Analysis applications of Schur polynomials

ILAS Invited Address
2023 Joint Math Meetings

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1. Entrywise positivity preservers and Schur polynomials

Introduction

Positivity (and preserving it) studied in many settings in the literature.

Different flavors of positivity:

- $\mathbb{P}_N$: Positive semidefinite $N \times N$ (real symmetric) matrices:

$$u^T A u \geqslant 0, \quad \forall u \in \mathbb{R}^N.$$

- Positive definite sequences/Toeplitz matrices
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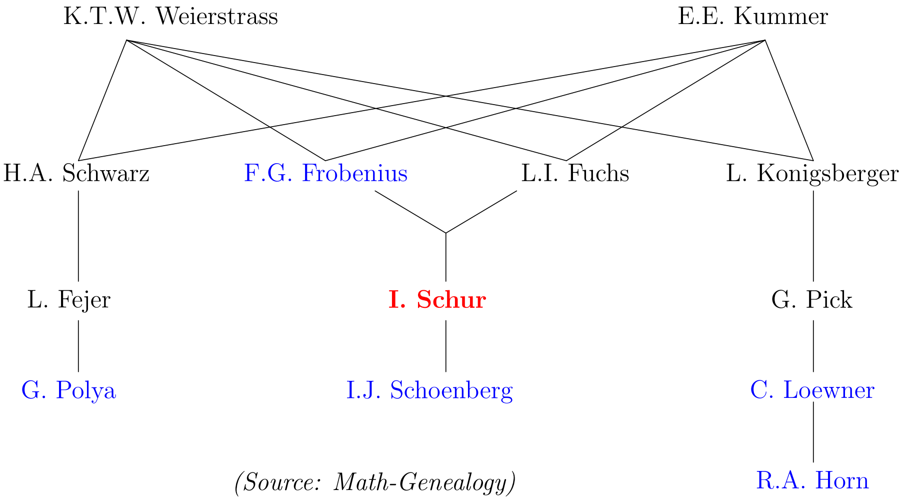
Question: Classify the positivity preservers in these settings.
Studied for the better part of a century.

Some contributors to entrywise functions



(Source: Math-Genealogy)

Entrywise functions preserving positivity

Definition: Given $N \geqslant 1$ and $I \subseteq \mathbb{R}$, let $\mathbb{P}_N(I)$ denote the $N \times N$ positive (semidefinite) matrices, with entries in I. (Say $\mathbb{P}_N = \mathbb{P}_N(\mathbb{R})$.)

Problem: For which functions $f : I \to \mathbb{R}$ is it true that

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- (*Long history!*) The Schur Product Theorem [Schur, *Crelle* 1911] says: If $A, B \in \mathbb{P}_N$, then so is $A \circ B := (a_{ij} b_{ij})$.
- As a consequence, $f(x) = x^k$ $(k \geqslant 0)$ preserves positivity on $\mathbb{P}_N$ for all N.

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- (Pólya–Szegö, 1925): Taking sums and limits, if $f(x) = \sum_{k=0}^{\infty} c_k x^k$ is convergent and $c_k \geqslant 0$, then $f[-]$ preserves positivity.

  Question: Anything else?

Schoenberg's theorem

Surprisingly, the answer is **no**, if $f[-]$ preserves positivity in *all* dimensions:

Theorem (Schoenberg, *Duke Math. J.* 1942)

Say $I = [-1,1]$ *and* $f : I \to \mathbb{R}$ *is continuous. The following are equivalent:*

1. $f[A] \in \mathbb{P}_N$ *for all* $A \in \mathbb{P}_N(I)$ *and all* N.
2. f *is analytic on* I *and has nonnegative Taylor coefficients.*
   In other words, $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *on* $[-1,1]$ *with all* $c_k \geqslant 0$.

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This continuity assumption was since removed, and the test set $\bigcup_{N \geqslant 1} \mathbb{P}_N(I)$ was greatly reduced, drawing from Fourier analysis and moment-problems:

Theorem (Rudin (*Duke* 1959); Belton–Guillot–K.–Putinar (*JEMS* 2022))

Suppose $0 < \rho \leqslant \infty$. *If* $f[-]$ *preserves positivity on all* ***Toeplitz*** *(resp.* ***Hankel****) matrices of rank* $\leqslant 3$ *with entries in* $I = (-\rho, \rho)$, *then* $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *with all* $c_k \geqslant 0$.

