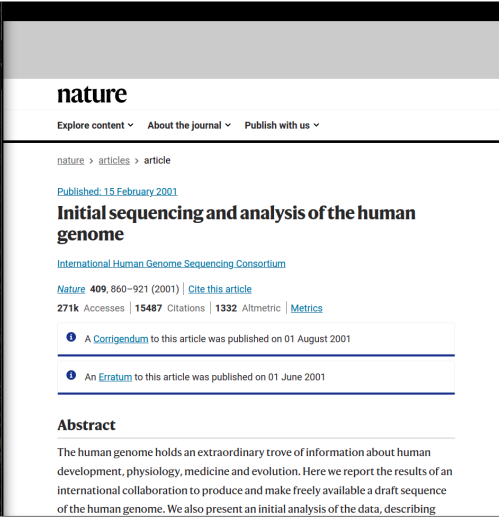
Polymath 14: A crowd-sourced, computer-aided analysis-definition of abelian groups

Apoorva Khare
Indian Institute of Science

Indian Academy of Sciences
Mid-Year Meeting (IISc), July 2022

Crowdsourced projects in the sciences

The sciences have seen very highly collaborative research projects – the Genome Project (Nobel Prize 2002, Breakthrough Prize in Life Sciences 2013):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Published: 15 February 2001

Initial sequencing and analysis of the human genome

International Human Genome Sequencing Consortium

Nature **409**, 860–921 (2001) | Cite this article

271k Accesses | **15487** Citations | **1332** Altmetric | Metrics

A Corrigendum to this article was published on 01 August 2001

An Erratum to this article was published on 01 June 2001

Abstract

The human genome holds an extraordinary trove of information about human development, physiology, medicine and evolution. Here we report the results of an international collaboration to produce and make freely available a draft sequence of the human genome. We also present an initial analysis of the data, describing

(200+ authors, some representing institutions!)

Crowdsourced projects in the sciences

The Higgs Boson discovery (Fundamental Physics Prize 2012, Nobel Prize 2013, Copley Medal 2015) – 3000+ authors:

ELSEVIER

Physics Letters B

Volume 716, Issue 1, 17 September 2012, Pages 1-29

Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC ☆

This paper is dedicated to the memory of our ATLAS colleagues who did not live to see the full impact and significance of their contributions to the experiment.

ATLAS Collaboration *, G. Aad [48], T. Abajyan [21], B. Abbott [111], J. Abdallah [12], S. Abdel Khalek [115], A.A. Abdelalim [49], O. Abdinov [11], R. Aben [105], B. Abi [112], M. Abolins [88], O.S. AbouZeid [158], H. Abramowicz [153], H. Abreu [136], B.S. Acharya [164a, 164b], L. Adamczyk [38], D.L. Adams [25], T.N. Addy [56] ... L. Zwalinski [30]

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https://doi.org/10.1016/j.physletb.2012.08.020

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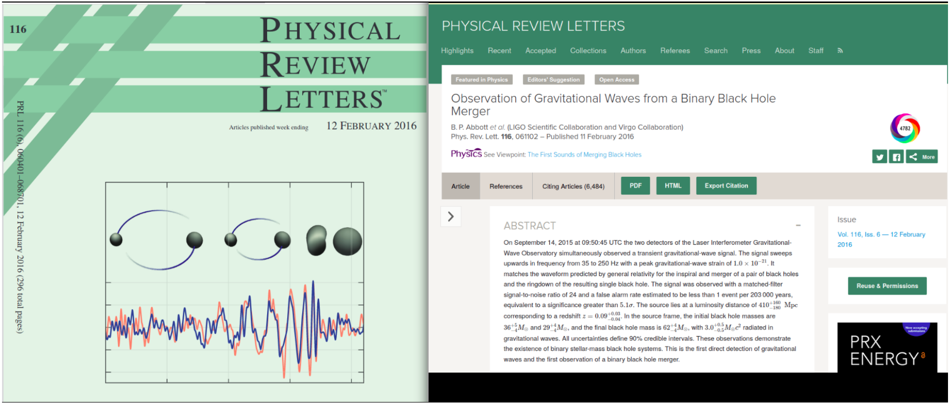
Crowdsourced projects in the sciences

Gravitational waves (Nobel Prize 2017):



(100+ institutions, 800+ authors)

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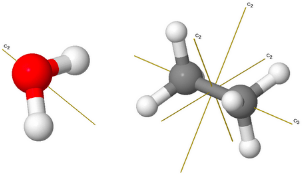
I will describe a "modern" collaboration mechanism – the Polymath project – in which I was involved, and which helped answer a basic question about groups.

