Matrix Analysis and Positivity Preservers

Apoorva Khare
Indian Institute of Science
1. Introduction

This text arose out of the course notes for Math 341: Matrix Analysis and Positivity, a one-semester course offered in Spring 2018 (and again in Fall 2019) at the Indian Institute of Science. Owing to the subsequent inclusion of additional topics, the text has now grown to cover roughly a two-semester course in analysis and matrix positivity preservers – or more broadly speaking, composition operators preserving various kinds of positive kernels. Thus in this text, we briefly describe some notions of positivity in matrix theory, followed by our main focus: a detailed study of the operations that preserve these notions (and in the process, understanding some aspects of real functions). Several different notions of positivity in analysis, studied for classical and modern reasons, are also touched upon in the text:

- Positive semidefinite (psd) and positive definite matrices.
- Entrywise positive matrices.
- A common strengthening of both of these notions, which involves totally positive (TP) and totally non-negative (TN) matrices.
- Settings somewhat outside matrix theory. For instance, consider discrete data associated to positive measures on locally compact abelian groups $G$. E.g. for $G = \mathbb{R}$, one obtains moment-sequences, which are intimately related to positive semidefinite Hankel matrices. For $G = S^1$, the circle group, one obtains Fourier–Stieltjes sequences, which are connected to positive definite Toeplitz matrices. (Works of Carathéodory, Hamburger, Hausdorff, Herglotz, and Stieltjes, among others.)
- More classically, functions and kernels with positivity structures have long been studied in analysis, including on locally compact groups and metric spaces. (Bochner, Schoenberg, von Neumann, Pólya, to name a few.) Some distinguished examples include positive definite functions, and Pólya frequency functions and sequences.

The text begins by discussing the above notions, focussing on their properties and some results in matrix theory. The remainder of the text then studies in detail the preservers of several of these notions of positivity. Among other things, this journey involves going through many beautiful classical results, by leading experts in analysis during the first half of the twentieth century. Apart from also covering several different tools required in proving these results, an interesting outcome also is that several classes of ‘positive’ matrices repeatedly get highlighted by way of studying positivity preservers – these include generalized Vandermonde matrices, Hankel moment matrices and kernels, and Toeplitz kernels on the line or the integers (aka Pólya frequency functions and sequences).

In this text we will study entrywise preservers of various notions of positivity. The question of why entrywise calculus – as compared to the usual holomorphic functional calculus – has a rich and classical history in the analysis literature, beginning with the work of Schoenberg, Rudin, Loewner, and Horn (these results are proved in Part 3 of the text); but also drawing upon earlier works of Menger, Schur, Bochner, and others. (In fact, the entrywise calculus was introduced, and the first such result proved, by Schur in 1911.) Interestingly, this entrywise calculus also arises in modern-day applications from high-dimensional covariance estimation; we elaborate on this in Section 13.1 and briefly also in Section 14. Furthermore, this evergreen area of mathematics continues to be studied in the literature, drawing techniques from – and also contributing to – symmetric function theory, statistics and graphical models, combinatorics, and linear algebra (in addition to analysis).

As a historical curiosity, the course and this text arose in a sense out of research carried out in significant measure by mathematicians at Stanford University (including their students) over the years. This includes Loewner and Karlin, and their students: FitzGerald, Horn,
1. Introduction

Micchelli, and Pinkus. Slightly less directly, there was also Katznelson, who had previously worked with Helson, Kahane, and Rudin, leading to Rudin’s strengthening of Schoenberg’s theorem. (Coincidentally, Pólya and Szegő, who made the original observation on entrywise preservers of positivity using the Schur product theorem, were again colleagues at Stanford.) On a personal note, the author’s contributions to this area also have their origins in his time spent at Stanford University, collaborating with Alexander Belton, Dominique Guillot, Mihai Putinar, Bala Rajaratnam, and Terence Tao (though the collaboration with the last-named colleague was carried out almost entirely at the Indian Institute of Science).

We now discuss the course, these notes, and their mathematical contents. The notes were scribed by the students taking the course in Spring 2018 at IISc, followed by extensive ‘homogenization’ by the author – and in several sections, addition of material. Each section was originally intended to cover the notes of roughly one 90-minute lecture, or occasionally two; that said, some material has subsequently been moved around for logical, mathematical, and expositional reasons. The notes, and the course itself, require an understanding of basic linear algebra and analysis, with a bit of measure theory as well. Beyond these basic topics, we have tried to keep these notes as self-contained as possible, with full proofs. To that end, we have included proofs of ‘preliminary’ results, including:

(i) early results of Schoenberg (following Menger, Fréchet, Bochner, von Neumann . . . ) connecting metric geometry and positive definite functions to matrix positivity;
(ii) Boas–Widder and Bernstein’s theorems on functions with positive forward differences;
(iii) Sierpiński’s result, on mid-convexity and measurability implying continuity;
(iv) an extension to normed linear spaces, of (a special case of) a classical result of Ostrowski on mid-convexity and local boundedness implying continuity;
(v) Whitney’s density of totally positive matrices inside totally non-negative matrices;
(vi) Descartes’ rule of signs – several variants;
(vii) a follow-up to Descartes, by Laguerre, on variation diminution in power series, and its follow-up by Fekete involving Pólya frequency sequences;
(viii) Fekete’s result on totally positive matrices, via positive contiguous minors;
(ix) (on a related note:) results on real polynomials, their ‘compositions’, and zeros of these: by Hermite–Poulain, Hermite–Biehler, Routh–Hurwitz, Laguerre, Maló, Jensen, Schur, Weyr, de Bruijn, and Pólya;
(x) a reasonably detailed sketch of Pólya and Schur’s characterizations of multiplier sequences;
(xi) Mercer’s lemma, identifying positive semidefinite kernels with kernels of positive type;
(xii) Perron’s theorem for matrices with positive entries (the precursor to Perron–Frobenius);
(xiii) compound matrices and Kronecker’s theorem on their spectra;
(xiv) Sylvester’s criterion and the Schur product theorem on positive (semi)definiteness (also, the Jacobi formula);
(xv) the Jacobi complementary minor formula;
(xvi) the Rayleigh–Ritz theorem;
(xvii) a special case of Weyl’s inequality on eigenvalues;
(xviii) matrix identities by Andréief and Cauchy–Binet, and a continuous generalization;
(xix) the discreteness of zeros of real analytic functions (and a sketch of the continuity of roots of complex polynomials);
(xx) the equivalence of Cauchy’s and Littlewood’s definitions of Schur polynomials (and of the Jacobi–Trudi and von Nágelsbach–Kostka identities) via Lindström–Gessel–Viennot bijections, among other results.
1. Introduction

Owing to considerations of time, we had to leave out some proofs, including of: theorems by Hamburger/Hausdorff/Stieltjes, Fubini, Tonelli, Cauchy, Montel, Morera, and Hurwitz; a Schur positivity phenomenon for ratios of Schur polynomials; Lebesgue’s dominated convergence theorem; as well as the closure of real analytic functions under composition. Most of these can be found in standard textbooks in mathematics. We also omit the proofs of several classical results on Laplace transforms and Pólya frequency functions, found in textbooks, in papers by Schoenberg and his co-authors, and in Karlin’s comprehensive monograph on total positivity. Nevertheless, as the previous and current paragraphs indicate, these notes cover many classical results by past experts, and acquaint the reader with a variety of tools in analysis (especially the study of real functions) and in matrix theory – many of these tools are not found in more ‘traditional’ courses on these subjects.

This text is broadly divided into six parts, with detailed bibliographic notes following each part. In Part 1, the key objects of interest – namely, positive semidefinite / totally positive / totally non-negative matrices – are introduced, together with some basic results as well as some important classes of examples. (The analogous kernels are also studied.) In Part 2, we begin the study of functions acting on such matrices entrywise, and preserving the relevant notion of positivity. Here we will mostly restrict ourselves to studying power functions that act on various sets of matrices of a fixed size. This is a long-studied question, including by Bhatia, Elsner, Fallat, FitzGerald, Hiai, Horn, Jain, Johnson, and Sokal; as well as by the author in collaboration with Guillot and Rajaratnam. In particular, an interesting highlight is the construction by Jain of individual (pairs of) matrices, which encode the entire set of entrywise powers preserving Loewner positivity, monotonicity, and convexity. We also obtain certain necessary conditions on general entrywise functions that preserve positivity, including multiplicative mid-convexity and continuity, as well as a classification of all functions preserving total non-negativity or total positivity in each fixed dimension. We explain some of the modern motivations, and end with some unsolved problems.

Part 3 deals with some of the foundational results on matrix positivity preservers. After mentioning some of the early history of the field – including work by Menger, Fréchet, Bochner, and Schoenberg – we classify the entrywise functions that preserve positive semidefiniteness (= positivity) in all dimensions, or total non-negativity on Hankel matrices of all sizes. This is a celebrated result of Schoenberg – later strengthened by Rudin – which is a converse to the Schur product theorem, and we prove a stronger version by using a rank-constrained test set. The proof given in these notes is different from the previous approaches of Schoenberg and Rudin, is essentially self-contained, and uses relatively less sophisticated machinery compared to loc. cit. Moreover, it goes through proving a variant by Vasudeva, for matrices with only positive entries; and it lends itself to a multivariate generalization (which will not be covered here). The starting point of these proofs is a necessary condition for entrywise preservers in fixed dimension, proved by Loewner and Horn in the late 1960s. To this day, this result remains essentially the only known condition in fixed dimension \( n \geq 3 \), and a proof of a (rank-constrained as above) stronger version is also provided in these notes. In addition to techniques and ingredients introduced by the above authors, the text also borrows from the author’s joint work with Belton, Guillot, and Putinar. This part ends with several appendices:

1. The first appendix covers a result by Boas and Widder, which shows a converse ‘mean value theorem’ for divided differences.

2. We next cover recent work by Vishwakarma on an ‘off-diagonal’ variant of the positivity preserver problem.
The third and fourth appendices classify preservers of Loewner positivity, monotonicity, and convexity on ‘dimension-free matrices’, or on kernels over infinite domains.

In Parts 4 and 5, we formulate the preserver problem in analysis terms, using composition operators on kernels. This allows one to consider such questions not only for matrices of a fixed or arbitrary size, but also over more general, infinite domains. It also makes available for use, the powerful analysis machinery developed by Bernstein, Pólya, Schoenberg, Widder, and others. Thus in Part 5, we provide characterizations of such composition operators preserving total positivity or non-negativity on structured kernels – specifically Toeplitz kernels on various sub-domains of \( \mathbb{R} \). Two distinguished classes of such kernels are Pólya frequency functions and Pólya frequency sequences, i.e., Toeplitz kernels on \( \mathbb{R} \times \mathbb{R} \) and \( \mathbb{Z} \times \mathbb{Z} \), respectively.

Before solving the preserver problem in this paradigm, we begin by developing some of the results by Schoenberg and others on Pólya frequency functions and sequences; we present additional modern results; and we provide connections to the Laguerre–Pólya–Schur program and its recent incarnations, as well as to the Wallach set in representation theory and probability.

In Part 5, we return to the question studied in Part 2 above, of classifying the preservers of all \( TN \) or \( TP \) kernels, on \( X \times Y \) for arbitrary totally ordered sets \( X, Y \). In this part we provide a complete resolution of this question. In a sense, the total positivity preserver problem is the culmination of all that has come before; it uses many of the tools and techniques from the previous parts. These include (i) Vandermonde kernels, \( TP \) kernel completions of \( 2 \times 2 \) matrices, and Fekete’s result, from Part 1; (ii) entrywise powers preserving positivity (via a trick of Jain to use Toeplitz cosine matrices), and the classification of total-positivity preservers in finite dimension, from Part 2; (iii) the stronger Vasudeva theorem classifying entrywise positivity preservers on low-rank Hankel matrices, from Part 3; and (iv) preservers of Toeplitz kernels, including of Pólya frequency functions and sequences, from Part 4. We also develop as needed, set-theoretic tools and Whitney-type density results – as well as understanding the structure and preservers of continuous Hankel kernels defined on an interval. This latter requires results of Mercer, Bernstein, Hamburger, and Widder.

We remark here that Part 4 contains – in addition to theorems found e.g. in Karlin’s book – more recent results on Pólya frequency functions (and not merely their preservers) from 2020+, which in particular are not found in previous treatments of the subject. To name a few:

- a characterization of Pólya frequency functions of order \( p \), for any \( p \geq 3 \);
- strengthenings of multiple results of Karlin (1955) and of Schoenberg (1964);
- a converse to the same result of Karlin (1955);
- a critical exponent phenomenon in total positivity;
- a closer look at a multiparameter family of density functions introduced by Hirschman and Widder (1949); and
- a connection between Pólya frequency functions and the Riemann hypothesis.

(Similarly, Part 2 of this text contains recent results on totally non-negative/positive matrices, which are not found in earlier books on total positivity.) Thus, in a sense Parts 4 and 5 can be viewed as ‘one possible sequel’ to Karlin’s book and to the body of work by Schoenberg on Pólya frequency functions, since they present novel material on these classes of functions, then use the ‘structural’ results by the aforementioned authors about various families of Toeplitz kernels to classify the preservers of various sub-families of these objects. In addition to results of Schoenberg and his coauthors (and an authoritative survey of these results, compiled in
Karlin’s majestic monograph), both parts borrow from the author’s 2020 works (one with Belton, Guillot, and Putinar).

In the final Part 6, we return to the study of entrywise functions preserving positivity in fixed dimension. This is a challenging problem – it is still open in general, even for $3 \times 3$ matrices – and we restrict ourselves in this part to studying polynomial preservers. By the Schur product theorem, if the polynomial has all coefficients non-negative then it is easily seen to be a preserver; but interestingly, until 2016 not a single example was known of any other entrywise polynomial preserver of positivity in a fixed dimension $n \geq 3$. Very recently, this question has been answered to some degree of satisfaction (by the author in collaboration – first with Belton, Guillot, and Putinar; and subsequently with Tao). The text ends by covering some of this recent progress, and comes back full circle to Schur, through symmetric function theory.

A quick note on the logical structure: Parts 1–5 are best read sequentially; note that some sections do not get used later in the text, e.g. Section 8 and the Appendices. That said, Part 4, which involves non-presenter results on Pólya frequency functions and sequences, can be read from scratch, requiring only Sections 6 and 12 and Lemma 26.3 as pre-requisites. The final Part 6 can be read following Section 9 (also see the Schoenberg–Rudin theorem 16.2). We also point out the occurrence of Historical notes and Further questions, which serve to acquaint the reader with past work(er)s as well as related areas; and possible avenues for future work – and which can be accessed from the Index at the end. (See also the Bibliographic notes at the end of each part of the text.)

To conclude, thanks are owed to the scribes (listed below), as well as to Alexander Belton, Shabarish Chenakkod, Projesh Nath Choudhury, Julian R. D’Costa, Dominique Guillot, Prakhar Gupta, Sarvesh Ravichandran Iyer, Poonendu Kumar, Gadadhar Misra, Aradhya Pandey, Paramita Pramanick, Mihai Putinar, Shubham Rastogi, Aditya Guha Roy, Siddhartha Sahi, Kartik Singh, Naren Sundaravaranadan, G.V. Krishna Teja, Raghavendra Tripathi, Prateek Kumar Vishwakarma, and Pranjal Warade for helpful suggestions that improved the text. I am of course deeply indebted to my collaborators for their support and all of their research efforts in positivity – but also for many stimulating discussions, which helped shape my thinking about the field as a whole and the structure of this text in particular. Finally, I am grateful to the University Grants Commission (UGC, Govt. of India); the Science and Engineering Research Board (SERB) and the Department of Science and Technology (DST) of the Govt. of India; the Infosys Foundation; and the Tata Trusts – for their support through a CAS-II grant; through a MATRICS grant and the Ramanujan and Swarnajayanti Fellowships; through a Young Investigator Award; and through their Travel Grants, respectively.

Department of Mathematics, Indian Institute of Science & Analysis and Probability Research Group Bangalore, India
**List of scribes:** Each section contains material originally covered in a 90-minute lecture, and subsequently augmented by the author. These lectures (in Spring 2018) did not cover Sections 4, 8 in Part 1, Section 15 in Part 2, the entirety of Parts 4 and 5, and the Appendices. The text also includes two sets of ‘general remarks’ by the author; see Sections 13.1 and 14.

**Part 1: Preliminaries**

1. January 03: (Much of the above text.)
2. January 08: Sarvesh Iyer
3. January 10: Prateek Kumar Vishwakarma
4. Apoorva Khare
5. January 17: Prakhar Gupta
7. January 24: Swarnalipa Datta, and Apoorva Khare
8. Apoorva Khare

**Part 2: Entrywise powers preserving (total) positivity in fixed dimension**

9. January 31: Raghavendra Tripathi
10. February 02: Pabitra Barman
11. February 07: Kartick Ghosh
12. February 09: K. Philip Thomas (and Feb 14: Pranab Sarkar), and Apoorva Khare
13. February 28: Ratul Biswas (with an introduction by Apoorva Khare)
14. March 02: Prateek Kumar Vishwakarma (with remarks by Apoorva Khare)
15. Apoorva Khare

**Part 3: Entrywise functions preserving positivity in all dimensions**

16. February 14, 16: Pranab Sarkar, Pritam Ganguly, and Apoorva Khare
17. March 07: Shubham Rastogi
18. March 09: Poornendu Kumar, Sarvesh Iyer, and Apoorva Khare
19. March 14: Poornendu Kumar and Shubham Rastogi
20. March 16: Lakshmi Kanta Mahata and Kartick Ghosh
21. March 19, 21: Kartick Ghosh, Swarnalipa Datta, and Apoorva Khare
22. March 23: Sarvesh Iyer and Raghavendra Tripathi
23. Appendix A: Poornendu Kumar, Shubham Rastogi, and Apoorva Khare
24. Appendix B: Prateek Kumar Vishwakarma and Apoorva Khare
25. Appendix C: Apoorva Khare
26. Appendix D: Apoorva Khare

**Part 4: Pólya frequency functions and sequences**

27–33 Apoorva Khare

**Part 5: Composition operators preserving totally positive kernels**

34–38 Apoorva Khare

**Part 6: Entrywise polynomials preserving positivity in fixed dimension**

39 March 26: Prakhar Gupta, Pranjal Warade, and Apoorva Khare
40 March 28: Ratul Biswas and K. Philip Thomas
41 April 02: Pritam Ganguly and Pranab Sarkar
42 April 11: Pabitra Barman, Lakshmi Kanta Mahata, and Apoorva Khare
43 Appendix E: Apoorva Khare
## Contents

1. Introduction 3

### Part 1: Preliminaries

2. The cone of positive semidefinite matrices. Examples. Schur complements. 13
3. Schur product theorem. Totally positive/non-negative matrices. Examples. 21
4. Fekete’s result. Hankel moment matrices are $TN$. Hankel positive semidefinite and $TN$ matrices. 27
5. Generalized Vandermonde matrices. Cauchy–Binet formula and its generalization. 31
6. Hankel moment matrices are $TP$. Andréief’s identity. Density of $TP$ in $TN$. 35
7. (Non-)Symmetric $TP$ completion problems. 39
8. Theorems of Perron and Kronecker. Spectra of $TP$ and $TN$ matrices. 43

### Bibliographic notes and references 47

### Part 2: Entrywise powers preserving (total) positivity in fixed dimension

9. Entrywise powers preserving positivity in fixed dimension: I. 51
10. Entrywise powers preserving total positivity: I. 55
11. Entrywise powers preserving total positivity: II. 59
12. Entrywise functions preserving total positivity. Mid-convex implies continuous. The test set of Hankel $TN$ matrices. 63
13. Entrywise powers (and functions) preserving positivity: II. Matrices with zero patterns. 73
15. Loewner convexity. Single matrix encoders of entrywise power-preservers of Loewner properties. 85

### Bibliographic notes and references 93

### Part 3: Entrywise functions preserving positivity in all dimensions

18. The stronger Horn–Loewner theorem. Mollifiers. 111
19. The stronger Vasudeva and Schoenberg theorems. Bernstein’s theorem. Moment-sequence transforms. 115
20. Proof of stronger Schoenberg Theorem: I. Continuity. The positivity-certificate trick. 121
21. Proof of stronger Schoenberg Theorem: II. Smoothness implies real analyticity. 127
22. Proof of stronger Schoenberg Theorem: III. Complex analysis. Further remarks. 131
23. Appendix A: The Boas–Widder theorem on functions with positive differences. 133
24. Appendix B: Functions acting outside forbidden diagonal blocks. Dimension-free non-absolutely-monotonic preservers. 143
25. Appendix C. Preservers of positivity on kernels. 149
26. Appendix D. Preservers of Loewner monotonicity and convexity on kernels. 153

### Bibliographic notes and references 159
Part 1: Preliminaries

Part 1: Preliminaries


A kernel is a function $K : X \times Y \to \mathbb{R}$. Broadly speaking, the goal of this text is to understand: Which functions $F : \mathbb{R} \to \mathbb{R}$, when applied to kernels with some notion of positivity, preserve that notion? To do so, we first study the test sets of such kernels $K$ themselves, and then the post-composition operators $F$ that preserve these test sets. We begin by understanding such kernels when the domains $X, Y$ are finite, i.e. matrices.

