t(c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}) - x^M,$$

and the partitions $\mathbf{n} = (n_0, \ldots, n_{N-1})$ and $\mathbf{n}_j = (n_0, \ldots, \widehat{n_j}, \ldots, n_{N-1}, M)$ as above.

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and the partitions $\mathbf{n} = (n_0, \ldots, n_{N-1})$ and $\mathbf{n}_j = (n_0, \ldots, \widehat{n_j}, \ldots, n_{N-1}, M)$ as above. The following identity holds for all $\mathbf{u}, \mathbf{v} \in \mathbb{F}^N$:

$$\det p_t[\mathbf{u}\mathbf{v}^T] = t^{N-1} V(\mathbf{u}) V(\mathbf{v}) s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \prod_{j=0}^{N-1} c_{n_j} \times \left(t - \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u}) s_{\mathbf{n}_j}(\mathbf{v})}{c_{n_j} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v})} \right).$$

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- If $p_t[\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_N$ for all $\mathbf{u} \in (0, \sqrt{\rho})^N_{\neq}$, and $t, c_{n_0}, \dots, c_{n_{N-1}} > 0$, then

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- In previous talk / 'baby case', we have $\mathbf{n} = (0, 1, \dots, N-1)$.
  Thus the denominator is $c_{n_j} \cdot 1^2 \rightsquigarrow$ maximize $s_{\mathbf{n}_j}(\mathbf{u})^2$ over $[0, \sqrt{\rho}]^N$.
- This is easy, since every Schur polynomial is a *sum* of monomials. *What to do in the general case?*

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  If the ratios $s_{\mathbf{n}_j}/s_{\mathbf{n}}$ were coordinate-wise non-decreasing, the maximum would again occur (like last time) at $(\sqrt{\rho}, \dots, \sqrt{\rho})$, and we could proceed.

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- Need to take a closer look at (ratios of) Schur polynomials.
  Toy example: use $\mathbf{n}_j = (0, 2, 4)$ and $\mathbf{n} = (0, 2, 3)$, worked out above.

Schur Monotonicity Lemma

Example: The ratio $s_{\mathbf{n}_j}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ for $\mathbf{n}_j = (0,2,4)$, $\mathbf{n} = (0,2,3)$ is:

$$f(u_1,u_2,u_3) = \frac{(u_1+u_2)(u_2+u_3)(u_3+u_1)}{u_1u_2+u_2u_3+u_3u_1}, \qquad u_1,u_2,u_3 > 0.$$

Note: both numerator and denominator are monomial-positive (in fact *Schur-positive*, obviously) – hence non-decreasing in each coordinate.

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is non-decreasing in each coordinate.

Schur Monotonicity Lemma (cont.)

Claim: *The ratio* $f(u_1, u_2, u_3) = \dfrac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1 u_2 + u_2 u_3 + u_3 u_1}$,

treated as a ***function*** *on the orthant* $(0, \infty)^3$, *is coordinatewise non-decreasing.*

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Now if we write this as $\sum_{j \geqslant 0} p_j(u_1, u_2) u_3^j$, then each p_j is Schur-positive, i.e. a sum of Schur polynomials:

$$p_0(u_1, u_2) = 0,$$

$$p_1(u_1, u_2) = 2u_1 u_2^2 + 2u_1^2 u_2 = 2 \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} + 2 \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} = 2 s_{(1,3)}(u_1, u_2),$$

$$p_2(u_1, u_2) = (u_1 + u_2)^2 = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$= s_{(0,3)}(u_1, u_2) + s_{(1,2)}(u_1, u_2).$$

Proof-sketch of Schur Monotonicity Lemma

The proof for general $\mathbf{m} \geqslant \mathbf{n}$ is similar:

By symmetry, and the quotient rule of differentiation, it suffices to show that

$$s_{\mathbf{n}} \cdot \partial_{u_N}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_N}(s_{\mathbf{n}})$$

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Our Schur Monotonicity Lemma in fact shows that the coefficient of each u_N^j is (also) Schur-positive.

Key ingredient: Schur-positivity result by Lam–Postnikov–Pylyavskyy (2007). (In turn, this emerged out of Skandera's 2004 results on determinant inequalities for totally non-negative matrices.) □

Proof-sketch of main result (concl.)