Groups

Groups are ubiquitous in science... as *symmetries*.



(Credit: Symmetry @ Otterbein site)

- The group of rotations in the plane.
  The rigid body motions (rotations, reflections, translations).
- Physical laws of nature/spacetime $\longrightarrow$ Lorentz group of symmetries.
- The space of eigenvectors of a matrix (for some eigenvalue λ) $\longrightarrow$ group under $+, -$.
- Permutations of a set $\{1, 2, \ldots, n\}$ $\longrightarrow$ symmetric group S_n (size $= n!$).

Group notation + Abelian groups

Given two symmetries of a system α, β, one can:

- *compose* them: $\alpha \circ \beta,\ \beta \circ \alpha$,
- *reverse* them: $\alpha^{-1},\ \beta^{-1}$.
- Composing α and α^{-1} does "nothing", i.e. yields the *identity* symmetry e.

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Do all compositions of symmetries always give the same answer? No:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix},$$
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$$\alpha \circ \beta = \beta \circ \alpha \qquad \iff \qquad \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1} = e.$$

The quantity $\alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$ is called the commutator of α, β.

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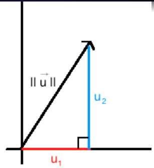

(Credit: Wikidot)

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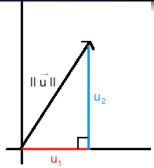

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Formal definition: A norm on a group G is a function $\|\cdot\| : G \to \mathbb{R}$, satisfying:

- $\|\alpha\| > 0$ for all non-identity elements/symmetries α in G. And $\|e\| = 0$.
- $\|\alpha \circ \beta\| \leq \|\alpha\| + \|\beta\|$ (triangle inequality).
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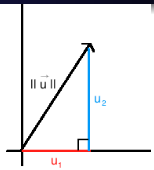

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Example: (As above!) $\mathbb{R}^2 =$ all vectors $\mathbf{u}$ in the plane (or translations by $\mathbf{u}$, under $+$).
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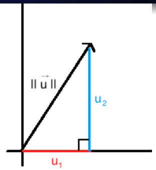

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Question (Khare): Does there exist a *non-abelian* group with a norm?

Asking around...

- I came to the above question (non-abelian group with a norm?) in 2015, motivated by some work in probability theory.
  [Khare & Rajaratnam, published in *Annals of Probability* 2017]

- Starting from April 2015, I emailed several experts, in Canada, India, Poland, USA (Courant, Harvard, Northwestern, Wisconsin, ...).

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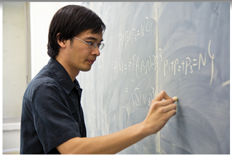
- **No** such examples were found.
- **No** such theorems were known.

What is the answer? Find at least one such group? Or prove that such a group can *never* exist!

December 15–16, **2017:** Then I visited a collaborator, Terence Tao (UCLA).



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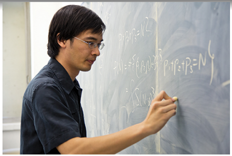

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Our discussion:

- Suppose G is any group with a norm.
- Take any two elements α, β in G.
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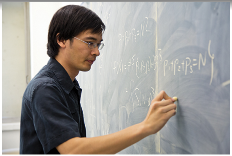

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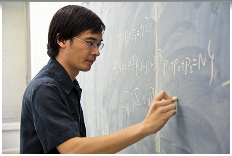

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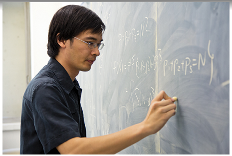

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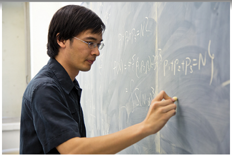

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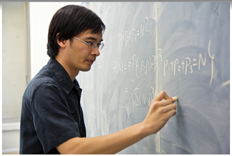

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https://terrytao.wordpress.com/2017/12/16/bi-invariant-metrics-o

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Bi-invariant metrics of linear growth on the free group

16 December, 2017 in math.GR, math.MG, question | Tags: free group, geometric group theory, polymath14

Here is a curious question posed to me by Apoorva Khare that I do not know the answer to. Let F_2 be the free group on two generators a, b. Does there exist a metric d on this group which is

- bi-invariant, thus $d(xg, yg) = d(gx, gy) = d(x, y)$ for all $x, y, g \in F_2$; and
- linear growth in the sense that $d(x^n, 1) = nd(x, 1)$ for all $x \in F_2$ and all natural numbers n?