In this text, we will assume familiarity with linear algebra and a first course in calculus/analysis. To set notation: an uppercase letter with a two-integer subscript (such as $A_{m \times n}$) represents a matrix with $m$ rows and $n$ columns. If $m, n$ are clear from context or unimportant, then they will be omitted. Three examples of real matrices are $0_{m \times n}, 1_{m \times n}, \text{Id}_{n \times n}$, which are the (rectangular) matrix consisting of all zeros, all ones, and the identity matrix, respectively. The entries of a matrix $A$ will be denoted $a_{ij}, a_{jk}$, etc. Vectors are denoted by lowercase letters (occasionally in bold), and are columnar in nature. All matrices, unless specified otherwise, are real; and similarly, all functions, unless otherwise specified otherwise, are defined on $-$ and take values in $- \mathbb{R}^m$ for some $m$.

2.1. Preliminaries. We begin with several basic definitions.

Definition 2.1. A matrix $A_{n \times n}$ is said to be symmetric if $a_{jk} = a_{kj}$ for all $1 \leq j, k \leq n$. A symmetric matrix $A_{n \times n}$ is said to be positive semidefinite (psd) if the real number $x^T Ax$ is non-negative for all $x \in \mathbb{R}^n$ -- in other words, the quadratic form given by $A$ is positive semidefinite. If, furthermore, $x^T Ax > 0$ for all $x \neq 0$ then $A$ is said to be positive definite.

The spectrum of $A$, denoted $\sigma(A) \subset \mathbb{C}$, is the multiset of all eigenvalues of $A$ (counted with multiplicities).

We state the spectral theorem for symmetric (i.e. self-adjoint) operators without proof.

Theorem 2.2 (Spectral theorem for symmetric matrices). For $A_{n \times n}$ a symmetric matrix, $A = U^T DU$ for some orthogonal matrix $U$ (that is, $U^T U = \text{Id}$) and real diagonal matrix $D$. $D$ contains all the eigenvalues of $A$ (and counting multiplicities) along its diagonal.

As a consequence, $A = \sum_{j=1}^{n} \lambda_j v_j v_j^T$, where each $v_j$ is an eigenvector for $A$ with real eigenvalue $\lambda_j$, and the $v_j$ together form an orthonormal basis of $\mathbb{R}^n$. (Note that the $v_j$ are the columns of $U^T$.)

We also have the following consequence for two commuting matrices:

Theorem 2.3 (Spectral theorem for commuting symmetric matrices). Let $A_{n \times n}$ and $B_{n \times n}$ be two commuting symmetric matrices. Then $A$ and $B$ are simultaneously diagonalizable, i.e., for the same orthogonal $U$, $A = U^T D_1 U$ and $B = U^T D_2 U$ for $D_1$ and $D_2$ diagonal matrices (whose diagonal entries comprise the spectra $\sigma(A), \sigma(B)$ respectively).

2.2. Criteria for positive (semi)definiteness. We write down several equivalent criteria for positive (semi)definiteness. There are three initial criteria which are easy to prove, and a final criterion which requires separate treatment.

Theorem 2.4 (Criteria for positive (semi)definiteness). Given $A_{n \times n}$ a real symmetric matrix of rank $0 \leq r \leq n$, the following are equivalent:

1. $A$ is positive semidefinite (respectively, positive definite).
2. $\sigma(A) \subset [0, \infty)$. (Respectively, $\sigma(A) \subset (0, \infty)$.)
Lemma 2.8. If \( A \in \mathbb{R}^{r \times n} \) of rank \( r \) such that \( B^T B = A \). (In particular, if \( A \) is positive definite then \( B \) is square and non-singular.)

**Proof.** We prove only the positive semidefinite statements. The corresponding positive definite statements will follow with minor changes.

If (1) holds and \( \lambda \) is any eigenvalue, with corresponding (nonzero) eigenvector \( x \), then \( x^T A x = \lambda \|x\|^2 \geq 0 \). Hence \( \lambda \geq 0 \), proving (2). Conversely, if (2) holds then by the spectral theorem, \( A = \sum_j \lambda_j v_j v_j^T \), with all \( \lambda_j \geq 0 \). Now for any \( x \in \mathbb{R}^n \),

\[
x^T A x = \sum_j \lambda_j x^T v_j v_j^T x = \sum_j \lambda_j (x^T v_j)^2 \geq 0,
\]

and so \( A \) is positive semidefinite.

Next if (1) holds then write \( A = U^T D U \) by the spectral theorem; note that \( D = U A U^T \) has the same rank as \( A \). Since \( D \) has non-negative diagonal entries \( d_{jj} \), it has a square root \( \sqrt{D} \), which is a diagonal matrix with diagonal entries \( \sqrt{d_{jj}} \). Write \( D = \begin{pmatrix} D'_{r \times r} & 0 \\ 0 & 0_{(n-r) \times (n-r)} \end{pmatrix} \), where \( D' \) is a diagonal matrix with positive diagonal entries. Correspondingly, write \( U = \begin{pmatrix} P_{r \times r} & Q \\ R & S_{(n-r) \times (n-r)} \end{pmatrix} \). If we set \( B := \sqrt{D' P \mid \sqrt{D' Q}}_{r \times n} \), then it is easily verified that

\[
B^T B = \begin{pmatrix} P^T D' P & P^T D' Q \\ Q^T D' P & Q^T D' Q \end{pmatrix} = U^T D U = A.
\]

Hence (1) \( \implies \) (3). Conversely, if (3) holds then \( x^T A x = \|B x\|^2 \geq 0 \) for all \( x \in \mathbb{R}^n \). Hence \( A \) is positive semidefinite. Moreover, we claim that \( B \) and \( B^T B \) have the same null space and hence the same rank. Indeed, if \( B x = 0 \) then \( B^T B x = 0 \), while

\[
B^T B x = 0 \implies x^T B^T B x = 0 \implies \|B x\|^2 = 0 \implies B x = 0. \quad \square
\]

**Corollary 2.5.** For any real symmetric matrix \( A_{n \times n} \), the matrix \( A - \lambda_{\min} \text{Id}_{n \times n} \) is positive semidefinite, where \( \lambda_{\min} \) denotes the smallest eigenvalue of \( A \).

We now state Sylvester’s criterion for positive (semi)definiteness. (Incidentally, Sylvester is believed to have first introduced the use of ‘matrix’ in mathematics, in the 19th century.) This requires some additional notation.

**Definition 2.6.** Given an integer \( n > 0 \), define \( [n] := \{1, \ldots, n\} \). Now given a matrix \( A_{m \times n} \) and subsets \( J \subset [m], K \subset [n] \), define \( A_{J \times K} \) to be the submatrix of \( A \) with entries \( a_{jk} \) for \( j \in J, k \in K \) (always considered to be arranged in increasing order in this text). If \( J, K \) have the same size then \( \det A_{J \times K} \) is called a minor of \( A \). If \( A \) is square and \( J = K \) then \( A_{J \times K} \) is called a principal submatrix of \( A \), and \( \det A_{J \times K} \) is a principal minor. The principal submatrix (and principal minor) are leading if \( J = K = \{1, \ldots, m\} \) for some \( 1 \leq m \leq n \).

**Theorem 2.7** (Sylvester’s criterion). A symmetric matrix is positive semidefinite (respectively, positive definite) if and only if all its principal minors are non-negative (respectively, positive).

We will prove this with the help of some lemmas.

**Lemma 2.8.** If \( A_{n \times n} \) is a positive semidefinite (respectively, positive definite) matrix, then so are all principal submatrices of \( A \).

Proof. Fix a subset \( J \subset [n] = \{1, \ldots, n\} \) (so \( B := A_{J\times J} \) is the corresponding principal submatrix of \( A \)), and let \( x \in \mathbb{R}^{|J|} \). Define \( x' \in \mathbb{R}^n \) to be the vector such that \( x'_j = x_j \) for all \( j \in J \) and 0 otherwise. It is easy to see that \( x^T B x = (x')^T A x' \). Hence \( B \) is positive (semi)definite if \( A \) is.

As a corollary, all the principal minors of a positive semidefinite (positive definite) matrix are non-negative (positive), since the corresponding principal submatrices have non-negative (positive) eigenvalues and hence non-negative (positive) determinants. So one direction of Sylvester’s criterion holds trivially.

**Lemma 2.9.** Sylvester’s criterion is true for positive definite matrices.

Proof. We induct on the dimension of the matrix \( A \). Suppose \( n = 1 \). Then \( A \) is just an ordinary real number, whence its only principal minor is \( A \) itself, and so the result is trivial.

Now, suppose the result is true for matrices of dimension \( \leq n - 1 \). We claim that \( A \) has at least \( n - 1 \) positive eigenvalues. To see this, let \( \lambda_1, \lambda_2 \leq 0 \) be eigenvalues of \( A \). Let \( W \) be the \( n - 1 \) dimensional subspace of \( \mathbb{R}^n \) with last entry 0. If \( v_j \) are orthogonal eigenvectors for \( \lambda_j \), \( j = 1, 2 \), then the span of \( v_j \) must intersect \( W \) non-trivially, since the sum of dimensions of these two subspaces of \( \mathbb{R}^n \) exceeds \( n \). Define \( u := c_1 v_1 + c_2 v_2 \in W \); then \( u^T A u > 0 \) by Lemma 2.8. However,

\[
u^T A u = (c_1 v_1^T + c_2 v_2^T) A (c_1 v_1 + c_2 v_2) = c_1^2 \lambda_1 ||v_1||^2 + c_2^2 \lambda_2 ||v_2||^2 \leq 0\]

giving a contradiction, and proving the claim.

Now since the determinant of \( A \) is positive (it is the minor consisting of \( A \) itself), it follows that all eigenvalues are positive, completing the proof.

We will now prove the Jacobi formula, an important result in its own right. A corollary of this result will be used, along with the previous result and the idea that positive semidefinite matrices can be expressed as entrywise limits of positive definite matrices, to prove Sylvester’s criterion for all positive semidefinite matrices.

**Theorem 2.10 (Jacobi formula).** Let \( A_t : \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \) be a matrix-valued differentiable function. Then,

\[
\frac{d}{dt} (\det A_t) = \text{tr} \left( \text{adj}(A_t) \frac{dA_t}{dt} \right),
\]

where \( \text{adj}(A_t) \) denotes the adjugate matrix of \( A_t \).

Proof. The first step is to compute the differential of the determinant. We claim that for any \( n \times n \) real matrices \( A, B \),

\[
d(\det)(A)(B) = \text{tr}(\text{adj}(A)B).
\]

As a special case, at \( A = \text{Id}_{n \times n} \), the differential of the determinant is precisely the trace.

To show the claim, we need to compute the directional derivative

\[
\lim_{\epsilon \to 0} \frac{\det(A + \epsilon B) - \det A}{\epsilon}.
\]

The fraction is a polynomial in \( \epsilon \) with vanishing constant term (e.g., set \( \epsilon = 0 \) to see this); and we need to compute the coefficient of the linear term. Expand \( \det(A + \epsilon B) \) using the Laplace expansion as a sum over permutations \( \sigma \in S_n \); now each individual summand \((-1)^\sigma \prod_{k=1}^n (a_{\sigma(k)} + b_{\sigma(k)})\) splits as a sum of \( 2^n \) terms. (It may be illustrative to try and work out the \( n = 3 \) case by hand.) From these \( 2^n \cdot n! \) terms, choose the ones that are linear in \( \epsilon \). For each \( 1 \leq i, j \leq n \), there are precisely \( (n - 1)! \) terms corresponding to \( eb_{ij} \); and
added together, they equal the \((i,j)\)th cofactor \(C_{ij}\) of \(A\) — which equals \(\text{adj}(A)_{ji}\). Thus, the coefficient of \(\epsilon\) is:

\[
d(d(\det)(A)(B)) = \sum_{i,j=1}^{n} C_{ij}b_{ij},
\]

and this is precisely \(\text{tr(adj}(A)B)\), as claimed.

More generally, the above argument shows that if \(B(\epsilon)\) is any family of matrices, with limit \(B(0)\) as \(\epsilon \to 0\), then

\[
\lim_{\epsilon \to 0} \frac{\det(A + \epsilon B(\epsilon)) - \det A}{\epsilon} = \text{tr(adj}(A)B(0)). \tag{2.12}
\]

Returning to the proof of the theorem, for \(\epsilon \in \mathbb{R}\) small and \(t \in \mathbb{R}\) we write:

\[
A_{t+\epsilon} = A_t + \epsilon B(\epsilon)
\]

where \(B(\epsilon) \to B(0) := \frac{dA_t}{dt}\) as \(\epsilon \to 0\), by definition. Now compute using (2.12):

\[
\frac{d}{dt} (\det A_t) = \lim_{\epsilon \to 0} \frac{\det(A_t + \epsilon B(\epsilon)) - \det A_t}{\epsilon} = \text{tr(adj}(A_t) \frac{dA_t}{dt}),
\]

as desired. \(\square\)

With these results in hand, we can finish the proof of Sylvester’s criterion for positive semidefinite matrices.

Proof of Theorem 2.7. For positive definite matrices, the result was proved in Lemma 2.9. Now suppose \(A_{n \times n}\) is positive semidefinite. We show the result by induction on \(n\), with an easy argument for \(n = 1\) similar to the positive definite case.

Now suppose the result holds for matrices of dimension \(\leq n - 1\), and let \(A_{n \times n}\) have all principal minors non-negative. Let \(B\) be any principal minor of \(A\), and define \(f(t) := \det(B + t \text{Id}_{n \times n})\). Note that \(f'(t) = \text{tr(adj}(B + t \text{Id}_{n \times n}))\) by the Jacobi formula (2.11).

We claim that \(f'(t) > 0\) for all \(t > 0\). Indeed, all diagonal entries of \(\text{adj}(B + t \text{Id}_{n \times n})\) are proper principal minors of \(A + t \text{Id}_{n \times n}\), which is positive definite since \(x^T(A + t \text{Id}_{n \times n})x = x^TAx + t\|x\|^2\) for \(x \in \mathbb{R}^n\). The claim now follows using Lemma 2.8 and the induction hypothesis.

From the claim, we obtain: \(f(t) > f(0) = \det B \geq 0\) for all \(t > 0\). Therefore, all principal minors of \(A + tI\) are positive, and by Sylvester’s criterion for positive definite matrices, \(A + tI\) is positive definite for all \(t > 0\). Now note that \(x^TAx = \lim_{t \to 0^+} x^T(A + t \text{Id}_{n \times n})x\); therefore the non-negativity of the right-hand side implies that of the left-hand side for all \(x \in \mathbb{R}^n\), completing the proof. \(\square\)

2.3. Examples of positive semidefinite matrices. We next discuss several examples of positive semidefinite matrices.

2.3.1. Gram matrices.

Definition 2.13. For any finite set of vectors \(x_1, \ldots, x_n \in \mathbb{R}^m\), their Gram matrix is given by \(\text{Gram}(\{x_j\}) := (\langle x_j, x_k \rangle)_{1 \leq j, k \leq n}\).

A correlation matrix is a positive semidefinite matrix with ones on the diagonal.

In fact we need not use \(\mathbb{R}^m\) here; any inner product space/Hilbert space is sufficient.

Proposition 2.14. Given a real symmetric matrix \(A_{n \times n}\), it is positive semidefinite if and only if there exist an integer \(m > 0\) and vectors \(x_1, \ldots, x_n \in \mathbb{R}^m\) such that \(A = \text{Gram}(\{x_j\})\).
As a special case, correlation matrices precisely correspond to those Gram matrices for which the \( x_j \) are unit vectors. We also remark that a ‘continuous’ version of this result is given by a well-known result of Mercer \[201\]. See Theorem 37.9.

**Proof.** If \( A \) is positive semidefinite, then by Theorem 2.4 we can write \( A = B^T B \) for some matrix \( B_{m \times n} \). It is now easy to check that \( A \) is the Gram matrix of the columns of \( B \).

Conversely, if \( A = \text{Gram}\{x_1, \ldots, x_n\} \) with all \( x_j \in \mathbb{R}^m \), then to show that \( A \) is positive semidefinite, we compute for any \( u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n \):

\[
    u^T A u = \sum_{j,k=1}^{n} u_j u_k \langle x_j, x_k \rangle = \left\| \sum_{j=1}^{n} u_j x_j \right\|^2 \geq 0,
\]

using the bilinearity of the inner product. \( \square \)

### 2.3.2. (Toeplitz) Cosine matrices.

**Definition 2.15.** A matrix \( A = (a_{jk}) \) is **Toeplitz** if \( a_{jk} \) depends only on \( j - k \).

**Lemma 2.16.** Let \( \theta_1, \ldots, \theta_n \in [0, 2\pi] \). Then the matrix \( C := (\cos(\theta_j - \theta_k))_{j,k=1}^{n} \) is positive semidefinite, with rank at most 2. In particular, \( \alpha 1_{n \times n} + \beta C \) has rank at most 3 (for scalars \( \alpha, \beta \)), and it is positive semidefinite if \( \alpha, \beta \geq 0 \).

This family of Toeplitz matrices was used by Rudin in a 1959 paper on entrywise positivity preservers; we will discuss his result below.

**Proof.** Define the vectors \( u, v \in \mathbb{R}^n \) via:

\[
    u^T = (\cos \theta_1, \ldots, \cos \theta_n), \quad v^T = (\sin \theta_1, \ldots, \sin \theta_n).
\]

Then \( C = uu^T + vv^T \) via the identity \( \cos(a - b) = \cos a \cos b + \sin a \sin b \), and clearly the rank of \( C \) is at most 2. (For instance, it can have rank 1 if the \( \theta_j \) are equal.) As a consequence,

\[
    \alpha 1_{n \times n} + \beta C = \alpha 1_n 1_n^T + \beta uu^T + \beta vv^T
\]

has rank at most 3; the last assertion is straightforward. \( \square \)

As a special case, suppose \( \theta_1, \ldots, \theta_n \) are in arithmetic progression, i.e., there exists \( \theta \) such that \( \theta_{j+1} - \theta_j = \theta \) for \( j = 1, \ldots, n - 1 \). The above construction now yields the positive semidefinite matrix

\[
    C = \begin{pmatrix}
        1 & \cos \theta & \cos 2\theta & \ldots \\
        \cos \theta & 1 & \cos \theta & \cos 2\theta & \ldots \\
        \cos 2\theta & \cos \theta & 1 & \cos \theta & \cos 2\theta & \ldots \\
        \vdots & \ddots & \ddots & \ddots & \ddots
    \end{pmatrix},
\]

which is Toeplitz.

### 2.3.3. Hankel matrices.

**Definition 2.17.** A matrix \( A = (a_{jk}) \) is **Hankel** if \( a_{jk} \) depends only on \( j + k \).

**Example 2.18.** \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is Hankel but not positive semidefinite.
Example 2.19. For each \( x \geq 0 \), the matrix
\[
\begin{pmatrix}
1 & x & x^2 & \\
x & x^2 & x^3 & \\
x^2 & x^3 & x^4 & \\
x^3 & x^4 & \ldots & \\
\end{pmatrix}
\begin{pmatrix}
1 & x & x^2 & \\
\end{pmatrix}
\]
(1 \( x \) \( x^2 \)) is Hankel and positive semidefinite of rank 1.