Returning to the proof of the main result:

- If $p_t[\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_N$ for all $\mathbf{u} \in (0, \sqrt{\rho})^N$, and $t, c_{n_0}, \dots, c_{n_{N-1}} > 0$, then

$$0 \leqslant \frac{\det p_t[\mathbf{u}\mathbf{u}^T]}{t^{N-1} V(\mathbf{u})^2 s_{\mathbf{n}}(\mathbf{u})^2 c_{n_0} \cdots c_{n_{N-1}}} = t - \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_{n_j} s_{\mathbf{n}}(\mathbf{u})^2}.$$

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- By the Schur Monotonicity Lemma, this is if and only if

$$t \geqslant \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\sqrt{\rho}, \dots, \sqrt{\rho})^2}{c_{n_j} s_{\mathbf{n}}(\sqrt{\rho}, \dots, \sqrt{\rho})^2} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}} = \mathcal{K}_{\rho, \mathbf{n}, M},$$

by the Weyl Dimension Formula. □

Outstanding questions: 2. Real powers

Analogue of Loewner's necessary condition implies:
Suppose $c_0, c_e, c_\pi \neq 0$ are real, $M \in (\pi, \infty)$, and

$$c_0 + c_e x^e + c_\pi x^\pi + c_M x^M$$

entrywise preserves positivity on $\mathbb{P}_3((0, \rho))$.
Then $c_0, c_e, c_\pi > 0$. **Can c_M be negative? (Not known.)**

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General case:
Fix an integer $N \geqslant 3$ and *real powers* $0 \leqslant n_0 < \cdots < n_{N-1} < M$.
Also fix positive real scalars $\rho, c_{n_0}, \ldots, c_{n_{N-1}} > 0$. Suppose

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Can c_M be negative? How about a sharp bound, as above?
(More generally, which coefficients in such a preserver can be negative?)

Generalized Vandermonde determinants

The technical heart of the proof is similar:

Theorem (K.–Tao, in press)

Let $N \in \mathbb{N}$ *and* $0 \leqslant n_0 < \cdots < n_{N-1} < M$ *be* ***real****. Suppose* $c_0, \dots, c_{N-1} \in (0, \infty)$, *and define*

$$p_t(x) := t(c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}) - x^M.$$

Then for $\mathbf{u} \in (0,\infty)^N_{\neq}$,

$$\det p_t[\mathbf{u}\mathbf{u}^T] = t^{N-1} \det(\mathbf{u}^{\circ \mathbf{n}})^2 \prod_{j=0}^{N-1} c_{n_j} \times \left(t - \sum_{j=0}^{N-1} \frac{\det(\mathbf{u}^{\circ \mathbf{n}_j})^2}{\det(\mathbf{u}^{\circ \mathbf{n}})^2} \right),$$

where $\mathbf{u}^{\circ \mathbf{n}} := (u_i^{n_{j-1}})_{i,j=1}^N$ *is a generalized Vandermonde matrix.*

Now need to maximize a ratio of Vandermonde determinants, again with $\mathbf{n}_j \geqslant \mathbf{n}$ coordinate-wise.

Schur–Vandermonde Monotonicity Lemma

Theorem (K.–Tao, in press)

For real tuples $n_0 < \cdots < n_{N-1}$ *and* $m_0 < \cdots < m_{N-1}$ *such that* $n_j \leqslant m_j \ \forall j$,

$$f : (0,\infty)^N_{\neq} \to \mathbb{R}, \qquad f(\mathbf{u}) := \frac{\det \mathbf{u}^{\circ \mathbf{m}}}{\det \mathbf{u}^{\circ \mathbf{n}}}$$

defined over 'pairwise distinct' u_j, *is non-decreasing in each coordinate.*

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- By multiplying by $(u_1 \cdots u_N)^{-n_0}$, we may assume all $m_j, n_j \geqslant 0$.

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Proof:

- By multiplying by $(u_1 \cdots u_N)^{-n_0}$, we may assume all $m_j, n_j \geqslant 0$.
- W.l.o.g., $u_1 < \cdots < u_N$. Now if m_j, n_j are rational, say with common denominator $K \in \mathbb{N}$, work with $y_j = u_j^{1/K}$:
$$f(\mathbf{u}) = \frac{\det \mathbf{u}^{\circ \mathbf{m}}}{\det \mathbf{u}^{\circ \mathbf{n}}} = \frac{\det \mathbf{y}^{\circ (K \cdot \mathbf{m})}}{\det \mathbf{y}^{\circ (K \cdot \mathbf{n})}} = \frac{V(\mathbf{y}) \cdot s_{K \cdot \mathbf{m}}(\mathbf{y})}{V(\mathbf{y}) \cdot s_{K \cdot \mathbf{n}}(\mathbf{y})}.$$
This is coordinate-wise non-decreasing in $\mathbf{y}$ by the Schur Monotonicity Lemma, hence in $\mathbf{u}$.