By defining the "norm" of an element $x \in F_2$ to be $\|x\| := d(x, 1)$, an equivalent formulation of the problem asks if there exists a non-negative norm function $\|\| : F_2 \to \mathbf{R}^+$ that obeys the conjugation invariance

$$\|gxg^{-1}\| = \|x\| \qquad (1)$$

for all $x, g \in F_2$, the triangle inequality

$$\|xy\| \leq \|x\| + \|y\| \qquad (2)$$

for all $x, y \in F_2$, and the linear growth

$$\|x^n\| = |n|\|x\| \qquad (3)$$

Scott Aaronson – Ten signs a claimed mathematical proof is wrong

Timothy Gowers – Elsevier — my part in its downfall

Timothy Gowers – The two cultures of mathematics

William Thurston – On proof and progress in mathematics

What is not clear to me is if one can keep arguing like this to continually improve the upper bounds on the norm $\|g\|$ of a given non-trivial group element g to the point where this norm must in fact vanish, which would demonstrate that no metric with the above properties on F_2 would exist (and in fact would impose strong constraints on similar metrics existing on other groups as well). It is also tempting to use some ideas from geometric group theory (e.g. asymptotic cones) to try to understand these metrics further, though I wasn't able to get very far with this approach. Anyway, this feels like a problem that might be somewhat receptive to a more crowdsourced attack, so I am posing it here in case any readers wish to try to make progress on it.

Timeline of the reducing bound

(All times are in Indian Standard Time, in December 2017.
All progress is as recorded on Tao's blog.)

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  which Siddhartha Gadgil and his computer broke through!

Timeline of the reducing bound (cont.)

- *20 Dec, 11:12 am:* $KQ = 22/23 \approx 0.956522$.
- *21 Dec, 9:30 am:* (Siddhartha Gadgil and his computer) $KQ \approx 0.816$. (Understood by Pace Nielsen.)

20 December, 2017 at 8:00 pm
Siddhartha Gadgil

One can get a bound on the commutator of 0.816 with a brutal proof (not optimized by any means). The proof (computer generated but formatted to be readable) is at

https://github.com/siddhartha-gadgil/Superficial/wiki/A-commutator-bound

Comments welcome. Someone cleverer may extract useful inequalities from this (or find an error).

Reply

20 December, 2017 at 9:40 pm
Pace Nielsen

This was beautiful! My intuition was, before scouring your file, that we now have strong evidence that the norm will be forced to equal zero on commutators. Thus, elements like $xyx^{-1}y^{-1}x$ should have norm the same as x. Our job is then to bound the norm of words like $(xyx^{-1}y^{-1})^k x$ (for larger and larger values of k) nearer and nearer the norm of x.

I was surprised, when reading your file, that this seems to be the path that the computer has taken, with a few additional ideas thrown in. The first 43 lines establish

$$\|(xyx^{-1}y^{-1})^2x\| \leq \frac{1}{18}(20\|x\| + 18\|y\|)$$

and lines 44 to 73 establish

$$\|xyx^{-1}y^{-1}x\| \leq \frac{1}{18}(18\|x\| + 10\|y\|)$$

(which, incidentally, gives a new $(\gamma,\delta) = (1, 5/9)$ value). Lines 74 to 119, using these previous bounds, establishe a similar type of bound on $\|(xyx^{-1}y^{-1})^6x\|$. And the last few lines use this information to bound the norm of a commutator.

This was very well done!

We can improve the first computation to the bound

$$\|(xyx^{-1}y^{-1})^2x\| \leq \|x\| + \|y\|$$

A commutator bound

Siddhartha Gadgil edited this page on Dec 21, 2017 · 2 revisions

Gross proof time

Here is a computer generated proof of a bound on the length of the commutator abāb̄ for a *linear* norm on the free group with the lengths of the generators bounded above by 1.

1. ○ |ā| ≤ 1.0
2. ○ |b̄ab| ≤ 1.0 using |ā| ≤ 1.0
3. ○ |b̄| ≤ 1.0
4. ○ |ab̄ā| ≤ 1.0 using |b̄| ≤ 1.0

124. ○ |b̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄| ≤ 12.859649122807017 using |b̄abāb̄abāb̄abāb̄abāb̄abāb̄ab| ≤ 4.9298245614035086 and |āb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄| ≤ 7.9298245614035086
125. ○ |abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄| ≤ 13.859649122807017 using |abā| ≤ 1.0 and |b̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄| ≤ 12.859649122807017
126. ○ |abāb̄| ≤ 0.8152734778121775 using |abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄abāb̄| ≤ 13.859649122807017 by taking 17th power