A more general perspective is as follows. Define
\[
H_x := \begin{pmatrix}
1 & x & x^2 & \\
x & x^2 & x^3 & \\
x^2 & x^3 & x^4 & \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
and let \( \delta_x \) be the Dirac measure at \( x \in \mathbb{R} \). The moments of this measure are given by
\[
s_k(\delta_x) = \int_{\mathbb{R}} y^k \, d\delta_x(y) = x^k, \quad k \geq 0.
\]
Thus \( H_x \) is the moment matrix of \( \delta_x \). More generally, given any non-negative measure \( \mu \) supported on \( \mathbb{R} \), with all moments finite, the corresponding Hankel moment matrix is the bi-infinite ‘matrix’ given by
\[
H_\mu = \begin{pmatrix}
s_0 & s_1 & s_2 & \\
s_1 & s_2 & s_3 & \\
s_2 & s_3 & s_4 & \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
where \( s_k = s_k(\mu) := \int_{\mathbb{R}} y^k \, d\mu(y) \).

Lemma 2.21. The matrix \( H_\mu \) is positive semidefinite. In other words, every finite principal submatrix is positive semidefinite.

Notice that this condition is equivalent to every leading principal submatrix being positive semidefinite.

Proof. Fix \( n \geq 1 \) and consider the finite principal (Hankel) submatrix \( H_\mu' \) with the first \( n \) rows and columns. Let \( H_\delta_x' \) be the Hankel matrix defined in a similar manner for the measure \( \delta_x, x \in \mathbb{R} \). Now to show that \( H_\mu' \) is positive semidefinite, we compute for any vector \( u \in \mathbb{R}^n \):
\[
u^T H_\mu' u = \int_{\mathbb{R}} u^T H_\delta_x' u \, d\mu(x) = \int_{\mathbb{R}} ((1, x, \ldots, x^{n-1})u)^2 \, d\mu(x) \geq 0,
\]
where the final equality holds because \( H_\delta_x' \) has rank one, and factorizes as in Example 2.19. (Note that the first equality holds because we are taking finite linear combinations of the integrals in the entries of \( H_\mu' \)).

Remark 2.22. Lemma 2.21 is (the easier) half of a famous classical result by Hamburger. The harder converse result says that if a semi-infinite Hankel matrix is positive semidefinite, with \( (j,k) \)-entry \( s_{j+k} \) for \( j,k \geq 0 \), then there exists a non-negative Borel measure on the real line whose \( k \)th moment is \( s_k \) for all \( k \geq 0 \). This theorem will be useful later; it was shown by Hamburger in 1920–21, when he extended the Stieltjes moment problem to the entire real line in the series of papers [122]. These works established the moment problem in its own right, as opposed to being a tool used to determine the convergence or divergence of continued fractions (as previously developed by Stieltjes – see Remark 4.4).

There is also a multivariate version of Lemma 2.21 which is no harder, modulo notation:

Lemma 2.23. Given a measure \( \mu \) on \( \mathbb{R}^d \) for some integer \( d \geq 1 \), we define its moments for tuples of non-negative integers \( n = (n_1, \ldots, n_d) \) via:
\[
s_n(\mu) := \int_{\mathbb{R}^d} x^n \, d\mu(x) = \int_{\mathbb{R}^d} \prod_{j=1}^d x_j^{n_j} \, d\mu,
\]
if these integrals converge. Now suppose \( \mu \geq 0 \) on \( \mathbb{R}^d \) and let \( \Psi : (\mathbb{Z}^\geq)^d \to \mathbb{Z}^\geq \) be any bijection such that \( \Psi(0) = 0 \) (although this restriction is not really required). Define the semi-infinite matrix \( H_\mu := \sum_{j,k=0}^\infty \psi_j \psi_k \) via \( a_{jk} := \psi_{\mu \psi^{-1}(j) + \psi^{-1}(k)} \), where we assume that all moments of \( \mu \) exist. Then \( H_\mu \) is positive semidefinite.

**Proof.** Given a real vector \( \mathbf{u} = (u_0, u_1, \ldots)^T \) with finitely many nonzero coordinates, we have:

\[
\mathbf{u}^T H_\mu \mathbf{u} = \sum_{j,k \geq 0} \int_{\mathbb{R}^d} u_j u_k \Psi^{-1}(j) \Psi^{-1}(k) d\mu(x) = \int_{\mathbb{R}^d} ((1, x^{\Psi^{-1}(1)}, x^{\Psi^{-1}(2)}, \ldots) \mathbf{u})^2 d\mu(x) \geq 0.
\]

\( \square \)

### 2.3.4. Matrices with sparsity

Another family of positive semidefinite matrices involves matrices with a given zero pattern, i.e. structure of (non)zero entries. Such families are important in applications, as well as in combinatorial linear algebra, spectral graph theory, and graphical models / Markov random fields.

**Definition 2.24.** A graph \( G = (V, E) \) is **simple** if the sets of vertices/nodes \( V \) and edges \( E \) are finite, and \( E \) contains no self-loops \( (v, v) \) or multi-edges. In the sequel, all graphs will be finite and simple. Given such a graph \( G = (V, E) \), with node set \( V = [n] = \{1, \ldots, n\} \), define

\[
\mathbb{P}_G := \{ A \in \mathbb{P}_n : a_{jk} = 0 \text{ if } j \neq k \text{ and } (j, k) \notin E \},
\]

where \( \mathbb{P}_n \) comprises the (real symmetric) positive semidefinite matrices of dimension \( n \).

**Remark 2.26.** The set \( \mathbb{P}_G \) is a natural mathematical generalization of the cone \( \mathbb{P}_n \) (and shares several of its properties). In fact, two ‘extreme’ special cases are: (i) \( G \) is the complete graph, in which case \( \mathbb{P}_G \) is the full cone \( \mathbb{P}_n \) for \( n = |V| \); and (ii) \( G \) is the empty graph, in which case \( \mathbb{P}_G \) is the cone of \( |V| \times |V| \) diagonal matrices with non-negative entries.

Akin to both of these cases, for all graphs \( G \), the set \( \mathbb{P}_G \) is in fact a closed convex cone.

**Example 2.27.** Let \( G = \{ \{v_1, v_2, v_3\}, \{(v_1, v_3), (v_2, v_3)\} \} \). The adjacency matrix is given by

\[
A_G = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}.
\]

This is not in \( \mathbb{P}_G \) (but \( A_G - \lambda_{\min}(A_G) \text{Id}_{3 \times 3} \in \mathbb{P}_G \), see Corollary 2.5).

**Example 2.28.** For any graph with node set \([n]\), let \( D_G \) be the diagonal matrix with \( (j,j) \) entry the degree of node \( j \), i.e. the number of edges adjacent to \( j \). Then the **graph Laplacian**, defined to be \( L_G := D_G - A_G \) (where \( A_G \) is the adjacency matrix), is in \( \mathbb{P}_G \).

**Example 2.29.** An important class of examples of positive semidefinite matrices arises from the Hessian matrix of (suitably differentiable) functions. In particular, if the Hessian is positive definite at a point, then this is an isolated local minimum.

### 2.4. Schur complements

We mention some more preliminary results here; these may be skipped for now, but will get used in Lemma 2.5 below.

**Definition 2.30.** Given a matrix \( M = \begin{pmatrix}
P & Q \\
R & S
\end{pmatrix} \), where \( P \) and \( S \) are square and \( S \) is non-singular, the **Schur complement** of \( M \) with respect to \( S \) is given by

\[
M/S := P - QS^{-1}R.
\]

Schur complements arise naturally in theory and applications. As an important example, suppose \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_m \) are random variables with covariance matrix
The cone of positive semidefinite matrices. Examples. Schur complements.

\[ \Sigma = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \text{ with } C \text{ non-singular. Then the conditional covariance matrix of } X \text{ given } Y \text{ is } \text{Cov}(X|Y) := A - BC^{-1}B^T = \Sigma/C. \text{ That such a matrix is also positive semidefinite is implied by the following folklore (1969) result by Albert in SIAM J. Appl. Math.} \]

**Theorem 2.31.** Given a symmetric matrix \( \Sigma = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \), with \( C \) positive definite, the matrix \( \Sigma \) is positive (semi)definite if and only if the Schur complement \( \Sigma/C \) is thus.

**Proof.** We first write down a more general matrix identity: for a non-singular matrix \( D \) and a square matrix \( A \), one uses a factorization shown by Schur in 1917 in J. reine angew. Math.:

\[
\begin{pmatrix} A & B \\ B' & D \end{pmatrix} = \begin{pmatrix} \text{Id} & BD^{-1} \\ 0 & \text{Id} \end{pmatrix} \begin{pmatrix} A - BD^{-1}B' & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ D^{-1}B' & \text{Id} \end{pmatrix}.
\] (2.32)

(Note here that the identity matrices on the right-hand side are of two different sizes.) Specializing to \( B' = B^T \) and \( D = C \) yields: \( \Sigma = X^T Y X \), where \( X = \begin{pmatrix} \text{Id} & 0 \\ 0 & C \text{Id} \end{pmatrix} \) is non-singular, and \( Y = \begin{pmatrix} A - BC^{-1}B^T & 0 \\ 0 & C \end{pmatrix} \) is block diagonal (and real symmetric). The result is not hard to show from here. \(\)

Akin to Sylvester’s criterion, the above characterization has a variant for when \( C \) is positive semidefinite; however, this is not as easy to prove, and requires a more flexible ‘inverse’:

**Definition 2.33** (Moore–Penrose inverse). Given any real \( m \times n \) matrix \( A \), the pseudo-inverse or Moore–Penrose inverse of \( A \) is an \( n \times m \) matrix \( A^\dagger \) satisfying: \( AA^\dagger A = A \), \( A^\dagger AA^\dagger = A^\dagger \), and \((AA^\dagger)_{m \times m}, (A^\dagger A)_{n \times n} \) are symmetric.

The following fact can be found in standard texts, and is presented here without proof.

**Lemma 2.34.** For every \( A_{m \times n} \), the matrix \( A^\dagger \) exists and is unique.

**Example 2.35.** Here are some examples of the Moore–Penrose inverse of square matrices.

(a) If \( D = \text{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0) \), with all \( \lambda_j \neq 0 \), then \( D^\dagger = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_r^{-1}, 0, \ldots, 0) \).

(b) If \( A \) is positive semidefinite, then \( A = U^T DU \) where \( D \) is a diagonal matrix. It is easy to verify that \( A^\dagger = U^T D^\dagger U \).

(c) If \( A \) is non-singular then \( A^\dagger = A^{-1} \).

We now mention the connection between the positivity of a matrix and its Schur complement with respect to a singular submatrix. First note that the Schur complement is now defined in the expected way (here, \( S \) is square, as is \( M \)):

\[
M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \implies M/S := P - QS^\dagger R, \tag{2.36}
\]

Now the proof of the following result can be found in standard textbooks on matrix analysis.

**Theorem 2.37.** Given a symmetric matrix \( \Sigma = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \), with \( C \) not necessarily invertible, the matrix \( \Sigma \) is positive semidefinite if and only if the following conditions hold:

(a) \( C \) is positive semidefinite,

(b) the Schur complement \( \Sigma/C \) is positive semidefinite, and

(c) \((\text{Id} - CC^\dagger)B^T = 0\).

3.1. The Schur product. We now make some straightforward observations about the set $\mathbb{P}_n$. The first is that $\mathbb{P}_n$ is topologically closed, convex, and closed under scaling by positive multiples (a ‘cone’):

**Lemma 3.1.** $\mathbb{P}_n$ is a closed convex cone in $\mathbb{R}^{n \times n}$.

**Proof.** All properties are easily verified using the definition of positive semidefiniteness. □

If $A$ and $B$ are positive semidefinite matrices, then we expect the product $AB$ to also be positive semidefinite. This is true if $AB$ is symmetric.

**Lemma 3.2.** For $A, B \in \mathbb{P}_n$, if $AB$ is symmetric then $AB \in \mathbb{P}_n$.

**Proof.** In fact $AB = (AB)^T = B^T A^T = BA$, whence $A$ and $B$ commute. Writing $A = U^T D_1 U$ and $B = U^T D_2 U$ as per the Spectral Theorem 2.3 for commuting matrices, we have:

$$
x^T (AB) x = x^T (U^T D_1 U \cdot U^T D_2 U) x = x^T U^T (D_1 D_2) U x = \| \sqrt{D_1} D_2 U x \|^2 \geq 0.
$$

Hence $AB \in \mathbb{P}_n$. □

Note however that $AB$ need not be symmetric even if $A$ and $B$ are symmetric. In this case, the matrix $AB$ certainly cannot be positive semidefinite; however, it still satisfies one of the equivalent conditions for positive semidefiniteness (shown above for symmetric matrices), namely, having a non-negative spectrum. We prove this with the help of another result, which shows the ‘tracial’ property of the spectrum:

**Lemma 3.3.** For $A_{n \times m}, B_{m \times n}$, we have $\sigma(AB) \cup \{0\} = \sigma(ABA) \cup \{0\}$.

(Here, ‘tracial’ suggests that the expression for $AB$ equals that for $BA$, as does the trace.)

**Proof.** Assume without loss of generality that $1 \leq m \leq n$. The result will follow if we can show that $\det(\lambda \text{Id}_{n \times n} - AB) = \lambda^{n-m} \det(\lambda \text{Id}_{n \times m} - BA)$ for all $\lambda$. In turn, this follows from the equivalence of characteristic polynomials of $AB$ and $BA$ up to a power of $\lambda$, which is why we must take the union of both spectra with zero. (In particular, the sought-for equivalence would also imply that the nonzero eigenvalues of $AB$ and $BA$ are equal up to multiplicity).

The proof finishes by considering the two following block matrix identities:

$$
\begin{pmatrix}
\text{Id}_{n \times n} & -A \\
0 & \lambda \text{Id}_{m \times m}
\end{pmatrix}
\begin{pmatrix}
\lambda \text{Id}_{n \times n} & A \\
B & \text{Id}_{m \times m}
\end{pmatrix}
= 
\begin{pmatrix}
\lambda \text{Id}_{n \times n} - AB & 0 \\
\lambda B & \lambda \text{Id}_{m \times m}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\text{Id}_{n \times n} & 0 \\
-B & \lambda \text{Id}_{m \times m}
\end{pmatrix}
\begin{pmatrix}
\lambda \text{Id}_{n \times n} & A \\
B & \text{Id}_{m \times m}
\end{pmatrix}
= 
\begin{pmatrix}
\lambda \text{Id}_{n \times n} & A \\
0 & \lambda \text{Id}_{m \times m} - BA
\end{pmatrix}
$$

Note that the determinants on the two left-hand sides are equal. Now, equating the determinants on the right-hand sides and cancelling $\lambda^m$ shows the desired identity:

$$
\det(\lambda \text{Id}_{n \times n} - AB) = \lambda^{n-m} \det(\lambda \text{Id}_{m \times m} - BA)
$$

for $\lambda \neq 0$. But since both sides here are polynomial (hence continuous) functions of $\lambda$, taking limits implies the identity for $\lambda = 0$ as well. (Alternately, $AB$ is singular if $n > m$, which shows the identity for $\lambda = 0$.) □

With Lemma 3.3 in hand, we can prove:

**Proposition 3.4.** For $A, B \in \mathbb{P}_n$, $AB$ has non-negative eigenvalues.
Proof. Let $X = \sqrt{A}$ and $Y = \sqrt{AB}$, where $A = U^T DU \implies \sqrt{A} = U^T \sqrt{D} U$. Then $XY = AB$ and $YX = \sqrt{AB} \sqrt{A}$. In fact, $YX$ is (symmetric and) positive semidefinite, since $x^T YX x = \| \sqrt{B} \sqrt{A} x \|^2$, $\forall x \in \mathbb{R}^n$.

It follows that $YX$ has non-negative eigenvalues, whence the same holds by Lemma 3.3 for $XY = AB$, even if $AB$ is not symmetric. □

We next introduce a different multiplication operation on matrices (possibly rectangular, including row or column matrices), which features extensively below.

Definition 3.5. Given positive integers $m, n$, the Schur product of $A_{m \times n}$ and $B_{m \times n}$ is the matrix $C_{m \times n}$ with $c_{jk} = a_{jk} b_{jk}$ for $1 \leq j \leq m, 1 \leq k \leq n$. We denote the Schur product by $\circ$ (to distinguish it from the conventional matrix product).

Lemma 3.6. Given integers $m, n > 0$, the space $(\mathbb{R}^{m \times n}, +, \circ)$ forms a commutative associative algebra.

Proof. The easy proof is omitted. More formally, $\mathbb{R}^{m \times n}$ under coordinatewise addition and multiplication is simply the direct sum (or direct product) of copies of $\mathbb{R}$ under usual addition and multiplication. □

Remark 3.7. Schur products occur in a variety of settings in mathematics. This includes the theory of Schur multipliers, association schemes in combinatorics, characteristic functions in probability theory, and the weak minimum principle in partial differential equations. Yet another application is to products of integral equation kernels and the connection to Mercer’s theorem (see Theorem 37.9). A final connection mentioned here is to trigonometric moments. Suppose $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are continuous and $2\pi$-periodic, with Fourier coefficients $a_j^{(k)}(j) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} f_j(\theta) d\theta$, $j = 1, 2, k \in \mathbb{Z}$.

If one defines the convolution product of $f_1, f_2$, via

$$(f_1 * f_2)(\theta) := \int_0^{2\pi} f_1(t) f_2(\theta - t) \, dt,$$

then this function has corresponding $k$th Fourier coefficient $a^{(k)}_1 a^{(k)}_2$. Thus, the bi-infinite Toeplitz matrix of Fourier coefficients for $f_1 * f_2$ equals the Schur product of the Toeplitz matrices $(a_1^{(p-q)})_{p,q \in \mathbb{Z}}$ and $(a_2^{(p-q)})_{p,q \in \mathbb{Z}}$.

Remark 3.8. The Schur product is also called the entrywise product or the Hadamard product in the literature; the latter is likely owing to the famous paper by Hadamard in Acta Math. (1899), in which he shows (among other things) the Hadamard multiplication theorem. This relates the radii of convergence and singularities of two power series $\sum_{j \geq 0} a_j z^j$ and $\sum_{j \geq 0} b_j z^j$ with those of $\sum_{j \geq 0} a_j b_j z^j$. This ‘coefficientwise product’ will recur later in this text, when we discuss the Laguerre–Pólya class of entire functions.

Before proceeding further, we define another product operator on matrices of any dimensions:
**Definition 3.9.** Given matrices $A_{m \times n}, B_{p \times q}$, the Kronecker product of $A$ and $B$, denoted $A \otimes B$ is the $mp \times nq$ block matrix, described as:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

While the Kronecker product (as defined) is asymmetric in its arguments, it is easily seen that the analogous matrix $B \otimes A$ is obtained from $A \otimes B$ by permuting its rows and columns.

The next result, by Schur in *J. reine angew. Math.* (1911), is important in later sections. We provide four proofs.

**Theorem 3.10** (Schur product theorem). $\mathbb{P}_n$ is closed under $\circ$.

**Proof.** Suppose $A, B \in \mathbb{P}_n$; we present four proofs that $A \circ B \in \mathbb{P}_n$.

1. Let $A, B \in \mathbb{P}_n$ have eigenbases $(\lambda_j, v_j)$ and $(\mu_k, w_k)$, respectively. Then,

$$(A \otimes B)(v_j \otimes w_k) = \lambda_j \mu_k(v_j \otimes w_k), \quad \forall 1 \leq j, k \leq n. \quad (3.11)$$

It follows that the Kronecker product has spectrum $\{\lambda_j \mu_k\}$, and hence is positive (semi)definite if $A, B$ are positive (semi)definite. Hence every principal submatrix is also positive (semi)definite by Lemma 2.8. But now observe that the principal submatrix of $A \otimes B$ with entries $a_{jk}b_{jk}$ is precisely the Schur product $A \circ B$.

2. By the spectral theorem and the bilinearity of the Schur product,

$$A = \sum_{j=1}^n \lambda_j v_j v_j^T, \quad B = \sum_{k=1}^n \mu_k w_k w_k^T \implies A \circ B = \sum_{j,k=1}^n \lambda_j \mu_k (v_j \circ w_k)(v_j \circ w_k)^T.$$  

This is a non-negative linear combination of rank-one positive semidefinite matrices, hence lies in $\mathbb{P}_n$ by Lemma 3.1.