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  This is coordinate-wise non-decreasing in $\mathbf{y}$ by the Schur Monotonicity Lemma, hence in $\mathbf{u}$.
- Finally, extend to real tuples $\mathbf{m}, \mathbf{n}$ by rational approximation. □

Main result (for real powers)

This helps show:

Theorem (K.–Tao, *Amer. J. Math.*, in press)

Fix $N \in \mathbb{N}$ *and real scalars*

$$n_0 < \cdots < n_{N-1} < M, \qquad \rho > 0, \qquad c_{n_0}, \dots, c_{n_{N-1}}.$$

For $t > 0$, *define* $p_t(x) := t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M$. *The following are equivalent.*

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1. $p_t[-]$ *preserves positivity on rank-one matrices in* $\mathbb{P}_N((0,\rho))$.
2. *All coefficients* $c_{n_j} > 0$, *and*
$$t \geqslant \mathcal{K}_{\rho,\mathbf{n},M} := \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}.$$
3. $p_t[-]$ *preserves positivity on Hankel rank-one matrices in* $\mathbb{P}_N((0,\rho))$.

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3. *$p_t[-]$ preserves positivity on Hankel rank-one matrices in $\mathbb{P}_N((0,\rho))$.*

If all $n_j \in \mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$, then the rank-constraint in (1) can be removed.

Extension to power series

The above results say that if $f(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j}$ and $g(x) := x^M$ for an integer $M > n_{N-1}$, then we have the **linear matrix inequality**

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Proposition (K.–Tao)

Yes.

Further applications

1. In fact we work with more general '*Laplace transforms*'

$$g(x) = \int_{n_{N-1}+\varepsilon}^{\infty} x^t \, d\mu(t), \qquad \varepsilon > 0,$$

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2. These results lead to (sharp) linear matrix inequalities, for Hadamard powers.

3. Application to *spectrahedra and matrix cubes*:
   Upper and lower bounds, which are asymptotically equal.

4. Reformulation in terms of *generalized Rayleigh quotients*.

Weak majorization through Schur polynomials

- Our Schur Monotonicity Lemma implies in particular:

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1,\dots,1)}{s_{\mathbf{n}}(1,\dots,1)}, \qquad \forall \mathbf{u} \in [1,\infty)^N.$$

  if $\mathbf{m}$ dominates $\mathbf{n}$ coordinatewise.
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Theorem (K.–Tao, *Amer. J. Math.*, in press)

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This problem was studied originally by Skandera and others in the 2010s, for integer powers, and *on the entire positive orthant* $(0,\infty)^N$:

Cuttler–Greene–Skandera conjecture

Theorem (Cuttler–Greene–Skandera and Sra, *Eur. J. Comb.*, 2011, 2016)

Fix integers $0 \leqslant n_0 < \cdots < n_{N-1}$ *and* $0 \leqslant m_0 < \cdots < m_{N-1}$. *Then*

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Yes, and Yes:

Majorization via Vandermonde determinants

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Our preceding result: $-\mathbf{m} \succ_w -\mathbf{n}$; and $\mathbf{m} \succ_w \mathbf{n} \iff \mathbf{m}$ majorizes $\mathbf{n}$.

Open question: Optimizing over $[-1,1]^N$?

- The previous talk and this talk concerned polynomials/power series that entrywise preserve positive semidefiniteness in a fixed dimension.
- The maximization of $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ over $(0,1]^N$ reveals tight bounds on certain polynomial preservers, acting on $\mathbb{P}_N([0,1])$. (By homogeneity and continuity, maximize only over the cube-boundary $(0,1]^N \cap \partial(0,1]^N$.)

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- What about on *all* correlation matrices? Need to upper-bound $s_{\mathbf{m}}(\mathbf{u})^2/s_{\mathbf{n}}(\mathbf{u})^2$ over all of $[-1,1]^N \setminus \{0\}$.
- For this, need to ensure $s_{\mathbf{n}}(\mathbf{u})$ does not vanish except at 0. **Facts:**
  (1) The only such $\mathbf{n} = (0,1,\dots,N-2,N-1+2r)$ for $r \in \mathbb{Z}^{\geqslant 0}$.
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Question: *Say* $m_j \geqslant j$ *for* $j = 0,1,\dots,N-2$, *and* $m_{N-1} \geqslant N-1+2r$.
Maximize $\dfrac{s_{\mathbf{m}}(\mathbf{u})^2}{h_{2r}(\mathbf{u})^2}$ *on* $[-1,1]^N \setminus \{0\}$ *– or just on its cube-boundary.*

Selected publications

A. Belton, D. Guillot, A. Khare, and M. Putinar:

[1] *Matrix positivity preservers in fixed dimension. I,* Advances in Math., 2016.
[2] *Moment-sequence transforms,* J. Eur. Math. Soc., accepted.
[3] *A panorama of positivity* (survey), Shimorin volume + Ransford-60 proc.

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[4] *On the sign patterns of entrywise positivity preservers in fixed dimension,* (With T. Tao) Amer. J. Math., in press.

[5] *Matrix analysis and preservers of (total) positivity,* 2020+, Lecture notes (website); forthcoming book – Cambridge Press + TRIM.