Conclusion

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$$f(m,k) \leqslant \frac{1}{2}\left(f(m+1,k) + f(m-1,k-1)\right).$$

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21 December, 2017 at 2:27 pm
Terence Tao

I think this does indeed work to give arbitrarily small bounds on $\|[x,y]\|$, killing off the problem! One can work with some m between $\sqrt{k}$ and k, say $m \approx k^{2/3}$. A simple random walk that starts at (m,k) and repeatedly moves by a step $(-1,0)$ or $(1,-1)$ with a 1/2 probability of each will end up hitting $(m',0)$ for some $m' \approx m$ with exponentially high probability by the Chernoff bound, which implies the bound $\|x^m[x,y]^k\| \ll m\|x\| + \|[x,y]\|$, which by the triangle inequality gives $\|[x,y]\| \ll \frac{m}{k}\|x\|$, and sending $k \to \infty$ we win. I'll write the details on the Wiki. EDIT: now available at http://michaelnielsen.org/polymath1/index.php?title=Linear_norm#Solution

3 0 Rate This

The theorem and the paper

Theorem (D.H.J. Polymath, *Algebra & Number Th.* 2018)

Let G be a group. Then G has a norm if and only if G is abelian and torsion free,

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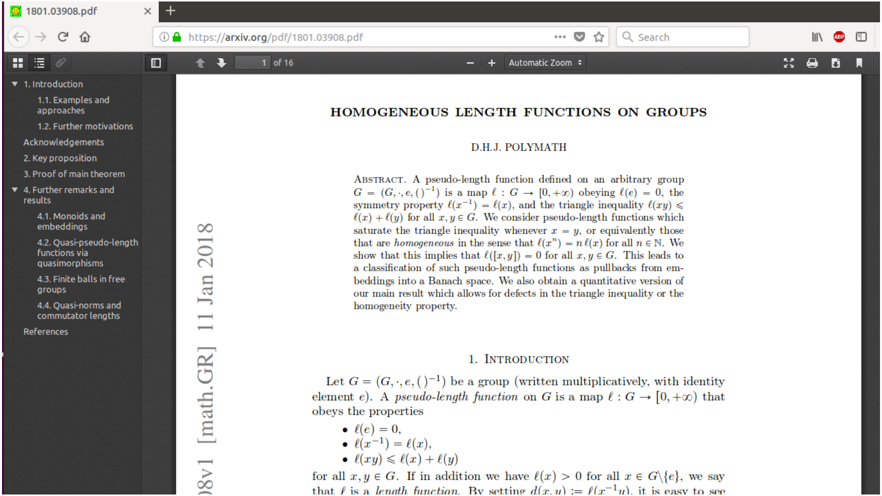
Let G be a group. Then G has a norm if and only if G is abelian and torsion free, if and only if G is an additive subgroup of a Banach space.



HOMOGENEOUS LENGTH FUNCTIONS ON GROUPS

D.H.J. POLYMATH

[math.GR] 11 Jan 2018

ABSTRACT. A pseudo-length function defined on an arbitrary group $G = (G, \cdot, e, ()^{-1})$ is a map $\ell : G \to [0, +\infty)$ obeying $\ell(e) = 0$, the symmetry property $\ell(x^{-1}) = \ell(x)$, and the triangle inequality $\ell(xy) \leqslant \ell(x) + \ell(y)$ for all $x, y \in G$. We consider pseudo-length functions which saturate the triangle inequality whenever $x = y$, or equivalently those that are *homogeneous* in the sense that $\ell(x^n) = n\,\ell(x)$ for all $n \in \mathbb{N}$. We show that this implies that $\ell([x,y]) = 0$ for all $x, y \in G$. This leads to a classification of such pseudo-length functions as pullbacks from embeddings into a Banach space. We also obtain a quantitative version of our main result which allows for defects in the triangle inequality or the homogeneity property.

1. INTRODUCTION

Let $G = (G, \cdot, e, ()^{-1})$ be a group (written multiplicatively, with identity element e). A *pseudo-length function* on G is a map $\ell : G \to [0, +\infty)$ that obeys the properties

- $\ell(e) = 0$,
- $\ell(x^{-1}) = \ell(x)$,
- $\ell(xy) \leqslant \ell(x) + \ell(y)$

for all $x, y \in G$. If in addition we have $\ell(x) > 0$ for all $x \in G \backslash \{e\}$, we say that ℓ is a *length function*. By setting $d(x, y) := \ell(x^{-1}y)$, it is easy to see