Entrywise positivity preservers in fixed dimension

Preserving positivity for fixed N:

- Natural refinement of original problem of Schoenberg.
- In applications: dimension of the problem is known.
  Unnecessarily restrictive to preserve positivity in all dimensions.

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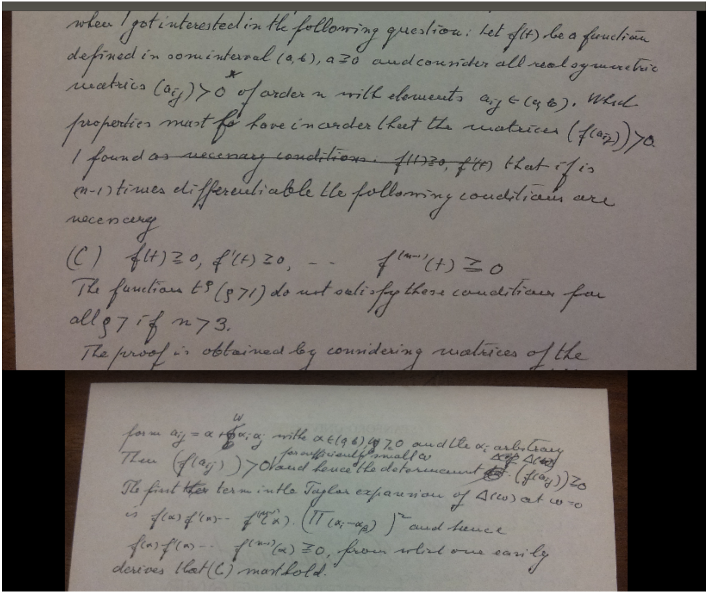
What is known in fixed dimension?

Essentially the only result is a necessary condition, by C. Loewner (in the PhD thesis of his student Roger A. Horn, in *Trans. AMS* 1969).
Loewner had initially summarized these computations in a letter to Josephine Mitchell (Penn. State) in 1967:

Loewner's computations



when I got interested in the following question: Let $f(t)$ be a function defined in some interval (a, b), $a \geq 0$ and consider all real symmetric matrices $(a_{ij}) > 0$* of order n with elements $a_{ij} \in (a, b)$. What properties must f have in order that the matrices $(f(a_{ij})) > 0$.
I found as ~~necessary conditions: $f(t) \geq 0$, $f'(t)$~~ that if f is $(n-1)$ times differentiable the following conditions are necessary

(C) $f(t) \geq 0, f'(t) \geq 0, \ldots f^{(n-1)}(t) \geq 0$

The functions t^ρ $(\rho > 1)$ do not satisfy these conditions for all $\rho > 1$ if $n > 3$.
The proof is obtained by considering matrices of the form $a_{ij} = \alpha + \omega \alpha_i \alpha_j$ with $\alpha \in (a, b)$, $\omega \geq 0$ and the α_i arbitrary. Then $(f(a_{ij})) > 0$ for sufficiently small ω and hence the determinant $\Delta(\omega) = (f(a_{ij})) \geq 0$. The first term in the Taylor expansion of $\Delta(\omega)$ at $\omega = 0$ is $f(\alpha) f'(\alpha) \ldots f^{(n-1)}(\alpha) \cdot \left(\prod (\alpha_i - \alpha_\beta)\right)^2$ and hence $f(\alpha) f'(\alpha) \ldots f^{(n-1)}(\alpha) \geq 0$, from which one easily derives that (C) must hold.

Special case: Polynomials

Following Schoenberg (1942) and Rudin (1959), suppose

$$f(t) = \sum_{j=1}^{N} c_j t^{j-1} + c' t^M, \qquad c_j \in \mathbb{R}, \ M \geqslant N$$

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More generally, the first N *nonzero* Maclaurin coefficients must be positive.
Q2: *Can the next one be negative?*

Polynomials preserving positivity in fixed dimension

Theorem (K.–Tao, *Amer. J. Math.* 2021)

Fix $\rho > 0$ *and integers* $0 \leqslant n_1 < \cdots < n_N < M$, *and let*

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$$\mathcal{C} := \sum_{j=1}^{N} \frac{\rho^{M-n_j}}{c_j} \prod_{i=1, i \neq j}^{N} \frac{(M-n_i)^2}{(n_j-n_i)^2}.$$

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- This holds even if n_j, M are not integers.
- "Baby case (Q1)": Belton–Guillot–K.–Putinar in [*Adv. Math.* 2016].