3. This proof uses a clever computation. Given any commutative ring $R$, square matrices $A, B \in R^{n \times n}$, and vectors $u, v \in R^n$, we have

$$u^T (A \circ B)v = \text{tr}(B^T D_u A D_v), \quad (3.12)$$

where $D_v$ denote the diagonal matrix with diagonal entries the coordinates of $v$ (in the same order). Now if $R = \mathbb{R}$ and $A, B \in \mathbb{P}_n(\mathbb{R})$, then

$$v^T (A \circ B)v = \text{tr}(AD_v B^T D_v) = \text{tr}(A^{1/2} D_u B^T D_v A^{1/2}).$$

But $A^{1/2} D_u B^T D_v A^{1/2}$ is positive semidefinite, so its trace is non-negative, as desired.

4. Given $t > 0$, let $X, Y$ be independent multivariate normal vectors centered at 0 and with covariance matrices $A + t \text{Id}_{n \times n}, B + t \text{Id}_{n \times n}$ respectively. (Note that these always exist.) The Schur product of $X$ and $Y$ is then a random vector with mean zero, and covariance matrix precisely $(A + t \text{Id}_{n \times n}) \circ (B + t \text{Id}_{n \times n})$. Now the result follows directly from the fact that covariance matrices are positive semidefinite, upon taking the limit as $t \to 0^+$.

**Remark 3.13.** The first of the above proofs also shows the Schur product theorem for (complex) positive definite matrices: If $A, B \in \mathbb{P}_n(\mathbb{C})$ are positive definite, then so is their Kronecker product $A \otimes B$, whence its principal submatrix $A \circ B$, the Schur product.
Note that the Schur product is ‘qualitative’, in that it says $M \circ N \geq 0$ if $M, N \geq 0$. In the century after its formulation, ‘tight’ quantitative nonzero lower bounds have been discovered. For instance, in *Linear Algebra Appl.*, Friedler–Markham (1995) and Reams (1999) showed:

$$M \circ N \geq \lambda_{\min}(N)(M \circ \text{Id}_n),$$

$$M \circ N \geq \frac{1}{1^T N^{-1} 1} M, \text{ if } \det(N) > 0.$$  \hfill (3.14)

Here we state and prove a recent new lower bound, first obtained in a weaker form by Vybíral in 2019:

**Theorem 3.15.** Fix integers $a, b, n \geq 1$ and nonzero matrices $A \in \mathbb{C}^{n \times a}$, $B \in \mathbb{C}^{n \times b}$. Then we have the (rank $\leq 1$) lower bound:

$$AA^* \circ BB^* \geq \frac{1}{\min(\text{rk}(A^*), \text{rk}(BB^*))} d_{ABP}d_{ABP}^*,$$

where $P := \text{proj}_{\ker(A)^\perp} |\text{im}(B^*)|$, and given a square matrix $M = (m_{jk})$, the vector $d_M := (m_{jj})$. Here the coefficient $1/\min(\cdot, \cdot)$ is best possible. Also, if $a = b$, one can use $P = \text{Id}_n$ instead.

We make several remarks here. First, if $A, B$ are rank-one, then a stronger result holds: we get equality above (say unlike in (3.14), for instance). Second, reformulating the result in terms of $M = AA^*, N = BB^*$ says that $M \circ N$ is bounded below by many possible rank-one submatrices, one for every (square) matrix decomposition $M = AA^*, N = BB^*$. Third, the result extends to Hilbert–Schmidt operators, in which case it again provides nonzero positive lower bounds – and in a more general form even on $\mathbb{P}_n(\mathbb{C})$; see a recent 2019 preprint by the author (following its special case by Vybíral in 2019). Finally, specializing the result to the positive semidefinite square roots $A = \sqrt{M}, B = \sqrt{N}$ yields a novel connection between the matrix functional calculus and entrywise operations on matrices:

$$M \circ N \geq \frac{1}{\min(\text{rk}(M), \text{rk}(N))} \sqrt{\sqrt{N} \sqrt{M}} \sqrt{N} \sqrt{M} \quad \forall M, N \in \mathbb{P}_n(\mathbb{C}), \ n \geq 1.$$

**Proof.** We write down the proof as it serves to illustrate another important tool in matrix analysis: the *trace form* on matrix space $\mathbb{C}^{a \times b}$. Compute using (3.12):

$$u^*(AA^* \circ BB^*)u = \text{tr}(BB^T D_{AB}AA^* D_{AB}) = \text{tr}(T^* T), \quad \text{where } T := A^* D_u B.$$

Now use the inner product on $\mathbb{C}^{a \times b}$, given by $\langle X, Y \rangle := \text{tr}(X^* Y)$. Let $P$ be as above. Then,

$$\langle P, P \rangle \leq \min(\dim(\ker(A))^\perp, \dim(\text{im}(B^T))) = \min(\text{rk}(A^*), \text{rk}(B^*)) = \min(\text{rk}(AA^*), \text{rk}(BB^*)).$$

Here we use that $AA^*$ and $A^*$ have the same null space:

$$A^* x = 0 \Rightarrow AA^* x = 0 \Rightarrow \|A^* x\|^2 = 0 \Rightarrow A^* x = 0,$$

and hence the same rank. Now using the Cauchy–Schwarz inequality (for this tracial inner product) and the above computations, we have:

$$u^*(AA^* \circ BB^*)u = \langle T, T \rangle \geq \frac{|\langle T, P \rangle|^2}{\langle P, P \rangle} = \frac{|\text{tr}(APB^T D_{AB})|^2}{\langle P, P \rangle} = \frac{|u^* d_{ABP}^T|^2}{\langle P, P \rangle} \geq \frac{1}{\min(\text{rk}(AA^*), \text{rk}(BB^*))} u^* d_{ABP}d_{ABP}^* u.$$

As this holds for all vectors $u$, the result follows. If moreover $a = b$, then one easily verifies (given the choice of $P$) that $APB^T = AB^T$, which enables replacing $P$ by $\text{Id}_n$ in the result.

Finally, to see the tightness of the lower bound, choose integers $1 \leq r \leq a$, $1 \leq s \leq b$ with $r, s \leq n$, and complex block diagonal matrices

$$A_{n \times a} := \begin{pmatrix} D_{r \times r} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{n \times b} := \begin{pmatrix} D'_{s \times s} & 0 \\ 0 & 0 \end{pmatrix},$$

with both $D, D'$ non-singular. Now $P := \begin{pmatrix} \text{Id}_{\min(r,s)} & 0 \\ 0 & 0 \end{pmatrix}$, and the inequality is indeed tight, as can be verified using the Cauchy–Schwarz identity.

□

Remark 3.16. A simpler lower bound for Schur products is given by Vybíral for square matrix decompositions: if $A, B \in \mathbb{C}^{n \times n}$, then

$$AA^* \circ BB^* \geq (A \circ B)(A \circ B)^*,$$

where both sides are positive semidefinite. Indeed, if $v_j, w_k$ denote the columns of $A, B$ respectively, then $AA^* = \sum_j v_jv_j^*$ and $BB^* = \sum_k w_kw_k^*$, so

$$AA^* \circ BB^* = \sum_{j,k=1}^n (v_jv_j^*) \circ (w_kw_k^*) \geq \sum_{j=1}^n (v_j \circ w_j)(v_j^* \circ w_j^*) = (A \circ B)(A \circ B)^*.$$

3.2. Totally Positive (TP) and Totally Non-negative (TN) matrices.

Definition 3.17. Given an integer $p > 0$, we say a matrix is totally positive (totally non-negative) of order $p$, denoted $TP_p$ ($TN_p$), if all its $1 \times 1, 2 \times 2, \ldots, p \times p$ minors are positive (non-negative). We will also abuse notation and write $A \in TP_p$ ($A \in TN_p$) if $A$ is $TP_p$ ($TN_p$). A matrix is totally positive (TP) (respectively, totally non-negative (TN)) if $A \in TP_p$ (respectively $TN_p$) for all $p \geq 1$.

Remark 3.18. In classical works, as well as the books by Karlin and Pinkus, totally non-negative and totally positive matrices were referred to, respectively, as totally positive and strictly totally positive matrices.

Here are some distinctions between $TP/TN$ matrices and positive (semi)definite ones:

- For $TP/TN$ matrices we consider all minors, not just the principal ones.
- As a consequence of considering the $1 \times 1$ minors, it follows that the entries of $TP$ ($TN$) matrices are all positive (non-negative).
- $TP/TN$ matrices need not be symmetric, unlike positive semidefinite matrices.

Example 3.19. The matrix \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\] is totally non-negative but not totally positive, while the matrix \[
\begin{pmatrix} 1 & 2 \\ 3 & 16 \end{pmatrix}
\] is totally positive.

Totally positive matrices and kernels have featured in the mathematics literature in a variety of classical and modern topics. A few of these topics are now mentioned, as well as a few of the experts who have worked/written on them.

- Interacting particle systems (Gantmacher, Krein), see [99, 100] and follow-up papers.
- Analysis (Aissen, Edrei, Pólya, Schoenberg, Whitney), see Part 4 for more details. See also Loewner [187]. (Also Belton–Guillot–Khare–Putinar [23].)
- Differential equations – see e.g. Loewner [187], Schwarz [261], the book of Karlin [154], and numerous follow-ups.
- Probability and statistics (Efron, Karlin, McGregor, Pitman, Rinott), see e.g. [78, 153, 154, 155, 156, 157, 158, 165, 219] and numerous follow-ups.
Theorem 3.20 (Motzkin, 1936). The following are equivalent for a matrix $K \in \mathbb{R}^{m \times n}$:

1. $K$ is variation diminishing: $S^-(Kx) \leq S^-(x) \forall x \in \mathbb{R}^n$. Here, $S^-(x)$ for a vector $x$ denotes the number of changes in sign, after removing all zero entries in $x$.
2. Let $K$ have rank $r$. Then $K$ should not have two minors of equal size $< r$ but opposite signs; and $K$ should not have two minors of equal size $= r$ but opposite signs if these minors come from the same rows or columns of $K$.

In the remainder of this section and the next, we discuss several examples of $TN$ matrices. Specifically, we show that the (positive semidefinite) Toeplitz cosine matrices and Hankel moment matrices considered above, are in fact totally non-negative!

Example 3.21 (Toeplitz cosine matrices). We claim that the following matrices are $TN$:

$$C(\theta) := (\cos(j-k)\theta)_{j,k=1}^n,$$

where $\theta \in [0, \frac{\pi}{2(n-1)}]$. Indeed, all $1 \times 1$ minors are non-negative, and as discussed above, $C(\theta)$ has rank at most 2, and so all $3 \times 3$ and larger minors vanish. It remains to consider all $2 \times 2$ minors. Now a $2 \times 2$ sub-matrix of $C(\theta)$ is of the form $C' = \begin{pmatrix} C_{ab} & C_{ac} \\ C_{db} & C_{dc} \end{pmatrix}$, where $1 \leq a < d \leq n$, and $1 \leq b < c \leq n$, and $C_{ab}$ denotes the matrix entry $\cos((a-b)\theta)$. Writing $a,b,c,d$ in place of $a\theta, b\theta, c\theta, d\theta$ in the next computation ease of exposition, the corresponding minor is:

$$\det C' = \frac{1}{2} \left\{ 2 \cos(a-b) \cos(d-c) - 2 \cos(a-c) \cos(d-b) \right\}$$

$$= \frac{1}{2} \left\{ \cos(a-b+d-c) + \cos(a-b-d+c) - \cos(a-c+d-b) - \cos(a-c-d+b) \right\}$$

$$= \frac{1}{2} \left\{ \cos(a-b-d+c) - \cos(a-c-d+b) \right\} = \frac{1}{2} (-2) \sin(a-d) \sin(c-b).$$

Thus, $\det C' = \sin(d\theta - a\theta) \sin(c\theta - b\theta)$, which is non-negative because $a < d, b < c$, and $\theta \in [0, \frac{\pi}{2(n-1)}]$. This shows that $C(\theta)$ is totally non-negative.
4. Fekete’s result. Hankel moment matrices are $TN$.

Continuing from the previous section, we show that the Hankel moment matrices $H_\mu$ studied above are not only positive semidefinite, but more strongly, totally non-negative (TN). Akin to the Toeplitz cosine matrices (where the angle $\theta$ is restricted to ensure the entries are non-negative), we restrict the support of the measure to $[0, \infty)$, which guarantees that the entries are non-negative.

To achieve these goals, we prove the following result, which is crucial in relating preservers of positive semidefinite matrices (and kernels) to those of Hankel $TN$ matrices (and kernels), later in this text.

**Theorem 4.1.** If $1 \leq p \leq n$ are integers, and $A_{n\times n}$ is a real Hankel matrix, then $A$ is $TP_p$ ($TN_p$) if and only if all contiguous principal minors of both $A$ and $A^{(1)}$, of order $\leq p$, are positive (semi)definite, Here $A^{(1)}$ is obtained from $A$ by removing the first row and last column, and by a contiguous minor we mean the determinant of a square submatrix corresponding to successive rows and to successive columns.

In particular, $A$ is $TP$ ($TN$) if and only if $A$ and $A^{(1)}$ are positive (semi)definite.

From this theorem, we derive the following two consequences, both of which are useful in later sections. The first follows from the fact that all contiguous minors of a Hankel matrix are Hankel, whence symmetric:

**Corollary 4.2.** For all integers $1 \leq p \leq n$, the set of Hankel $TN_p$ $n \times n$ matrices is a closed convex cone, further closed under taking Schur products.

The second corollary of Theorem 4.1 provides a large class of examples of such Hankel $TN$ matrices:

**Corollary 4.3.** Suppose $\mu$ is a non-negative measure supported in $[0, \infty)$, with all moments finite. Then $H_\mu$ is $TN$.

The proofs are left as easy exercises, say using Lemma 2.21.

**Remark 4.4.** Akin to Lemma 2.21 and the remark following its proof, Corollary 4.3 is also the easy half of a well-known classical result on moment problems – this time, by Stieltjes. The harder converse of Stieltjes’ result says (in particular) that if a semi-infinite Hankel matrix $H$ is TN, with $(j, k)$-entry $s_{j+k} \geq 0$ for $j, k \geq 0$, then there exists a non-negative Borel measure $\mu$ on $\mathbb{R}$ with support in $[0, \infty)$, whose $k$th moment is $s_k$ for all $k \geq 0$. By Theorem 4.1, this is equivalent to both $H$ as well as $H^{(1)}$ being positive semidefinite, where $H^{(1)}$ is obtained by truncating the first row (or the first column) of $H$.

In his work in the 1890s, Stieltjes was working on continued fractions and divergent series, following Euler, Laguerre, Hermite, and others. One result that is relevant here is that Stieltjes produced a nonzero function $\varphi : [0, \infty) \to \mathbb{R}$ such that $\int_0^\infty x^k \varphi(x) \, dx = 0$ for all $k = 0, 1, \ldots$ – an indeterminate moment problem. (The work in this setting also led him to develop the Stieltjes integral; see e.g. [167] for a detailed historical account.) This marks the beginning of his exploration of the moment problem, which he resolved in his well-known memoir [273].

A curious follow-up, by Boas in *Bull. Amer. Math. Soc.* in 1939, is that if one replaces the non-negativity of the Borel measure $\mu$ by the weaker hypothesis of being of the form $d\alpha(t)$ on $[0, \infty)$ with $\alpha$ of bounded variation, then this recovers all real sequences! See [38].
The remainder of this section is devoted to showing Theorem 4.1. The proof uses a sequence of lemmas shown by Gantmacher–Krein, Fekete, Schoenberg, and others. The first of these lemmas may be (morally) attributed to Laplace.

**Lemma 4.5.** Let \( r \geq 1 \) be an integer and \( U = (u_{jk}) \) an \((r+2) \times (r+1)\) matrix. Given subsets \([a,b],[c,d] \subset (0,\infty)\), let \( U_{[a,b],[c,d]} \) denote the submatrix of \( U \) with entries \( u_{jk} \) such that \( j,k \) are integers and \( a \leq j \leq b, \ c \leq k \leq d \). Then the following identity holds:

\[
\begin{align*}
\det U_{[1,r]}[r+2] \times [1,r+1] & \cdot \det U_{[2,r+1]}[1,r] \\
= & \det U_{[2,r+2]}[1,r+1] \cdot \det U_{[1,r]}[1,r] \\
& + \det U_{[1,r+1]}[1,r+1] \cdot \det U_{[2,r]}[r+2] \times [1,r].
\end{align*}
\]

(4.6)

Note that in each of the three products of determinants, the second factor in the subscript for the first (respectively second) determinant terms is the same: \([1, r+1] \) (respectively \([1, r] \)).

To give a feel for the result, the special case of \( r = 1 \) asserts that

\[
u_{11} \begin{vmatrix} u_{21} & u_{22} \\ u_{31} & u_{32} \end{vmatrix} - u_{21} \begin{vmatrix} u_{11} & u_{12} \\ u_{31} & u_{32} \end{vmatrix} + u_{31} \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} = 0.
\]

But this is precisely the Laplace expansion along the third column of the singular matrix

\[
\det \begin{pmatrix} u_{11} & u_{12} & u_{11} \\ u_{21} & u_{22} & u_{21} \\ u_{31} & u_{32} & u_{31} \end{pmatrix} = 0.
\]

**Proof.** Consider the \((2r+1) \times (2r+1)\) block matrix of the form

\[
M = \begin{pmatrix} b^T & A \\ a^T & c^T \\ d^T & u_{r+2,r+1} \\ A & 0_{(r-1) \times r} \end{pmatrix} = \begin{pmatrix} (u_{jk})_{j \in [1,r+2], k \in [1,r+1]} & (u_{jk})_{j \in [1,r+2], k \in [1,r]} \\ (u_{jk})_{j \in [2,r], k \in [1,r+1]} & 0_{(r-1) \times r} \end{pmatrix};
\]

that is, where

\[
a = (u_{2,r+1}, \ldots, u_{r,r+1})^T, \quad b = (u_{1,1}, \ldots, u_{1,r})^T, \quad c = (u_{r+1,1}, \ldots, u_{r+1,r})^T, \\
d = (u_{r+2,1}, \ldots, u_{r+2,r})^T, \quad A = (u_{jk})_{2 \leq j \leq r, 1 \leq k \leq r}.
\]

Notice that \( M \) is a square matrix whose first \( r+2 \) rows have column space of dimension at most \( r+1 \); hence \( \det M = 0 \). Now we compute \( \det M \) using the (generalized) Laplace expansion by complementary minors: choose all possible \((r+1)\)-tuples of rows from the first \( r+1 \) columns to obtain a submatrix \( M'_{(r+1)} \); and deleting these rows and columns from \( M \) yields the complementary \( r \times r \) submatrix \( M''_{(r)} \) from the final \( r \) columns. The generalized Laplace expansion says that if one multiplies \( \det M'_{(r+1)} \cdot \det M''_{(r)} \) by \((-1)^{\Sigma}\), with \( \Sigma \) the sum of the row numbers in \( M'_{(r+1)} \); then summing over all such products (running over subsets of rows) yields \( \det M \) – which vanishes for this particular matrix \( M \).

Now in the given matrix, to avoid obtaining zero terms, the rows in \( M'_{(r+1)} \) must include all entries from the final \( r-1 \) rows (and the first \( r+1 \) columns). But then it moreover cannot include entries from the rows of \( M \) labelled \( 2, \ldots, r \); and it must include two of the remaining three rows (and entries from only the first \( r+1 \) columns).

Thus, we obtain three product terms that sum to: \( \det M = 0 \). Upon carefully examining the terms and computing the companion signs (by row permutations), we obtain (4.6). \( \square \)
4. Fekete’s result. Hankel moment matrices are $TN$.

Hankel positive semidefinite and $TN$ matrices.

The next two results are classical facts about totally positive matrices, first shown by Fekete in *Rend. Circ. Mat. Palermo* (1912).

**Lemma 4.7.** Given integers $m \geq n \geq 1$ and a real matrix $A_{m \times n}$ such that

(a) all $(n - 1) \times (n - 1)$ minors $\det A_{J \times [1,n-1]} > 0$ for $J \subset [1,m]$ of size $n - 1$, and

(b) all $n \times n$ minors $\det A_{[j+1,j+n] \times [1,n]} > 0$ for $0 \leq j \leq m - n$,

we have that all $n \times n$ minors of $A$ are positive.