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- Essentially, the *Weyl dimension formula* in representation theory – or the *principal specialization formula* for Schur polynomials.
- Schur polynomials (algebraic characters) – now treated as *functions* on the positive orthant $(0, \infty)^N$ – are the key tool used to prove the theorem.

Schur polynomials

Given an increasing N-tuple of integers $0 \leqslant n_1 < \cdots < n_N$,
the corresponding **Schur polynomial** over a field $\mathbb{F}$ is the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{\mathbf{n}}(u_1, \ldots, u_N) := \frac{\det(u_i^{n_j})_{i,j=1}^N}{\det(u_i^{j-1})} = \frac{\det(u_i^{n_j})_{i,j=1}^N}{V(\mathbf{u})}$$

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- Basis of homogeneous symmetric polynomials in $u_1, \ldots, u_N$.
- Characters of irreducible polynomial representations of $GL_N(\mathbb{C})$, usually defined in terms of semi-standard Young tableaux.

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- Weyl Character (Dimension) Formula in Type A:

$$s_{\mathbf{n}}(1, \dots, 1) = \prod_{1 \leqslant i < j \leqslant N} \frac{n_j - n_i}{j - i} = \frac{V(\mathbf{n})}{V((0, 1, \dots, N-1))}.$$

Schur polynomials via semi-standard Young tableaux

Schur polynomials are also defined using semi-standard Young tableaux:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (0, 2, 4)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

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Note: Both polynomials are coordinate-wise non-decreasing on $(0, \infty)^N$.

Schur Monotonicity Lemma

Example: The ratio $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ for $\mathbf{m} = (0,2,4)$, $\mathbf{n} = (0,2,3)$ is:

$$f(u_1,u_2,u_3) = \frac{(u_1+u_2)(u_2+u_3)(u_3+u_1)}{u_1u_2+u_2u_3+u_3u_1}, \qquad u_1,u_2,u_3 > 0.$$

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Theorem (K.–Tao, *Amer. J. Math.*, 2021)

For integer tuples $0 \leqslant n_1 < \cdots < n_N$ and $0 \leqslant m_1 < \cdots < m_N$ such that $n_j \leqslant m_j \ \forall j$, the function

$$f : (0, \infty)^N \to \mathbb{R}, \qquad f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$$

is non-decreasing in each coordinate.
(In fact, a stronger Schur positivity phenomenon holds.)

Schur Monotonicity Lemma (cont.)

Claim: *The ratio* $f(u_1, u_2, u_3) = \dfrac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1 u_2 + u_2 u_3 + u_3 u_1}$,

treated as a ***function*** *on the orthant* $(0, \infty)^3$, *is coordinate-wise non-decreasing.*

Schur Monotonicity Lemma (cont.)

Claim: *The ratio* $f(u_1, u_2, u_3) = \dfrac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1 u_2 + u_2 u_3 + u_3 u_1}$,

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and this is monomial-positive (hence numerically positive).

In fact, upon writing this as $\sum_{j\geqslant 0} p_j(u_1,u_2)u_3^j$, each p_j is Schur-positive, i.e. a sum of Schur polynomials:

$$p_0(u_1,u_2) = 0,$$

$$p_1(u_1,u_2) = 2u_1u_2^2 + 2u_1^2u_2 = 2\,\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array} + 2\,\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array} = 2s_{(1,3)}(u_1,u_2),$$

$$p_2(u_1,u_2) = (u_1+u_2)^2 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$$

$$= s_{(0,3)}(u_1,u_2) + s_{(1,2)}(u_1,u_2).$$

Proof-sketch of Schur Monotonicity Lemma

The proof for general $\mathbf{m} \geqslant \mathbf{n}$ is similar:

By symmetry, and the quotient rule of differentiation, it suffices to show that

$$s_{\mathbf{n}} \cdot \partial_{u_N}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_N}(s_{\mathbf{n}})$$

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Key ingredient: Schur-positivity result by Lam–Postnikov–Pylyavskyy [*Amer. J. Math.* 2007]. □

2. (Weak) Majorization inequalities

Weak majorization through Schur polynomials

- Our Schur Monotonicity Lemma implies in particular:

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1,\dots,1)}{s_{\mathbf{n}}(1,\dots,1)} = \frac{V(\mathbf{m})}{V(\mathbf{n})}, \qquad \forall \mathbf{u} \in [1,\infty)^N.$$

  if $\mathbf{m}$ dominates $\mathbf{n}$ coordinate-wise.
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Now extended to *real* tuples (generalized Vandermonde determinants):

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

Given reals $n_1 < \cdots < n_N$ and $m_1 < \cdots < m_N$, TFAE:

1. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct-coordinate" tuples* $\mathbf{u} \in [1,\infty)^N_{\neq}$.
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Ingredients of proof: (a) "First-order" approximation of Schur polynomials; (b) Harish-Chandra–Itzykson–Zuber integral; (c) Schur convexity result.

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Yes, and Yes:

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Precursors to Cuttler-Greene-Skandera (and Sra, . . .)

Instead of using Schur polynomials, what if one uses other symmetric functions?

C-G-S: $\frac{s_{\mathbf{m}}(u_1, \dots, u_N)}{s_{\mathbf{m}}(1, \dots, 1)} \geqslant \frac{s_{\mathbf{n}}(u_1, \dots, u_N)}{s_{\mathbf{n}}(1, \dots, 1)}$ on $(0, \infty)^N \iff \mathbf{m}$ majorizes $\mathbf{n}$.

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Question: What if one restricts to $\mathbf{u} \in [1,\infty)^N$?

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The C-G-S–Sra inequality (and its follow-up by K.–Tao)
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Other majorization inequalities have been shown by:

- Maclaurin (1729)
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Recent vast generalization by McSwiggen–Novak [*IMRN* 2022] to all Weyl groups $W \quad \rightsquigarrow \quad W$-*majorization*.

3. Symmetric function identities

Going back. . . to Loewner, 1967

In his 1967 letter to Josephine Mitchell, Loewner's approach was as follows:

- Suppose $f[-]$ entrywise preserves positivity on $\mathbb{P}_N([0,\infty))$.
- Fix $\mathbf{u} = (u_1, \dots, u_N)^T$ with $u_i > 0$ pairwise distinct.
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where $V(\mathbf{u}) = \prod_{i<j}(u_j - u_i)$ is the Vandermonde determinant.

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$$\frac{\Delta^{(\binom{N}{2}+1)}(0)}{(\binom{N}{2}+1)!} = V(\mathbf{u})^2 \cdot (u_1 + \cdots + u_N)^2 \cdot \frac{f(0)}{0!} \frac{f'(0)}{1!} \cdots \frac{f^{(N-2)}(0)}{(N-2)!} \cdot \frac{f^{(N)}(0)}{N!}.$$

Hidden inside this derivative is a Schur polynomial!

From each smooth function to all Schur polynomials

This provides a novel bridge, between analysis and symmetric function theory:

Given $f : [0, \epsilon) \to \mathbb{R}$ *smooth, and* $u_1, \ldots, u_N > 0$ *pairwise distinct (for* $\epsilon > 0$ *and* $N \geqslant 1$*),*
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Uncovers *all* Schur polynomials:

Theorem (K., *Trans. Amer. Math. Soc.* 2022)

Suppose f, ϵ, N *are as above. Fix* $\mathbf{u}, \mathbf{v} \in (0, \infty)^N$ *and set* $\Delta(t) := \det f[t\mathbf{u}\mathbf{v}^T]$*. Then for all* $M \geqslant 0$,

$$\frac{\Delta^{(M)}(0)}{M!} = \sum_{\mathbf{n}=(n_N, \dots, n_1) \vdash M} V(\mathbf{u})V(\mathbf{v}) \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \cdot \prod_{j=1}^{N} \frac{f^{(n_j)}(0)}{n_j!}.$$

- All Schur polynomials "occur" inside each smooth function.