**Proof.** Define the *gap*, or ‘index’ of a subset of integers $J = \{j_1 < j_2 < \cdots < j_n\} \subset [1,m]$, to be $g_J := j_n - j_1 - (n - 1)$. Thus, the gap is zero if and only if $J$ consists of successive integers, and in general it counts precisely the number of integers between $j_1$ and $j_n$ that are missing from $J$.

We will prove that $\det A_{J \times [1,n]} > 0$ for $|J| = n$, by induction on the gap $g_J \geq 0$; note that the base case $g_J = 0$ is given as hypothesis. For the induction step, suppose $j^0$ is a missing index (or row number) in $J = \{j_1 < \cdots < j_n\}$. By suitably specializing the identity \((4.6)\), we obtain:

\[
det A_{\{j_1,\ldots,j_n\} \times [1,n]} \cdot det A_{\{j_2,\ldots,j_n-1,j^0\} \times [1,n-1]} = det A_{\{j_1,\ldots,j_n-1,j^0\} \times [1,n]} \cdot det A_{\{j_2,\ldots,j_n\} \times [1,n-1]} - det A_{\{j_2,\ldots,j_n,j^0\} \times [1,n]} \cdot det A_{\{j_1,\ldots,j_n-1\} \times [1,n-1]}.
\]

Consider the six factors in serial order. The first, fourth, and sixth factors have indices listed in increasing order, while the other three factors have $j^0$ listed at the end, so their indices are not listed in increasing order. For each of the six factors, the number of ‘bubble sorts’ required to rearrange indices in increasing order (by moving $j^0$ down the list) equals the number of row exchanges in the corresponding determinants; label these numbers $n_1, \ldots, n_6$. Thus $n_1 = n_4 = n_6 = 0$ as above, while $n_2 = n_3$ (since $j_1 < j^0 < j_n$), and $|n_2 - n_5| = 1$. Now multiply the equation \((4.8)\) by $(-1)^{n_2}$, and divide both sides by

\[
c_0 := (-1)^{n_2} det A_{\{j_2,\ldots,j_n-1,j^0\} \times [1,n-1]} > 0.
\]

Using the given hypotheses as well as the induction step (since all terms involving $j^0$ have a gap equal to $g_J - 1$), it follows that

\[
det A_{\{j_1,\ldots,j_n\} \times [1,n]} = c_0^{-1} \left((-1)^{n_2} det A_{\{j_1,\ldots,j_n-1,j^0\} \times [1,n]} \cdot det A_{\{j_2,\ldots,j_n\} \times [1,n-1]} + (-1)^{n_2+1} det A_{\{j_2,\ldots,j_n,j^0\} \times [1,n]} \cdot det A_{\{j_1,\ldots,j_n-1\} \times [1,n-1]}\right)
\]

$> 0$.

This completes the induction step, and with it, the proof. $\Box$

We can now state and prove another 1912 result by Fekete for $TP$ matrices – extended to $TP_p$ matrices by Schoenberg in *Ann. of Math.* 1955:

**Lemma 4.9** (Fekete–Schoenberg lemma). Suppose $m, n \geq p \geq 1$ are integers, and $A \in \mathbb{R}^{m \times n}$ is a matrix, all of whose contiguous minors of order at most $p$ are positive. Then $A$ is $TP_p$.

Notice that the analogous statement for $TN_p$ is false, e.g. $p = 2$ and $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$. 
4. Fekete’s result. Hankel moment matrices are $TN$. Hankel positive semidefinite and $TN$ matrices.

**Proof.** We show that for any integer $s \in [1, p]$, every $s \times s$ minor of $A$ is positive. The proof is by induction on $s$ (and running over all real matrices satisfying the hypotheses); note that the base case of $s = 1$ is immediate from the assumptions. For the induction step, suppose

$$2 \leq s = |J| = |K| \leq p, \quad J \subset \mathbb{Z} \cap [1, m], \ K \subset \mathbb{Z} \cap [1, n].$$

First fix a subset $K$ that consists of consecutive rows, i.e. has gap $g_K = 0$ (as in the proof of Lemma 4.7). Let $B$ denote the submatrix $A_{[1, m] \times \mathbb{Z}}$. Then all $s \times s$ minors of $B$ are positive, by Lemma 4.7. In particular, it follows for all $J$ that all $s \times s$ minors $\det A_{J \times K'}$ are positive, whenever $K' \subset [1, n]$ has size $s$ and gap $g_{K'} = 0$. Now apply Lemma 4.7 to the matrix $B := (A_{J \times [1, n]})^T$ to obtain: $\det(A_{J,K})^T > 0$ for (possibly non-consecutive subsets) $K$. This concludes the proof. □

The final ingredient required to prove Theorem 4.1 is the following observation.

**Lemma 4.10.** If $A_{n \times n}$ is a Hankel matrix, then every contiguous minor of $A$ (see Lemma 4.9) is a contiguous principal minor of $A$ or of $A^{(1)}$.

**Recall that $A^{(1)}$ was defined in Theorem 4.1**

**Proof.** Let the first row (respectively, last column) of $A$ contain the entries $s_0, s_1, \ldots, s_{n-1}$ (respectively, $s_{n-1}, s_n, \ldots, s_{2n-2}$). Then every contiguous minor is the determinant of a submatrix of the form

$$M = \begin{pmatrix} s_j & \cdots & s_{j+m} \\ \vdots & \ddots & \vdots \\ s_{j+m} & \cdots & s_{j+2m} \end{pmatrix}, \quad 0 \leq j \leq j + m \leq n - 1.$$

It is now immediate that if $j$ is even (respectively odd) then $M$ is a contiguous principal submatrix of $A$ (respectively $A^{(1)}$). □

With these results in hand, we conclude by proving the above theorem.

**Proof of Theorem 4.7.** If the Hankel matrix $A$ is $TN_p \ (TP_p)$ then all contiguous minors of $A$ of order $\leq p$ are non-negative (positive), proving one implication. Conversely, suppose all contiguous principal minors of $A$ and $A^{(1)}$ of order $\leq p$ are positive. By Lemma 4.10 this implies every contiguous minor of $A$ of order $\leq p$ is positive. By the Fekete–Schoenberg Lemma 4.9, $A$ is $TP_p$, as desired.

Finally, suppose all contiguous principal minors of $A, A^{(1)}$ or size $\leq p$ are non-negative. It follows by Lemma 4.10 that every contiguous square submatrix of $A$ of order $\leq p$ is positive semidefinite. Also choose and fix a $n \times n$ Hankel $TP$ matrix $B$ (note by (5.9) or Lemma 6.9 below that such matrices exist for all $n \geq 1$). Applying Lemma 4.10, $B, B^{(1)}$ are positive definite, whence so is every contiguous square submatrix of $B$.

Now for $\epsilon > 0$, it follows (by Sylvester’s criterion, Theorem 2.7) that every contiguous principal minor of $A + \epsilon B$ of size $\leq p$ is positive. Again applying Lemma 4.9, the Hankel matrix $A + \epsilon B$ is necessarily $TP_p$, and taking $\epsilon \to 0^+$ finishes the proof.

The final statement is the special case $p = n$. □
5. **Generalized Vandermonde matrices. Cauchy–Binet formula and its generalization.**

The previous sections discussed examples (Toeplitz, Hankel) of totally non-negative (TN) matrices. These examples consisted of symmetric matrices. We now see some examples of non-symmetric matrices that are totally positive (TP).

- We now see some examples of non-symmetric matrices that are totally positive (TP).
- Let $H_\mu := \langle s_{j+k}(\mu) \rangle_{j,k \geq 0}$ denote the moment matrix associated to a non-negative measure $\mu$ supported on $[0, \infty)$. We have already seen that this matrix is Hankel and positive semidefinite – in fact, TN. We will show in this section and the next that $H_\mu$ is in fact TP in ‘many’ cases. The proof will use a ‘continuous’ generalization of the Cauchy–Binet formula.

5.1. **Generalized Vandermonde Matrices.** A generalized Vandermonde matrix is a matrix $(x_\alpha^k)_{j,k=1}^n$, where $x_j > 0$ and $\alpha_j \in \mathbb{R}$ for all $j$. If the $x_j$ are pairwise distinct, as are the $\alpha_k$, then the corresponding generalized Vandermonde matrix is non-singular. In fact, a stronger result holds:

**Theorem 5.1.** If $0 < x_1 < \cdots < x_m$ and $\alpha_1 < \cdots < \alpha_n$ are real numbers, then the generalized Vandermonde matrix $V_{m \times n} := (x_\alpha^k)$ is totally positive.

As an illustration, consider the special case $m = n$ and $\alpha_k = k - 1$, which recovers the usual Vandermonde matrix:

$$V = \begin{pmatrix}
1 & x_1 & \cdots & x_1^{n-1} \\
1 & x_2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \cdots & x_n^{n-1}
\end{pmatrix}.$$ 

This matrix has determinant $\prod_{1 \leq j < k \leq n} (x_k - x_j) > 0$. Thus if $0 < x_1 < x_2 < \cdots < x_n$ then $\det V > 0$. However, note this is not enough to prove that the matrix is totally positive. Thus, more work is required to prove total positivity, even for usual Vandermonde matrices. The following 1637 result by Descartes will help in proving the total positivity. (Curiously, this preliminary result is also the beginning of a mathematical journey that led to both total positivity and to the variation diminishing property! See Section 28.3 for the details.)

**Lemma 5.2 (Descartes’ rule of signs, weaker version).** Fix $n$ distinct real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and $n$ scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that not all scalars are 0. Then the function $f(x) := \sum_{k=1}^n c_k x^{\alpha_k}$ can have at most $(n-1)$ distinct positive roots.

The following proof is due to Laguerre (1883); that said, the trick of multiplying by a faster-decaying function and applying Rolle’s theorem was previously employed by Pouliain in 1867. See Theorem 31.3.

**Proof.** By induction on $n$. For $n = 1$, clearly $f(x)$ has no positive root. For the induction step, without loss of generality we may assume $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and that all $c_j$ are nonzero. If $f$ has $n$ distinct positive roots, then so does the function

$$g(x) := x^{-\alpha_1} f(x) = c_1 + \sum_{k=2}^n c_k x^{\alpha_k - \alpha_1}.$$
But then Rolle’s theorem implies that \( g'(x) = \sum_{k=2}^{n} c_k (\alpha_k - \alpha_1) x^{\alpha_k - \alpha_1} \) has \((n - 1)\) distinct positive roots. This contradicts the induction hypothesis, completing the proof. \( \square \)

With this result in hand, we can prove that generalized Vandermonde matrices are \( TP \).

**Proof of Theorem 5.1.** As any minor of \( V \) is also a generalized Vandermonde matrix, it suffices to show that the determinant of \( V \) is positive when \( m = n \).

We first claim that \( \det V \neq 0 \). Indeed, suppose for contradiction that \( V \) is singular. Then there is a vector \( c = (c_1, c_2, \ldots, c_n)^T \) such that \( Vc = 0 \). If it is so, then we have \( n \) distinct positive numbers \( x_1, x_2, \ldots, x_n \) such that \( \sum_{k=1}^{n} c_k x_k^{\alpha_k} = 0 \). This contradicts Lemma 5.2 for \( f(x) = \sum_{k=1}^{n} c_k x_k^{\alpha_k} \). Thus, the claim follows.

We now prove the theorem via a homotopy argument. Consider a (continuous) path \( \gamma : [0, 1] \to \mathbb{R}^n \) going from \( \gamma(0) = (0, 1, \ldots, n-1) \) to \( \gamma(1) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), such that at each timepoint \( t \in [0, 1] \), the coordinates of \( \gamma(t) \) are in increasing order. It is possible to choose such a path; indeed, the straight line ‘geodesic’ path is one such.

Now let \( W(t) := (x_j^{\gamma(t)}(t))_{j,k=1}^{n} \). Then \( W : [0, 1] \to \mathbb{R} \) is a nonzero, continuous map. Since \([0, 1]\) is connected and \( W(0) > 0 \) (see remarks above), it follows that \( W(1) = \det V > 0 \). \( \square \)

**Remark 5.3.** If we have \( 0 < x_n < x_{n-1} < \cdots < x_1 \) and \( \alpha_n < \alpha_{n-1} < \cdots < \alpha_1 \), then observe that the corresponding generalized Vandermonde matrix \( V' := (x_j^{\alpha_k})_{j,k=1}^{n} \) is also \( TP \). Indeed, once again we only need to show \( \det V' > 0 \), and this follows from applying the same permutation to the rows and to the columns of \( V' \) to reduce it to the situation in Theorem 5.1 (since then the determinant does not change in sign).

### 5.2. The Cauchy–Binet formula

The following is a recipe to construct ‘new’ examples of \( TP/TN \) matrices from known ones.

**Proposition 5.4.** If \( A_{m \times n}, B_{n \times k} \) are both \( TN \), then so is the matrix \((AB)_{m \times k} \). This assertion is also valid upon replacing ‘\( TN \)’ by ‘\( TP \)’, provided \( n \geq \max\{m, k\} \).

To prove this proposition, we require the following important result.

**Theorem 5.5** (Cauchy–Binet formula). Given matrices \( A_{m \times n} \) and \( B_{n \times m} \), we have

\[
\det(AB)_{m \times m} = \sum_{J \subseteq [n]} \det(A_{[m] \times J}) \det(B_{J^\uparrow \times [m]}),
\]

where \( J^\uparrow \) reiterates the fact that the elements of \( J \) are arranged in increasing order.

For example, if \( m = n \), this theorem just reiterates the fact that the determinant map is multiplicative on square matrices. If \( m > n \), the theorem says that determinants of singular matrices are zero. If \( m = 1 \), we obtain the inner product of a row and column vector.

**Proof.** Notice that

\[
\det(AB) = \det\left(\begin{array}{cccc}
\sum_{j_1=1}^{n} a_{1j_1} b_{j_11} & \cdots & \sum_{j_m=1}^{n} a_{1j_m} b_{j_m1} \\
\vdots & \ddots & \vdots \\
\sum_{j_1=1}^{n} a_{mj_1} b_{j_11} & \cdots & \sum_{j_m=1}^{n} a_{mj_m} b_{j_m1}
\end{array}\right)_{m \times m}.
\]
By the multilinearity of the determinant, expanding \( \det(AB) \) as a sum over all \( j_i \) yields:

\[
\det(AB) = \sum_{(j_1, j_2, \ldots, j_m) \in [n]^m} b_{j_1} b_{j_2} \cdots b_{j_m} \det \left( \begin{array}{cccc}
  a_{1j_1} & \cdots & a_{1j_m} \\
  \vdots & \ddots & \vdots \\
  a_{mj_1} & \cdots & a_{mj_m}
\end{array} \right).
\]

The determinant in the summand vanishes if \( j_k = j_m \) for any \( k \neq m \). Therefore,

\[
\det(AB) = \sum_{(j_1, j_2, \ldots, j_m) \in [n]^m, \text{ all } j_i \text{ are distinct}} b_{j_1} b_{j_2} \cdots b_{j_m} \det A_{[m] \times (j_1, j_2, \ldots, j_m)}.
\]

We split this sum into two sub-summations. One part runs over all collections of indices, while the other runs over all possible orderings – that is, permutations – of each fixed collection of indices. Thus for each ordering \( j = (j_1, \ldots, j_m) \) of \( J = \{j_1, \ldots, j_m\} \), there exists a unique permutation \( \sigma_j \in S_m \) such that \( (j_1, \ldots, j_m) = \sigma_j(J^\uparrow) \). Now,

\[
\det(AB) = \sum_{J = \{j_1, j_2, \ldots, j_m\} \subset [n], \text{ all } j_i \text{ distinct}} \sum_{\sigma = \sigma_j \in S_m} b_{j_1} b_{j_2} \cdots b_{j_m} (-1)^{\sigma_j} \det A_{[m] \times J^\uparrow} \det(B_{J^\uparrow \times [m]}).
\]

**Proof of Proposition 5.4**. Suppose two matrices \( A_{m \times n} \) and \( B_{n \times k} \) are both \( TN \). Let \( I \subset [m] \) and \( K \subset [k] \) be index subsets of the same size; we are to show \( \det(AB)_{I \times K} \) is non-negative. Define matrices, \( A' := A_{I \times [n]} \) and \( B' := B_{[n] \times K} \). Now it is easy to show that \( (AB)_{I \times K} = A'B' \). In particular, \( \det(AB)_{I \times K} = \det(A'B') \). Hence, the Cauchy–Binet theorem implies:

\[
det(AB)_{I \times K} = \sum_{J \subset [n], |J|=|I|} \det A'_{I \times J^\uparrow} \det B'_{J^\uparrow \times K} = \sum_{J \subset [n], |J|=|K|=|I|} \det A_{I \times J^\uparrow} \det B_{J^\uparrow \times K} \geq 0. \tag{5.7}
\]

It follows that \( AB \) is \( TN \) if both \( A \) and \( B \) are \( TN \). For the corresponding \( TP \)-version, the above proof works as long as the sums in the preceding equation are always over non-empty sets; but this happens whenever \( n \geq \max\{m, k\} \). \( \square \)

**Remark 5.8.** This proof shows that Proposition 5.4 holds upon replacing \( TN/TP \) by \( TN_p/TP_p \), for any \( 1 \leq p \leq \min\{m, n, k\} \).

### 5.3. Generalized Cauchy–Binet formula

We showed in the previous section that generalized Vandermonde matrices are examples of totally positive but non-symmetric matrices. Using these, we can construct additional examples of totally positive symmetric matrices: let \( V = (x_j^{\alpha_k})_{j,k=1}^n \) be a be a generalized Vandermonde matrix with \( 0 < x_1 < x_2 < \cdots < x_n \) and \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \). Then Proposition 5.4 implies that the symmetric matrices \( V^TV \) and \( VV^T \) are totally positive.

For instance, if we take \( n = 3 \) and \( \alpha_k = k - 1 \), then

\[
V = \begin{pmatrix}
1 & x_1 & x_1^2 \\
1 & x_2 & x_2^2 \\
1 & x_3 & x_3^2
\end{pmatrix}, \quad V^TV = \begin{pmatrix}
3 & \sum_{j=1}^3 x_j & \sum_{j=1}^3 x_j^2 \\
\sum_{j=1}^3 x_j & \sum_{j=1}^3 x_j^2 & \sum_{j=1}^3 x_j^3 \\
\sum_{j=1}^3 x_j^2 & \sum_{j=1}^3 x_j^3 & \sum_{j=1}^3 x_j^4
\end{pmatrix}. \tag{5.9}
\]
This is clearly the Hankel moment matrix $H_\mu$ for the counting measure on the set $\{x_1, x_2, x_3\}$. Moreover, $V^T V$ is (symmetric and) totally positive by the Cauchy–Binet theorem. More generally, for all increasing $\alpha_k$ which are in arithmetic progression, the matrix $V^T V$ (defined similarly as above) is a totally positive Hankel moment matrix for some non-negative measure on $[0, \infty)$ — more precisely, supported on $\{b^{\alpha_1}_0, b^{\alpha_2}_1, \ldots, b^{\alpha_m}_m\}$.

The following discussion aims to show (among other things) that the moment matrices $H_\mu$ defined in (2.20) are totally positive for ‘most’ non-negative measures $\mu$ supported in $[0, \infty)$. We begin with by studying functions that are $TP$ or $TN$.

**Definition 5.10.** Let $X, Y \subset \mathbb{R}$ and $K : X \times Y \to \mathbb{R}$ be a function. Given $p \in \mathbb{N}$, we say $K(x, y)$ is a totally non-negative/totally positive kernel of order $p$ (denoted $TN_p$ or $TP_p$) if for any integer $1 \leq n \leq p$ and elements $x_1 < x_2 < \cdots < x_n \in X$ and $y_1 < y_2 < \cdots < y_n \in Y$, we have $\det K(x_j, y_k)_{j,k=1}^n$ is non-negative (positive). Similarly, we say that the kernel $K : X \times Y \to \mathbb{R}$ is totally non-negative/totally positive if it is $TN_p$ (or $TP_p$) for all $p \geq 1$.

**Example 5.11.** The kernel $K(x, y) = e^{xy}$ is totally positive, with $X = Y = \mathbb{R}$. Indeed, choose real numbers $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$, the matrix $(e^{x_j y_k})_{j,k=1}^n$ is a generalized Vandermonde matrix, whence $TP$, so its determinant is positive.

We next generalize the Cauchy–Binet theorem to $TP/TN$ kernels. Let $X,Y,Z \subset \mathbb{R}$ and $\mu$ be a non-negative Borel measure on $Y$. Let $K(x, y)$ and $L(y, z)$ be ‘nice’ functions (i.e., Borel measurable with respect to $Y$), and assume the following function is well-defined:

$$M : X \times Z \to \mathbb{R}, \quad M(x, z) := \int_Y K(x, y)L(y, z)d\mu(y). \quad (5.12)$$

For example, consider $K(x, y) = e^{xy}$ and $L(y, z) = e^{yz}$. Take $X = Z = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $Y = \{\log(x_1), \log(x_2), \ldots, \log(x_n)\}$ such that $0 < x_1 < x_2 < \cdots < x_n$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Finally, $\mu$ denote the counting measure on $Y$. Then $M(\alpha_i, \alpha_k) = \sum_{j=1}^n x_j^{\alpha_i}x_j^{\alpha_k}$. So $(M(\alpha_i, \alpha_k))_{i,k=1}^n = V^T V$, where $V = (x_j^{\alpha_k})_{j,k=1}^n$ is a generalized Vandermonde matrix.

In this ‘discrete’ example (i.e., where the support of $\mu$ is a discrete set), $\det M$ is shown to be positive using the total positivity of $V, V^T$ and the Cauchy–Binet formula. Following Karlin, we note that this phenomenon extends to the above more general setting:

**Exercise 5.13** (Pólya–Szegö, Basic Composition Formula, or Generalized Cauchy–Binet formula). Suppose $X, Y, Z \subset \mathbb{R}$ and $K(x, y), L(y, z), M(x, z)$ are as above. Then using an argument similar to the above proof of the Cauchy–Binet formula, show that

$$\det \begin{pmatrix} M(x_1, z_1) & \cdots & M(x_1, z_m) \\ \vdots & \ddots & \vdots \\ M(x_m, z_1) & \cdots & M(x_m, z_m) \end{pmatrix} = \int_{y_1 < y_2 < \cdots < y_m} \prod_{j=1}^m \det(K(x_i, y_j))_{i,j=1}^n \cdot \det(L(y_j, z_k))_{j,k=1}^n \prod_{j=1}^m d\mu(y_j). \quad (5.14)$$

**Remark 5.15.** In the right-hand side, we may also integrate over the region $y_1 \leq \cdots \leq y_m$ in $Y$, since matrices with equal rows or columns are singular.
6. Hankel moment matrices are $TP$. Andréief’s identity.  
Density of $TP$ matrices in $TN$ matrices.


6.1. Total positivity of $H_\mu$ for ‘most’ measures; Andréief’s identity. Continuing from the previous section with the generalized Cauchy–Binet formula of Pólya–Szegő, from (5.14) we obtain the following consequence.

**Corollary 6.1.** (Notation as in the previous section.) If the kernels $K$ and $L$ are both $TN_p$ ($TP_p$) for some integer $p > 0$, then so is $M$, where $M$ was defined in (5.12). In particular, if $K$ and $L$ are $TP/ TN$ kernels, then so is $M$.

We will apply this result to the moment matrices $H_\mu$ defined in (2.20). We begin more generally: suppose $Y \subset \mathbb{R}$ and $u : Y \to (0, \infty)$ is a positive and strictly increasing function, all of whose moments exist with respect to some non-negative measure $\mu$:

$$\int_Y u(y)^n \, d\mu(y) < \infty, \quad \forall n \geq 0.$$ 

Then we claim:

**Proposition 6.2.** The kernel $M : \mathbb{R}^\geq \times \mathbb{R}^\geq \to \mathbb{R}$, given by:

$$M(n, m) := \int_Y u(y)^{n+m} \, d\mu(y)$$

is $TN$ as well as $TP_{|Y_+|}$, where $Y_+ := \text{supp}(\mu) \subset Y$.

**Proof.** To show $M$ is $TP_{|Y_+|}$, the first claim is that the kernel $K(n, y) := u(y)^n$ is $TP_{|Y_+|}$ on $\mathbb{R}^\geq \times Y$. Indeed, we can rewrite $K(n, y)$ as $K(n, y) = e^{n \log(u(y))}$. Now given increasing tuples of elements $n_j \in \mathbb{R}^\geq$ and $y_k \in Y_+$, the matrix $K(n_j, y_k)$ is $TP$, by the total positivity of $e^{\eta y}$ (shown in the previous section).

Similarly, $L(y, m) := u(y)^m$ is also $TP_{|Y_+|}$ on $Y \times \mathbb{R}^\geq$. The result now follows by Corollary 6.1. That $M$ is $TN$ follows from the same arguments, via Remark 5.15. \qed

This result implies the total positivity of the Hankel moment matrices (2.20). Indeed, setting $u(y) = y$ on domains $Y \subset [0, \infty)$, we obtain:

**Corollary 6.3.** Suppose $Y \subset [0, \infty)$ and $\mu$ is a non-negative measure on $Y$ with infinite support. Then the moment matrix $H_\mu$ is totally positive (of all orders).

We now show a result that will be used to provide another proof of the preceding corollary.

**Theorem 6.4** (Andréief’s identity, 1883). Suppose $Y \subset \mathbb{R}$ is a bounded interval, $n > 0$ is an integer, and $f_1, f_2, \ldots, f_n; g_1, g_2, \ldots, g_n : Y \to \mathbb{R}$ are integrable functions with respect to a positive measure $\mu$ on $Y$. Define $y := (y_1, \ldots, y_n)$, and

$$K(y) := (f_i(y_j))_{i,j=1}^n, \quad L(y) := (g_k(y_j))_{j,k=1}^n, \quad M' := \left(\int_Y f_i(y) g_k(y) \, d\mu(y)\right)_{i,j=1}^n.$$ 

Then,

$$\det M' = \frac{1}{n!} \int_{Y^n} \cdots \int_{Y^n} \det(K(y)) \det(L(y)) \prod_{j=1}^n d\mu(y_j).$$

(6.5)
Proof. We compute, beginning with the right-hand side:

\[
\int \cdots \int_{Y^n} \det(K(y)) \det(L(y)) \prod_{j=1}^n d\mu(y_j)
\]

\[
= \sum_{\sigma,\tau \in S_n} \text{sgn}(\sigma) \text{sgn}(\tau) \prod_{j=1}^n \int_Y f_{\sigma(j)}(y_j) g_{\tau(j)}(y_j) \, d\mu(y_j).
\]

Let \( \beta = \sigma^{-1} \tau \). Then a change of variables shows that this expression equals

\[
= \sum_{\tau,\beta \in S_n} \text{sgn}(\beta) \prod_{j=1}^n \int_Y g_{\tau(j)}(y) f_{\beta\tau(j)}(y) \, d\mu(y)
\]

\[
= \sum_{\tau \in S_n} \det \left( \int_Y f_{\tau(i)}(y) g_{\tau(k)}(y) \, d\mu(y) \right)_{i,k=1}^n
\]

\[
= n! \det M'.
\]

As a special case, let \( u : Y \to \mathbb{R} \) be positive and strictly increasing, and set \( f_i(y) = u(y)^{n_i}, g_k(y) = u(y)^{m_k} \) for all \( 1 \leq i, k \leq n \) and increasing sequences of integers \( n_1 < n_2 < \cdots \) and \( m_1 < m_2 < \cdots \). Then the matrix \( M' \) has \((i, k)\) entry \( \int_Y u(y)^{n_i+m_k} \, d\mu(y) \). Now using Andréief’s identity – and the analysis from earlier in this section – we obtain a second proof of Proposition 6.2

In particular, specializing to \( u(y) = y \) and \( Y \subset [0, \infty) \) reproves the total positivity of moment matrices \( H_\mu \) for measures \( \mu \geq 0 \) with infinite support in \( Y \). In this case we have \( n_j = m_j = j - 1 \) for \( j = 1, \ldots, n \). The advantage of this proof (over using the generalized Cauchy–Binet formula) is that we can compute \( \det M' \) ‘explicitly’ using Andréief’s identity:

\[
M' = (s_{i+k-2}(\mu))_{i,k=1}^n, \quad s_{i+k-2}(\mu) = \int_Y y^{i-1} y^{k-1} \, d\mu(y),
\]

\[
\det M' = \frac{1}{n!} \int_{Y^n} \prod_{1 \leq r<s \leq n} (y_s - y_r)^2 \, d\mu(y_1) d\mu(y_2) \cdots d\mu(y_n).
\]

(6.6)

(This uses the Vandermonde determinant identity \( \det K(y) = \det L(y) = \prod_{1 \leq r<s \leq n} (y_s - y_r) \).

6.2. Density of \( TP \) matrices in \( TN \) matrices. We will now prove an important density result due to Whitney in J. d’Analyse Math. (1952). Standard/well-known examples of such results are:

(1) Every square real matrix can be approximated by non-singular real matrices.

(2) Symmetric non-singular real matrices are dense in symmetric real matrices.

(3) \( n \times n \) positive definite matrices are dense in \( \mathbb{P}_n \).

The goal of this section is to prove the following

Theorem 6.7 (Whitney density). Given positive integers \( m, n \geq p \), the set of \( TP_p \) \( m \times n \) matrices is dense in the set of \( TN_p \) \( m \times n \) matrices.

In order to prove this theorem, we first prove a lemma by Pólya.

Lemma 6.8 (Pólya). For all \( \sigma > 0 \), the Gaussian kernel \( T_{G_\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) given by

\[
T_{G_\sigma}(x, y) := e^{-\sigma(x-y)^2}
\]
is totally positive.

Note that the function \( f(x) = e^{-\sigma x^2} \) is such that the TP kernel \( T_{G_\sigma}(x,y) \) can be rewritten as \( f(x-y) \). Such (integrable) functions on the real line are known as Pólya frequency (PF) functions, and we will study these functions – and their preservers – in detail in Sections 35 and 36.

**Proof.** Given real numbers \( x_1 < x_2 < \cdots < x_n \) and \( y_1 < y_2 < \cdots < y_n \), we have:

\[
(T_{G_\sigma}(x_j, y_k))_{j,k=1}^n = (e^{-\sigma x_j y_k} e^{-\sigma y_j^2})_{j,k=1}^n = \text{diag}(e^{-\sigma x_j y_1} \cdots e^{2\sigma x_j y_n}) \text{diag}(e^{-\sigma y_1^2} \cdots e^{-\sigma y_n^2})_{k=1}^n,
\]

and this has positive determinant by the previous section (see Example 5.11). □

In a similar vein, we have:

**Lemma 6.9.** For all \( \sigma > 0 \), the kernel \( H'_\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), given by

\[ H'_\sigma(x,y) := e^{\sigma(x+y)^2} \]

is totally positive. In particular, the kernels \( T_{G_\sigma} \) (from Lemma 6.8) and \( H'_\sigma \) provide examples of TP Toeplitz and Hankel matrices, respectively.

**Proof.** The proof of the total positivity of \( H'_\sigma \) is similar to that of \( T_{G_\sigma} \) above, and hence left as an exercise. To obtain TP Toeplitz and Hankel matrices, akin to Example 3.21 we choose any arithmetic progression \( x_1, \ldots, x_n \) of finite length, and consider the matrices with \( (j,k) \)th entry \( T_{G_\sigma}(x_j, x_k) \) and \( H'_\sigma(x_j, x_k) \), respectively. □

Now we come to the main proof of this section.

**Proof of Theorem 6.7.** Let \( A_{m \times n} \) be \( TN_p \) of rank \( r \). Define for each integer \( m > 0 \) the matrix

\[
(B_{\sigma,m})_{m \times m} = (e^{-\sigma(j-k)^2})_{j,k=1}^m,
A(\sigma) = G_{\sigma,m} A G_{\sigma,n}.
\]

(6.10)

Note that \( G_{\sigma,m} \) is TP by Lemma 6.8 and \( G_{\sigma,m} \to \text{Id}_{m \times m} \) as \( \sigma \to \infty \). Now as the product of totally non-negative matrices is totally non-negative (Proposition 5.4), and \( G_{\sigma,m} \) is non-singular for all \( m \), we have that \( A(\sigma) \) is \( TN_p \) of rank \( r \).

**Claim 6.11.** \( A(\sigma) \) is \( TP_{\min(r,p)} \).

**Proof.** For any \( s \leq \min(r,p) \), let \( J \subset [m] \), \( K \subset [n] \) of size \( s \). Using the Cauchy–Binet Formula, we compute:

\[
\det A(\sigma)_{J \times K} = \sum_{L,M \subset [\min(r,p)] \atop |L|=|M|=s} \det(G_{\sigma,m})_{J \times L} \det A_{L \times M} \det(G_{\sigma,n})_{M \times K}.
\]

Now note that all \( 1 \leq k \leq \min(r,p) \), at least one \( k \times k \) minor of \( A \) is positive, and all other \( k \times k \) minors are non-negative. Combined with the total positivity of \( G_{\sigma,m} \) and \( G_{\sigma,n} \), this shows that \( \det A(\sigma)_{J \times K} > 0 \). This concludes the proof. □
Returning to the proof of the theorem, if \( r \geq p \) then the \( TP \) matrices \( A(\sigma) \) approximate \( A \) as \( \sigma \to \infty \); thus the proof is complete.

For the remainder of the proof, assume that \( A \) and \( A(\sigma) \) both have rank \( r < p \). Define
\[
A^{(1)} := A(\sigma) + \frac{1}{\sigma} E_{11},
\]
where \( E_{11} \) is the elementary \( m \times n \) matrix with \((1,1)\) entry 1, and all other entries 0.

**Claim 6.12.** \( A^{(1)} \) is \( TN_p \) of rank \( r+1 \).

**Proof.** Fix an integer \( 1 \leq s \leq p \) and subsets \( J \subset [m], K \subset [n] \) of size \( s \). Now consider the \( s \times s \) minor \( A^{(1)}_{j \times K} \). If \( 1 \notin J \) or \( 1 \notin K \), then we have: \( \det A^{(1)}_{j \times K} = \det A(\sigma)_{j \times K} \geq 0 \), whereas if \( 1 \in J \cap K \), then expanding along the first row or column shows that \( A^{(1)} \) is \( TN_p \): \( \det A^{(1)}_{j \times K} = \det A(\sigma)_{j \times K} + \frac{1}{\sigma} \det (A(\sigma))_{j \{1\} \times K \{1\}} \geq 0 \).

As \( A, A(\sigma) \) have rank \( r \), and we are changing only one entry, all the \((r+2) \times (r+2)\) minors of \( A^{(1)} \) have determinant 0. Now an easy computation yields:
\[
\det A^{(1)} = \det A(\sigma) + \frac{1}{\sigma} \det (A(\sigma))_{\{1\} \times \{1\}} > 0,
\]
where the last inequality occurs because \( A(\sigma) \) is \( TP_p \) (shown above) and \( \det A(\sigma)_{\{1\} \times \{1\}} = 0 \). Thus the rank of \( A^{(1)} \) is \( r+1 \). \( \square \)

Returning to the proof: note that \( A^{(1)} \) also converges to \( A \) as \( \sigma \to \infty \). Inductively repeating this procedure, after \((p-r)\) iterations we obtain a matrix \( A^{(p-r)} \), via the procedure
\[
A^{(k)}(\sigma) := G_{\sigma,m} A^{(k)}(\sigma) G_{\sigma,n}, \quad A^{(k+1)} := A^{(k)}(\sigma) + \frac{1}{\sigma} E_{11}. \tag{6.13}
\]
Moreover, \( A^{(p-r)} \) is a \( TN_p \) matrix with rank \( p \). As \( \min(r,p) = p \) for this matrix, it follows that \( A^{(p-r)}(\sigma) \) is \( TP_p \) with \( A^{(p-r)}(\sigma) \to A^{(p-r-1)}(\sigma) \to \cdots \to A \) as \( \sigma \to \infty \). Thus, \( A \) can be approximated by \( TP_p \) matrices, and the proof is complete. \( \square \)

We now make some observations that further Whitney’s theorem. First, this density phenomenon also holds upon restricting to symmetric matrices:

**Proposition 6.14.** Given positive integers \( n \geq p \), the set of symmetric \( TP_p \) \( n \times n \) matrices is dense in the set of symmetric \( TN_p \) \( n \times n \) matrices.

**Proof.** The proof of Theorem 6.7 goes through verbatim; at each step, the resulting matrix is symmetric. \( \square \)

Second, a careful analysis of the above proof further shows that
\[
A(\sigma)_{jk} = \sum_{l,m} e^{-\sigma(j-l)^2} a_{lm} e^{-\sigma(m-k)^2} \geq a_{jk}.
\]
Thus, given \( A_{m \times n} \) that is \( TN_p \) (possibly symmetric) and \( \epsilon > 0 \), there exists \( B_{m \times n} \) that is \( TP_p \) (possibly symmetric) such that \( 0 \leq b_{jk} - a_{jk} < \epsilon \) for all \( j, k \).

**Remark 6.15.** In fact we can further refine this: by working with \( B(\sigma) \) for such \( B \), we can insist on \( 0 < b_{jk} - a_{jk} < \epsilon \). From this it follows that given a (possibly symmetric) \( TN_p \) matrix \( A_{m \times n} \), there exists a sequence \( B_l \) of \( m \times n \) matrices, all of them \( TP_p \) (and symmetric if \( A \) is), such that \( B_l \to A \) entrywise as \( l \to \infty \), and moreover for all \( j, k \),
\[
(B_1)_{jk} > (B_2)_{jk} > \cdots > (B_l)_{jk} > \cdots > a_{jk}.
\]
7. (Non-)Symmetric TP completion problems.

The main question in matrix completion problems is as follows. Given a partially filled matrix (that is, a partial matrix), do there exist choices for the ‘missing’ entries such that the resulting matrix has specified properties?

For example: can \( \begin{pmatrix} 1 & 0 &? \\ 2 &? &? \\ ? &? &? \end{pmatrix} \) be completed to a Toeplitz matrix? Yes: \( \begin{pmatrix} 1 & 0 & a \\ 2 & 1 & 0 \\ b & 2 & 1 \end{pmatrix} \), for arbitrary \( a, b \). Similarly, the above partial matrix can be completed to a non-singular, singular, or totally non-negative matrix. However, it cannot be completed to a Hankel or a symmetric Toeplitz matrix, nor to a positive (semi)definite or totally positive matrix. These are examples of some matrix completion problems. Similarly, one can ask if matrices with specified entries extend to kernels on more general domains.

In this section, we discuss three TP completion problems. The first is to understand which \( 2 \times 2 \) matrices can ‘embed in’ (or ‘extend to’) TP matrices – or even TP kernels:

**Theorem 7.1** \((2 \times 2 TP kernel completions)\). Suppose \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2} \). The following are equivalent:

1. For any \( m, n \geq 2 \) and specified pairs of rows and columns \( J \subset [m], K \subset [n] \) respectively, the matrix \( A \) can be completed to a multiple of a generalized Vandermonde (and hence TP) matrix \( \tilde{A}_{m \times n} \) such that \( (\tilde{A})_{J \times K} = A \).
2. For any totally ordered sets \( X, Y \) with sizes \( |X|, |Y| \geq 2 \) which admit a TP kernel on \( X \times Y \), and pairs of indices \( x_1 < x_2 \) in \( X \) and \( y_1 < y_2 \) in \( Y \), the matrix \( A \) can be completed to a TP kernel \( K \) on \( X \times Y \), such that \( K(\{x_1, x_2\}; \{y_1, y_2\}) = A \).
3. \( A \) is TP.

We will also show in Theorem 7.4 below, a ‘symmetric’ variant of this equivalence. To show these results, we first need to understand for which totally ordered sets \( X, Y \) do there exist TP kernels on \( X \times Y \). Note, this is not possible for all \( X, Y \). For instance, if \( |Y| > |\mathbb{R}| \) (e.g., \( Y \) is the power set of \( \mathbb{R} \)) and \( |X| \geq 2 \), then fix \( x_1 < x_2 \) in \( X \). Now any real kernel on \( X \times Y \) cannot be TP or even TP2, since when restricted to \( \{x_1, x_2\} \times Y \), it contains two equal columns by the pigeonhole principle as \( |Y| > |\mathbb{R}^2| \).