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This provides a novel bridge, between analysis and symmetric function theory:

Given $f : [0, \epsilon) \to \mathbb{R}$ *smooth, and* $u_1, \dots, u_N > 0$ *pairwise distinct (for* $\epsilon > 0$ *and* $N \geqslant 1$*),*
set $\Delta(t) := \det f[t\mathbf{u}\mathbf{v}^T]$ *and compute* $\Delta^{(M)}(0)$ *for all integers* $M \geqslant 0$.

Uncovers *all* Schur polynomials:

Theorem (K., *Trans. Amer. Math. Soc.* 2022)

Suppose f, ϵ, N *are as above. Fix* $\mathbf{u}, \mathbf{v} \in (0, \infty)^N$ *and set* $\Delta(t) := \det f[t\mathbf{u}\mathbf{v}^T]$. *Then for all* $M \geqslant 0$,

$$\frac{\Delta^{(M)}(0)}{M!} = \sum_{\mathbf{n}=(n_N,\dots,n_1) \vdash M} V(\mathbf{u})V(\mathbf{v}) \cdot s_{\mathbf{n}}(\mathbf{u})s_{\mathbf{n}}(\mathbf{v}) \cdot \prod_{j=1}^{N} \frac{f^{(n_j)}(0)}{n_j!}.$$

- All Schur polynomials "occur" inside each smooth function.
- If f is a power series, then so is Δ. What is its expansion? (*Long history!*)

Going further back…to Cauchy and Frobenius, 1800s

Theorem (Cauchy, 1841 memoir)

If $f(t) = (1-t)^{-1} = 1 + t + t^2 + \cdots = 1 \cdot t^0 + 1 \cdot t^1 + 1 \cdot t^2 + \cdots$, *then*

$$\det f[\mathbf{u}\mathbf{v}^T] = \sum_{M \geqslant 0} \sum_{\mathbf{n} \vdash M} V(\mathbf{u}) V(\mathbf{v}) \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \cdot 1^N.$$

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This is the $c = 0$ special case of:

Theorem (Frobenius, *J. reine Angew. Math.* 1882)

If $f(t) = \dfrac{1-ct}{1-t}$ *for a scalar* c, *then*

$$\det f[\mathbf{u}\mathbf{v}^T] = \det \left(\frac{1 - c u_i v_j}{1 - u_i v_j} \right)_{i,j=1}^{n}$$

$$= V(\mathbf{u}) V(\mathbf{v}) \left(\sum_{\mathbf{n} \,:\, n_1 = 0} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) (1-c)^{n-1} + \sum_{\mathbf{n} \,:\, n_1 > 0} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) (1-c)^{n} \right).$$

What is the expansion for a general power series $f(t)$*?*

Cauchy–Frobenius identity for all power series

Similar questions and follow-ups (on symmetric function identities) studied by

- Andrews–Goulden–Jackson [*Trans. Amer. Math. Soc.* 1988].
- Laksov–Lascoux–Thorup [*Acta Math.* 1989].
- Kuperberg [*Ann. of Math.* 2002].
- Ishikawa, Okado, and coauthors [*Adv. Appl. Math.* 2006, 2013].
- Also Krattenthaler, *Advanced determinantal calculus: I*, *II* in 1998, 2005.

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Theorem (K., *Trans. Amer. Math. Soc.* 2022)

Fix a commutative unital ring R and let t be an indeterminate. Let $f(t) := \sum_{M \geqslant 0} f_M t^M \in R[[t]]$ be an arbitrary formal power series. Given vectors $\mathbf{u}, \mathbf{v} \in R^N$ for some $N \geqslant 1$, we have:

$$\det f[t\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \sum_{M \geqslant \binom{N}{2}} t^M \sum_{\mathbf{n}=(n_N,\ldots,n_1) \vdash M} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \cdot \prod_{j=1}^{N} f_{n_j}.$$

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With Sahi [*Eur. J. Comb.* 2023] – extended to bosonic+fermionic identities, (a) for *all immanants*, (b) over all rings, (c) for all power series.

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Thank you for your attention.

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SERB INDIA

Indian Institute of Science

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DARPA

ICMS International Centre for Mathematical Sciences

ILAS

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