At the same time, TP1 kernels can exist on \( X \times Y \) for any totally ordered sets \( X, Y \) – e.g., the constant kernel \( 1_{X \times Y} \).

Thus, we begin by classifying all domains on which TP2 kernels exist. Interestingly, they always embed in the positive semi-axis:

**Lemma 7.2.** Given non-empty totally ordered sets \( X, Y \), the following are equivalent:

1. There exists a TP kernel on \( X \times Y \).
2. There exists a TP2 kernel on \( X \times Y \).
3. At least one of \( X, Y \) is a singleton, or there exist order-preserving maps from \( X, Y \) into \( (0, \infty) \).

The same equivalence holds if \( Y = X \) and \( K \) is symmetric: \( K(x, y) = K(y, x) \) for all \( x, y \in X \).

**Proof.** We prove a chain of cyclic implications. Clearly (1) implies (2). Now suppose (3) holds. If \( X \) or \( Y \) is a singleton then the kernel \( K \equiv 1_{X \times Y} \) proves (1). Otherwise, we identify \( X, Y \) with subsets of \( (0, \infty) \) as given; now (1) follows by considering the kernels \( K_{\pm}(x, y) := \exp \pm(x \pm y)^2 \) as in Lemmas 6.8 and 6.9.
Finally, we assume (2) and show (3). Suppose \(|X|, |Y| \geq 2\). Fix \(x_1 < x_2\) in \(X\); since \(K\) is TP2, it is an easy exercise to show that the ‘ratio function’

\[
\psi(y) := K(x_1, y)/K(x_2, y), \quad y \in Y
\]
is a strictly increasing function of \(y\). This yields the desired order-preserving injection \(\psi : Y \rightarrow (0, \infty)\); the same argument works for \(X\), implying (3).

This proof applies verbatim if \(Y = X\) and we consider symmetric kernels \(K\). □

**Proof of Theorem 7.2.** Clearly, (1) or (2) both imply (3); and (1) is a special case of (2), so it suffices to show (3) \(\Rightarrow\) (2). First use Lemma 7.2 to identify \(X, Y\) with subsets of \((0, \infty)\).

Now we will work with the TP matrix \(\begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix}\) for some scalar \(\beta > 0\). We claim there exists \(\beta > 0\) such that this matrix is of the form \((x_j^{\alpha_k})_{j,k=1}^2 = (\exp(\alpha_k \cdot \log x_j))^2_{j,k=1}\) – i.e. a generalized Vandermonde matrix – where either \(x_1 < x_2, \alpha_1 < \alpha_2\) or \(x_1 > x_2, \alpha_1 > \alpha_2\). But the latter case reduces to the former, by using \(1/x_j\) and \(-\alpha_k\) instead.

Thus, if the claim holds, then we may suppose \(A\) embeds in \(\psi\) and can be completed to the rescaled Vandermonde kernel \(K(x, y) = \beta^{-1}e^{xy}\) (noting that \(xy = e^{y \log x}\)). Now consider the two unique (increasing) linear maps \(\varphi_X, \varphi_Y : \mathbb{R} \rightarrow \mathbb{R}\), which change the ‘position’ of the chosen rows and columns to the specified positions, either to draw from \(K\) an \(m \times n\) TP matrix as in (1), or a TP kernel \(K\) on \(X \times Y\) as in (2): \(K(x, y) := \beta^{-1}\exp(\varphi_X(x)\varphi_Y(y))\).

Thus, it remains to show the above claim. For this, we repeatedly appeal to the total positivity of generalized Vandermonde matrices \((x_j^{\alpha_k})\) with \(x_j\) and \(\alpha_k\) either both increasing or both decreasing. See Theorem 5.1 and Remark 5.3.

**Case 1:** Suppose three of the four entries \(a, b, c, d\) are equal (note that all four cannot be equal). Then up to rescaling, the possible matrices are

\[
A_1 = \begin{pmatrix} \lambda & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ \mu & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & \mu \\ 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 \\ 1 & \lambda \end{pmatrix},
\]

where \(\lambda > 1 > \mu > 0\). Now \(A_1, \ldots, A_4\) are of the form \((x_j^{\alpha_k})\) with (respectively)

\[
(x_1, x_2, \alpha_1, \alpha_2) = (\lambda, 1, 1, 0), \quad (1, \mu, 1, 0), \quad (\mu, 1, 0, 1), \quad (1, \lambda, 0, 1).
\]

For \(A_1, A_2\), choosing any \(x_2 > x_3 > \cdots > x_m > 0\) and \(0 > \alpha_3 > \cdots > \alpha_n\), we are done. The other two cases are treated similarly.

**Case 2:** Suppose two entries in a row or column are equal (but three entries are not). Up to rescaling, the possible matrices are

\[
A_1' = \begin{pmatrix} 1 & 1 \\ \gamma & \delta \end{pmatrix}, \quad A_2' = \begin{pmatrix} \delta & \gamma \\ 1 & 1 \end{pmatrix}, \quad A_3' = \begin{pmatrix} \delta & 1 \\ \gamma & 1 \end{pmatrix}, \quad A_4' = \begin{pmatrix} 1 & \gamma \\ 1 & \delta \end{pmatrix},
\]

where \(0 < \gamma < \delta\), and \(\gamma, \delta \neq 1\). Now \(A_1', \ldots, A_4'\) are generalized Vandermonde matrices \((x_j^{\alpha_k})\) with (respectively)

\[
(x_1, x_2, \alpha_1, \alpha_2) = (1, e, \log \gamma, \log \delta), \quad (e, 1, \log \delta, \log \gamma), \quad (\delta, \gamma, 1, 0), \quad (\gamma, \delta, 0, 1).
\]

The result follows as in the previous case.

**Case 3:** In all remaining cases, \(\{a, d\}\) is disjoint from \(\{b, c\}\). Set \(\alpha_1 = 1\), and claim that there exist scalars \(\beta, x_1, x_2 > 0\) and \(\alpha_2 \in \mathbb{R}\) such that

\[
\beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x_1 & x_1^{\alpha_2} \\ x_2 & x_2^{\alpha_2} \end{pmatrix}.
\]
7. (Non-)Symmetric TP completion problems.

To see why, denote \( L := \log(\beta) \) for \( \beta > 0 \), as well as \( A = \log(a) \), \( B = \log(b) \) etc. Now applying \( \log \) entrywise to both sides of (7.3), we have
\[
\begin{pmatrix}
L + A & L + B \\
L + C & L + D
\end{pmatrix}
= 
\begin{pmatrix}
\log x_1 & \alpha_2 \log x_1 \\
\log x_2 & \alpha_2 \log x_2
\end{pmatrix},
\]
Taking determinants, we obtain:
\[
(L + A)(L + D) - (L + B)(L + C) = 0 \implies L = \frac{BC - AD}{(A + D) - (B + C)},
\]
where \( A + D > B + C \) since \( ad > bc \). Now check that \( x_1 = e^{L}a, x_2 = e^{L}c, \alpha_2 = \frac{L + B}{L + D} \) satisfies the conditions in (7.3). (Note here that by the assumptions on \( a, b, c, d \), the sum \( L + A \) is nonzero, as are \( L + B, L + C, L + D \) also.) This shows the claim.

To complete the proof, we need to check that \( \beta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \) is a generalized Vandermonde matrix. Since \( x_1 \neq x_2 \) by choice of \( a, c \), there are two cases. If \( x_1 < x_2 \), then \( a < c \), so
\[
(x_1/x_2)^{\alpha_2} = b/d < a/c = x_1/x_2 < 1 \implies \alpha_2 > 1.
\]
Hence we indeed get a generalized Vandermonde matrix \( (x_j^{\alpha_k})_{2 \times 2} \) with increasing \( x_j \) and increasing \( \alpha_k \). The case when \( x_1 > x_2 \) is similarly verified.

The second TP completion result embeds symmetric \( 2 \times 2 \) TP matrices into symmetric TP matrices or kernels, again in ‘any position’. This result, like Theorem 7.1, is used in a later part of this text to classify total-positivity preservers of kernels on general domains.

**Theorem 7.4** (Symmetric \( 2 \times 2 \) kernel completions). Suppose \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbb{R}^{2 \times 2} \). The following are equivalent:

1. For any \( n \geq 2 \) and specified pair \( J \subset [n] \) of rows and columns, the matrix \( A \) can be completed to a symmetric (in fact Hankel) TP matrix \( \bar{A}_{n \times n} \) such that \( (\bar{A})_{J \times J} = A \).
2. For any totally ordered set \( X \) of size at least \( 2 \) which admits a TP kernel on \( X \times X \), and any pair of indices \( (x_1 < x_2) \) in \( X \), the matrix \( A \) can be completed to a symmetric TP kernel \( K \) on \( X \times X \), such that \( K([x_1, x_2); (x_1, x_2)] = A \).
3. \( A \) is TP.

As a special case, consider the assertion (3) \( \implies \) (1), where we want to show that \( A \) embeds in the leading principal positions. It suffices to embed the matrix \( \begin{pmatrix} 1 & b \\ b & c \end{pmatrix} \), where \( 0 < b < \sqrt{c} \), inside the square matrix
\[
\frac{1}{K} V^T V = \frac{1}{K} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & x_1 & x_2 & \cdots & x_K \\
1 & \cdots & \cdots & \cdots & \cdots \\
1 & x_1^{n-1} & x_2^{n-1} & \cdots & x_K^{n-1}
\end{pmatrix} 
\begin{pmatrix}
1 & x_1 & \cdots & x_1^{n-1} \\
1 & x_2 & \cdots & x_2^{n-1} \\
\cdots & \cdots & \cdots & \cdots \\
1 & x_K & \cdots & x_K^{n-1}
\end{pmatrix},
\]
where \( K \geq n \) and \( V_{K \times n} \) is (part of) a ‘usual’ Vandermonde matrix \( (x_j^{k-1}) \). In terms of probability, this amounts to finding a uniform random variable, supported on \( \{x_1, \ldots, x_K\} \), with mean \( b \) and variance \( c - b^2 \). (See e.g. [[1]].) That is, a discrete inverse moment problem.

In fact this motivates the proof-strategy, even for the stronger result (3) \( \implies \) (2), which should similarly involve continuous distributions. Thus, the proof involves continuous random variables, and uses the Generalized Cauchy–Binet theorem (one can also use Andréief’s identity).
Proof. Clearly (1) or (2) implies (3), and (1) is a special case of (2), so it suffices to show
(3) \(\implies\) (2). The proof is similar to that of Theorem 7.1 except that we first embed \(X\) inside \(\mathbb{R}\) via Lemma 7.2. Now it suffices to embed any \(A\) as above in a continuous Hankel (whence symmetric) \(TP\) kernel \(K : \mathbb{R} \times \mathbb{R} \to (0, \infty)\), and then use an increasing linear map \(\varphi_X : X \to \mathbb{R}\) to ‘change locations’. To construct \(K\), we use Proposition 6.2. Thus, we need an increasing function \(u : \mathbb{R} \to (0, \infty)\), scalars \(s < t \in \mathbb{R}\), and a positive measure \(\mu\) on \(\mathbb{R}\) such that
\[
\int_{\mathbb{R}} u(y)^{2s} \, d\mu(y) = a, \quad \int_{\mathbb{R}} u(y)^{2t} \, d\mu(y) = b, \quad \int_{\mathbb{R}} u(y)^{2s} \, d\mu(y) = c.
\]
A solution is as follows, verified via direct computations. Define \(B := \log(b/a)\), \(C := \log(c/a)\) and note that \((C/2) - B = \frac{1}{2} \log(ac/b^2) > 0\). Now verify that the following works:
\[
s := \frac{4B - C}{8\sqrt{(C/2) - B}}, \quad t := \frac{3C - 4B}{8\sqrt{(C/2) - B}}, \quad u(y) := e^{2y}, \quad \mu(y) := \frac{a}{\sqrt{\pi}e^{4s^2} e^{-y^2}} \, dy. \quad \Box
\]

The third \(TP\) completion problem in this section extends Theorem 7.1 differently, to completions of matrices of arbitrary sizes that are totally positive of arbitrary order:

**Theorem 7.5.** Suppose \(m, n \geq 1\) and \(1 \leq p \leq \min(m, n)\) are integers, and \(J \subset [m]\), \(K \subset [n]\) are ‘sub-intervals’ containing \(m', n'\) consecutive integers, respectively. A real \(m' \times n'\) matrix \(A'\) can be completed to a \(TP\) real \(m \times n\) matrix, in positions \(J \times K\), if and only if \(A'\) is \(TP\).

Proof. One implication is obvious. Conversely, suppose \(A'_{m' \times n'}\) is \(TP\). It suffices to show that one can add an extra row either above or below \(A'\) and obtain a \(TP\) matrix. Then the result follows by induction and taking transposes.

We first show how to add a row \((a_1, \ldots, a_n)\) at the bottom. Choose any \(a_1 > 0\); having defined \(a_1, \ldots, a_k > 0\) for some \(0 < k < n'\), we inductively define \(a_{k+1}\) as follows. Define the \((m' + 1) \times (k + 1)\) matrix \(B_k := \begin{pmatrix} A'_{m' \times [k+1]} \\ a_1 \cdots a_{k+1} \end{pmatrix}\) (with unknown \(a_{k+1} > 0\)), and consider the \(1 \times 1, \ldots, p \times p\) submatrices \(B'\) of \(B_k\) which contain entries from the last row and last column, whence \(a_{k+1}\). Compute \(\det(B')\) by expanding along the last row, and from right to left. Requiring \(\det(B') > 0\) yields a strict lower bound for \(a_{k+1}\), since the cofactor corresponding to \(a_{k+1}\) is a lower-order minor of \(A'\), whence positive. Working over all such minors \(\det(B')\) yields a finite set of lower bounds, so that it is possible to define \(a_{k+1}\) and obtain all minors with ‘bottom corner’ \(a_{k+1}\) and size at most \(p \times p\) to be positive. By the induction hypothesis, all other minors of \(B_k\) of size at most \(p \times p\) are positive, so \(B_k\) is \(TP\). Proceeding inductively, we obtain the desired \((m' + 1) \times n'\) completion of \(A'\) that is \(TP\).

The argument is similar to add a row \((a_1, \ldots, a_n)\) on top of \(A'\); this time we proceed sequentially from right to left. First, \(a_n\) is arbitrary; then to define \(a_k\) (given \(a_{k+1}, \ldots, a_n\)), we require \(a_k\) to satisfy a finite set of inequalities (obtained by expanding \(\det(B')\) along the first row from left to right), and each inequality is again a strict lower bound. \(\Box\)

**Remark 7.6.** To conclude, we explain how Whitney density and the above \(TP\) completion problems are used in a later part of the text, in classifying the preservers \(F\) of \(TP\) matrices and kernels on arbitrary domains \(X \times Y\). The first step will be to deduce from Theorem 7.1 that any \(2 \times 2\) \(TP\) matrix can be embedded in a \(TP\) kernel on \(X \times Y\). This will help show that the preserver \(F\) must be continuous. We then use results akin to Whitney’s density theorem 6.7 to show that \(F\) preserves \(TN\) kernels on \(X \times Y\). These latter will turn out to be easier to classify. Similarly for the preservers of symmetric \(TP\) kernels on \(X \times X\).

We conclude this part by studying the spectra of TP/TN matrices. For real square symmetric matrices $A_{n \times n}$, recall Sylvester’s criterion [2.7] which says modulo Theorem 2.4 that such a matrix $A_{n \times n}$ has all principal minors non-negative (or positive), if and only if all eigenvalues of $A$ are non-negative (positive).

The goal in this section is to show a similar result for TP/TN matrices. More precisely, we will show the same statement as above, removing the words ‘symmetric’ and ‘principal’ from the preceding paragraph. In particular, not only are all minors of TP and TN matrices positive and non-negative respectively, but moreover, so are their eigenvalues. A slightly more involved formulation of this result is:

**Theorem 8.1.** Given positive integers $m, n \geq p$, and a real matrix $A_{m \times n}$, the following are equivalent:

1. For every square submatrix $B$ of $A$ of size $\leq p$, we have $\det(B)$ is non-negative (respectively positive). In other words, $A$ is $TN_p$ (respectively $TP_p$).

2. For every square submatrix $B$ of $A$ of size $\leq p$, the eigenvalues of $B$ are non-negative (respectively positive and simple).

Note that the analogous statement for positive semidefinite matrices clearly holds (as mentioned above), by Sylvester’s theorem.

We will follow the approach taken by Gantmacher and Krein in their 1937 paper in Compositio Math. This approach is also found in Chapter XIII.9 of F.R. Gantmacher’s book The theory of matrices; and in an expository account by A. Pinkus found in the conference proceedings Total positivity and its applications (of the 1994 Jaca meeting), edited by M. Gasca and C.A. Micchelli.

This approach relies on two well-known theorems, which are interesting in their own right. The first was shown by O. Perron in his 1907 paper in Math. Ann.:

**Theorem 8.2 (Perron).** Let $A_{n \times n}$ be a square, real matrix with all positive elements. Then $A$ has a simple, positive eigenvalue $\lambda$ with an eigenvector $u_0 \in \mathbb{R}^n$, such that:

(a) For the remaining $n - 1$ eigenvalues $\mu \in \mathbb{C}$, we have $|\mu| < \lambda$.

(b) The coordinates of $u_0$ are all nonzero and of the same sign.

This result has been studied and extended by many authors in the literature; notably, the *Perron–Frobenius theorem* is a key tool used in one of the approaches to studying discrete time Markov chains over finite state-space. As these extensions are not central to the present discussion, we do not pursue them further.

**Proof.** Write $v \geq u$ (or $v > u$) for $u, v \in \mathbb{R}^n$ to denote the (strict) coordinatewise ordering: $v_j \geq u_j$ (or $v_j > u_j$) for all $1 \leq j \leq n$. A first observation, used below, is:

$$u \leq v \text{ in } \mathbb{R}^n, u \neq v \implies Au < Av. \quad (8.3)$$

We now proceed to the proof. Define

$$\lambda := \sup\{\mu \in \mathbb{R} : Au \geq \mu u \text{ for some nonzero vector } 0 \leq u \in \mathbb{R}^n\}.$$

Now verify that

$$0 < n \min_{j,k} a_{jk} \leq \lambda \leq n \max_{j,k} a_{jk};$$

in particular, $\lambda$ is well-defined. Now for each $k \geq 1$, there exist (rescaled) vectors $u_k \geq 0$ in $\mathbb{R}^n$ whose coordinates sum to 1, and such that $Au_k \geq (\lambda - 1/k)u_k$. But then the $u_k$ belong to
a compact simplex $S$, whence there exists a subsequence converging to some vector $u_0 \in S$. It follows that $Au_0 \geq \lambda u_0$; if $Au_0 \neq \lambda u_0$, then an application of (8.3) leads to a contradiction to the maximality of $\lambda$.

Thus $Au_0 = \lambda u_0$ for nonzero $u_0 \geq 0$. But then $Au_0 > 0$, whence $u_0 = \lambda^{-1} Au_0$ has all positive coordinates. This proves part (b).

It remains to show part (a) and the simplicity of $\lambda$. First if $Av = \mu v$ for any eigenvalue $\mu$ of $A$ (and $v \neq 0$), then defining $|v| := (|v_1|, \ldots, |v_n|)^T$, we have:

$$A|v| \geq |Av| = |\mu v| = |\mu||v|.$$ 

From this, we obtain $\lambda \geq |\mu|$.

Suppose for the moment that $|\mu| = \lambda$. Then $A|v| = \lambda|v|$, else (as above) an application of (8.3) leads to a contradiction to the maximality of $\lambda$. But then $A|v| = |Av|$ from the preceding computation. By the triangle inequality over $\mathbb{C}$, this shows all coordinates of $v$ have the same argument, which we can take to be $e^{i0} = 1$ by normalizing $v$. It follows that $Av = \lambda v$ from above, since now $v = |v|$. Hence $\mu = \lambda$.

Thus we have shown that if $Av = \mu v$ for $|\mu| = \lambda$ (and $v \neq 0$), then $\mu = \lambda$ and we may rescale to get $v \geq 0$. In particular, this shows part (a) modulo the simplicity of the eigenvalue $\lambda$. Moreover, if $u_0, u'_0$ are linearly independent $\lambda$-eigenvectors for $A$, then one can come up with a linear combination $v \in \mathbb{R} u_0 + \mathbb{R} u'_0$ with at least one positive and one negative coordinate. This contradicts the previous paragraph, so it follows that $\lambda$ has geometric multiplicity one.

The final remaining task is to show that $\lambda$ is a simple eigenvalue of $A$. If not, then by the preceding paragraph there exists $u_1 \notin \mathbb{R} u_0$ such that $(Au_0 = \lambda u_0$ and) $Au_1 = \lambda u_1 + \mu u_0$ for some nonzero scalar $\mu$. Now since $A^T$ has the same eigenvalues as $A$, the above analysis there exists $v_0 \in \mathbb{R}^n$ such that $v_0^T A = \lambda v_0^T$. Hence:

$$\lambda v_0^T u_1 = v_0^T Au_1 = v_0^T (\lambda u_1 + \mu u_0).$$

But then $\mu \cdot v_0^T u_0 = 0$, which is impossible since $\mu \neq 0$ and $u_0, v_0 > 0$. This shows that $\lambda$ is simple, and concludes the proof. 

The second result we require is folklore: Kronecker’s theorem on compound matrices. We begin by introducing this family of auxiliary matrices, associated to each given matrix.

**Definition 8.4.** Fix a matrix $A_{m \times n}$ (which we take to be real, but the entries can lie in any unital commutative ring), and an integer $1 \leq r \leq \min(m, n)$.

1. Let $S_1, \ldots, S_{r(m)}$ denote the $r$-element subsets of $[m] = \{1, \ldots, m\}$, ordered lexicographically. (Thus $S_1 = \{1, \ldots, r\}$ and $S_{r(m)} = \{m-r+1, \ldots, m\}$.) Similarly, let $T_1, \ldots, T_{r(n)}$ denote the $r$-element subsets of $[n]$ in lexicographic order.

Now define the $r$th compound matrix of $A$ to be a matrix $C_r(A)$ of dimension $\binom{m}{r} \times \binom{n}{r}$, whose $(j, k)$th entry is the minor $\det(A_{S_j \times T_k})$.

2. For $r = 0$, define $C_0(A) := \text{Id}_{1 \times 1}$.

We now collect together some basic properties of compound matrices:

**Lemma 8.5.** Suppose $m, n \geq 1$ and $0 \leq r \leq \min(m, n)$ are integers, and $A_{m \times n}$ a matrix.

1. Then $C_1(A) = A$, and $C_r(cA) = c^r C_r(A)$ for all scalars $c$.

2. $C_r(A^T) = C_r(A)^T$.

3. $C_r(\text{Id}_{n \times n}) = \text{Id}_{\binom{n}{r} \times \binom{n}{r}}$. 


(4) The Cauchy–Binet formula essentially says:

$$C_r(AB) = C_r(A)C_r(B)$$

for matrices $A_{m \times n}$ and $B_{n \times p}$, where $p \geq 1$.

(5) As a consequence, $\det(C_r(AB)) = \det(C_r(A))\det(C_r(B))$ when $m = n = p$ (i.e., $A, B$ are square).

(6) As another consequence of the multiplicativity of $C_r$, if $A$ has rank $0 \leq r \leq \min(m, n)$, then $C_j(A)$ has rank $\binom{n}{r}$ for $j = 0, 1, r, r + 1, \ldots, \min(m, n)$.

(7) If $A$ is square, then $C_n(A) = \det(A)$; if $A$ is moreover invertible, then $C_r(A)^{-1} = C_r(A^{-1})$.

(8) If $A$ is upper/lower triangular, diagonal, symmetric, orthogonal, or normal, then $C_r(A)$ has the same property.

**Proof.** We only sketch a couple of the proofs, and leave the others as exercises. If $A$ has rank $r$, then one can write $A = M_{m \times r}N_{r \times n}$, where the columns of $M$ are linearly independent, as are the rows of $N$. But then $C_r(A)$ is the product of a nonzero column vector $C_r(M)$ and a nonzero row vector $C_r(N)$, hence has rank 1. (Here we require the underlying ground ring to be an integral domain.)

The other case we consider here is when $A$ is $(n \times n)$ and upper triangular. In this case let $J = \{j_1 < \cdots < j_r\}$ and $K = \{k_1 < \cdots < k_r\}$ be subsets of $[n]$, with $J > K$ in the lexicographic order. Hence there exists a unique $l \in [1, r]$ such that

$$j_1 = k_1, \ldots, j_{l-1} = k_{l-1}, \; k_l < j_l < j_{l+1} < \cdots < j_r.$$ 

It follows that $A_{J \times K}$ is a block triangular matrix of the form

$$\begin{pmatrix} C_{(l-1)\times(l-1)} & D \\ 0 & E_{(r-l+1)\times(r-l+1)} \end{pmatrix},$$

and that the leftmost column of $E$ is the zero vector. Hence $\det(A_{J\times K}) = 0$ if $J > K$. \hfill $\square$

With Lemma 8.5 in hand, one can state and prove

**Theorem 8.6** (Kronecker). Let $n \geq 1$ and suppose the complex matrix $A_{n \times n}$ has the multiset of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. For all $0 \leq r \leq n$, the $\binom{n}{r}$ eigenvalues of $C_r(A)$ are precisely of the form $\prod_{j \in S} \lambda_j$, where $S$ runs over all $r$-element subsets of $[n]$.

In words, the eigenvalues of $C_r(A)$ are precisely the $\binom{n}{r}$ products of $r$ distinct eigenvalues of $A$.

**Proof.** Let $J$ denote an (upper triangular) Jordan canonical form of $A$. That is, there exists an invertible matrix $M$ satisfying $MJM^{-1} = A$, with the diagonal entries of $J$ given by $\lambda_1, \ldots, \lambda_n$. Applying various parts of Lemma 8.5

$$C_r(A) = C_r(M) \; C_r(J) \; C_r(M)^{-1},$$

with $C_r(J)$ upper triangular. Thus the eigenvalues of $C_r(A)$ are precisely the diagonal entries of $C_r(J)$, and these are precisely the claimed set of scalars. \hfill $\square$

Finally, these ingredients help show that $TP$ square matrices have simple, positive eigenvalues.

**Proof of Theorem 8.7.** Clearly, (2) implies (1). Conversely, first note that by focussing on a fixed square submatrix $B$ and all of its minors, the implication $(1) \implies (2)$ for general $m, n \geq p$ reduces to the special case $m = p = n$, which we assume henceforth.
We first show the TP case; thus say $A_{n \times n}$ is TP. Relabel its eigenvalues $\lambda_1, \ldots, \lambda_n$ such that

$$|\lambda_1| \geq \cdots \geq |\lambda_n|.$$ 

Now let $1 \leq r \leq n$; then the compound matrix $C_r(A)$ has positive entries, so by Perron’s theorem [8.2], there exists a unique largest positive eigenvalue $\lambda_{\text{max},r}$, and all others are smaller in modulus. Hence by Kronecker’s theorem [8.6], $\lambda_{\text{max},r} = \lambda_1 \cdots \lambda_r$, and we have

$$\lambda_1 \cdots \lambda_r > 0, \quad \forall 1 \leq r \leq n.$$ 

It follows that each $\lambda_j$ is (real and) positive. Moreover, from Perron and Kronecker’s results it also follows for each $1 \leq r \leq n-1$ that

$$\lambda_1 \cdots \lambda_r > \lambda_1 \cdots \lambda_{r-1} \lambda_{r+1},$$

and so $\lambda_r > \lambda_{r+1}$, as desired.

This shows the result for TP matrices. Now suppose $A_{n \times n}$ is TN. By Whitney’s density theorem [6.7], we may approximate $A$ by a sequence $A_k$ of TP matrices. Hence the characteristic polynomials converge: $p_{A_k}(t) := \det(t \operatorname{Id}_{n \times n} - A_k) \to p_A(t)$ coefficientwise, as $k \to \infty$. Since $\deg(p_{A_k}) = n$ for all $k \geq 1$, it follows by the ‘continuity of roots’ – proved below – that the eigenvalues of $p_A$ also avoid the open set $C \setminus [0, \infty)$. This concludes the proof. \hfill \Box

Thus, it remains to show that the roots of a real or complex polynomial are continuous functions of its coefficients. This is in fact a consequence of Hurwitz’s theorem in complex analysis, but we restrict ourselves here to mentioning a simpler result. The following argument can be found online or in books.

**Proposition 8.7.** Suppose $p_k \in \mathbb{C}[t]$ is a sequence of polynomials, with $\deg(p_k)$ uniformly bounded over all $k \geq 1$. If $U \subset \mathbb{C}$ is an open set on which no $p_k$ vanishes, and $p_k(t) \to p(t)$ coefficientwise, then either $p \equiv 0$ on $U$, or $p$ is nonvanishing on $U$.

**Proof.** We restrict ourselves to outlining this argument, as this direction is not our main focus. Suppose $p|_U$ is not identically zero, and $p(w) = 0$ for some $w \in U$. Choose $\delta > 0$ such that the closed disc $\overline{D} := \overline{D}(w, \delta) \subset U$ and $p(t)$ has no roots in $\overline{D} \setminus \{w\}$. Then each $p_k$ is uniformly continuous on the compact boundary $\partial D = \overline{D} \setminus D$, where $D = D(w, \delta)$ is the open disc. For sufficiently large $k$, $\deg(p_k) = \deg(p) \geq 0$ by the hypotheses. This is used to show that the $p_k$ converge uniformly on $\partial D$ to $p$, and similarly, $p_k' \to p'$ uniformly on $\partial D$.

Since $p$ is nonvanishing on $\partial D$, we have

$$m := \min_{z \in \partial D} |p(z)| > 0,$$

and hence for sufficiently large $k$, we have

$$\min_{z \in \partial D} |p_k(z)| > \frac{m}{2}, \quad \forall k \gg 0.$$ 

Using this, one shows that the sequence $\{p_k'/p_k : k > 0\}$ converges uniformly on $\partial D$ to $p'/p$.

Now integrate on $\partial D$: since $p_k'/p_k$ equals $\sum_j 1/(z - \lambda_j(p_k))$ where one sums over the multiset of roots $\lambda_j$ of each $p_k$, and since $p_k$ does not vanish in $U \supset \overline{D}$, we have

$$0 = \oint_{\partial D} \frac{p_k'(z)}{p_k(z)} \, dz \to \oint_{\partial D} \frac{p'(z)}{p(z)} \, dz.$$ 

Hence the right-hand integral vanishes. On the other hand, that same integral equals a positive integer – namely, the multiplicity of the root $w$ of $p(t)$. This yields the desired contradiction, whence $p$ does not vanish on $U$. \hfill \Box
Part 1: Bibliographic notes and references

BIBLIOGRAPHIC NOTES AND REFERENCES

Most of the material in this part is standard and can be found in other textbooks on matrix theory; see e.g. Bapat–Raghavan [13], Bhatia [33], Fallat–Johnson [81], Gantmacher [98], Horn–Johnson [139, 140], Karlin [154], Pinkus [218], Zhan [300], and Zhang [301].

About the rest: the matrix factorization in (2.32) involving Schur complements was observed by Schur in [260]. Theorems 2.31 and 2.37 on the positivity of a block-matrix in terms of Schur complements, were shown by Albert [7]. Remark 3.7 on applications of Schur products to other areas is taken from discussions in the books [139, 140]. More broadly, a discussion of the legacy of Schur’s contributions in analysis can be found in the comprehensive survey [76].

The Schur product theorem 3.10 was shown by Schur [258], and its nonzero lower bound, Theorem 3.15 is by Khare [161] (following a prior bound by Vybíral [281]). Remark 3.16 is by Vybíral [281]; and the previous nonzero lower bounds on the Schur product in (3.14) are from [87] and [232]. For more on the Hamburger and Stieltjes moment problems (see Remarks 2.22 and 4.4 respectively), see the monographs [6, 245, 263].

The notion of $TN$ and $TP$ matrices and kernels was introduced by Schoenberg in [246], where he showed that $TN$ matrices satisfy the variation diminishing property. (Schoenberg then proved in [247] the Budan–Fourier theorem 10.6 using $TN$ matrices.) The characterization in Theorem 3.20 of this property is from Motzkin’s thesis [202]. (Instances of total non-negativity and of variation diminution had appeared in earlier works, e.g. by Fekete [86], Hurwitz [143], Laguerre [176], and others.) Theorem 4.1 relating positivity and total non-negativity for a Hankel matrix, appears first in [218] for $TP$ Hankel matrices, then in detail in [82] for the $TN, TP_p, TN_p$ variants. (Neither of these works uses contiguous minors, which have the advantage of only needing to work with Hankel submatrices.) The lemmas used in the proof above are given in [98, 99], and the result of Fekete and its extension by Schoenberg are in [80] and [254], respectively. Corollary 4.3 – on the total non-negativity of moment matrices of measures on [0, ∞) – is the easy half of the Stieltjes moment problem, and an alternate proof for this half was given by Heiligers [123].

Theorem 5.1 on the total positivity of generalized Vandermonde matrices is found in [98]. The ‘weak’ Descartes’ rule of signs (Lemma 5.2) was first shown by Descartes in 1637 [72] for polynomials; the proof given in this text via Rolle’s theorem is by Laguerre in 1883 [176], and holds equally well for the extension to real powers. The Basic Composition Formula (5.14) can be found in the books by Pólya–Szegö [225] (see also Karlin [154]), while Andréief’s identity is from [10]. Whitney’s density theorem is from [293]. Theorem 7.5 is due to Johnson and Smith [150]; all other results in Section 7 on $TP$ completions of $2 \times 2$ matrices to $TP$ matrices/kernels on arbitrary domains, are from Belton–Guillot–Khare–Putinar [23]. Theorem 8.1 on the eigenvalues of $TP$ and $TN$ matrices is due to Gantmacher and Krein [99], and is also found in numerous sources – to list a few, [98, 100, 103]. (The original result was for oscillatory matrices, and immediately follows from Theorem 8.1.) Perron’s theorem 8.2 is from [216]. Hurwitz’s theorem, or the continuity of zeros shown in Proposition 8.7 can be found in in standard textbooks on complex analysis, see e.g. [65].
Part 2:
Entrywise powers preserving (total) positivity in fixed dimension
Part 2: Entrywise powers preserving (total) positivity in fixed dimension

9. ENTRYWISE POWERS PRESERVING POSITIVITY IN FIXED DIMENSION

In the rest of this text, we discuss operations that preserve the notions of positivity that have been discussed earlier. Specifically, we will study functions that preserve positive semidefiniteness, or $TP/TN$, when applied entrywise to various classes of matrices — and in a later part, we explore this more generally via composition operators on positive kernels. This part of the text discusses the important special case of entrywise powers preserving positivity on matrices; to understand some of the modern and classical motivations behind this study, we refer the reader to Sections [13.1] and [16.1] below, respectively.

We begin with some preliminary definitions.

Definition 9.1. Given a subset $I \subset \mathbb{R}$, define $\mathbb{P}_n(I) := \mathbb{P}_n \cap I^{n \times n}$ to be the set of $n \times n$ positive semidefinite matrices, all of whose entries are in $I$.

A function $f : I \to \mathbb{R}$ acts entrywise on vectors/matrices with entries in $I$ via: $A = (a_{jk}) \mapsto f[A] := (f(a_{jk}))$. We say $f$ is Loewner positive on $\mathbb{P}_n(I)$ if $f[A] \in \mathbb{P}_n$ whenever $A \in \mathbb{P}_n(I)$.

Note that the entrywise operator $f[-]$ differs from the usual holomorphic calculus (except when acting on diagonal matrices by functions that vanish at the origin).

Remark 9.2. The entrywise calculus was initiated by Schur in the same paper in *J. reine angew. Math.* (1911) where he proved the Schur product theorem. Schur defined $f[A]$ (he called it $f^0(A)$) and proved the first result involving entrywise maps; see e.g. [76, Page cxii] for additional commentary.

We fix the following notation for future use. If $f(x) = x^\alpha$ for some $\alpha \geq 0$ and $I \subset [0, \infty)$, then we write $A^\alpha$ for $f[A]$, where $A$ is any vector or matrix. By convention we shall take $0^0 = 1$ whenever required, so that $A^0$ is the matrix $1$ of all ones, and this is positive semidefinite whenever $A$ is square.

At this point, one can ask the following question: Which entrywise power functions preserve positive semidefiniteness, total positivity or total-negativity on $n \times n$ matrices? (We will also study later, the case of general functions.) The first of these questions was studied by Loewner in connection with the Bieberbach conjecture. It was eventually answered in *J. Math. Anal. Appl.* (1977) by two of his students, C.H. FitzGerald and R.A. Horn.

**Theorem 9.3** (FitzGerald–Horn). Given an integer $n \geq 2$ and a scalar $\alpha \in \mathbb{R}$, $f(x) = x^\alpha$ preserves positive semidefiniteness on $\mathbb{P}_n((0, \infty))$ if and only if $\alpha \in \mathbb{Z}^{\geq 0} \cup [n - 2, \infty)$.

**Remark 9.4.** We will in fact show that if $\alpha$ is not in this set, there exists a rank two Hankel $TN$ matrix $A_{n \times n}$ such that $A^{\alpha} \notin \mathbb{P}_n$. (In fact, it is the (partial) moment matrix of a non-negative measure on two points.) Also notice that Theorem 9.3 holds for entrywise powers applied to $\mathbb{P}_n([0, \infty))$, since as we show, $\alpha < 0$ never works while $\alpha = 0$ always does so by convention; and for $\alpha > 0$ the power $x^\alpha$ is continuous on $[0, \infty)$, and we use the density of $\mathbb{P}_n((0, \infty))$ in $\mathbb{P}_n([0, \infty))$.

The 'phase transition' at $n - 2$ in Theorem 9.3 is a remarkable and oft-repeating phenomenon in the entrywise calculus (we will see additional examples of such events in Section 14). The value $n - 2$ is called the critical exponent for the given problem of preserving positivity.

To prove Theorem 9.3, we require a preliminary lemma, also by FitzGerald and Horn. Recall the preliminaries in Section 2.4.

**Lemma 9.5.** Given a matrix $A \in \mathbb{P}_n(\mathbb{R})$ with last column $\zeta$, the matrix $A - a_{nn}^\dagger \zeta \zeta^T$ is positive semidefinite with last row and column zero.
Here, \(a_{nn}^\dagger\) denotes the Moore–Penrose inverse of the \(1 \times 1\) matrix \((a_{nn})\).

**Proof.** If \(a_{nn} = 0\) then \(\zeta = 0\) by positive semidefiniteness, and \(a_{nn}^\dagger = 0\) as well. The result follows. Now suppose \(a_{nn} > 0\), and write \(A = \begin{pmatrix} B & \omega \\ \omega^T & a_{nn} \end{pmatrix} \). Then a straightforward computation shows that

\[
A - a_{nn}^{-1}\zeta\zeta^T = \begin{pmatrix} B - \frac{\omega\omega^T}{a_{nn}} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Notice that \(B - \frac{\omega\omega^T}{a_{nn}}\) is the Schur complement of \(A\) with respect to \(a_{nn} > 0\). Now since \(A\) is positive semidefinite, so is \(B - \frac{\omega\omega^T}{a_{nn}}\) by Theorem 2.31.

**Proof of Theorem 9.3.** Notice that \(x^\alpha\) preserves positivity on \(\mathbb{P}_n((0,\infty))\) for all \(\alpha \in \mathbb{Z}^\geq 0\), by the Schur product theorem 3.10. Now we prove by induction on \(n \geq 2\) that if \(\alpha \geq n - 2\), then \(x^\alpha\) preserves positivity on \(\mathbb{P}_n((0,\infty))\). If \(n = 2\) and \(A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbb{P}_2((0,\infty))\), then \(ac \geq b^2 \implies (ac)^\alpha \geq b^{2\alpha}\) for all \(\alpha \geq 0\). It follows that \(A^{\alpha \alpha} \in \mathbb{P}_2((0,\infty))\), proving the base case.

For the induction step, assume the result holds for \(n - 1 \geq 2\). Suppose \(A \in \mathbb{P}_n((0,\infty))\) and \(\alpha \geq n - 2\). If \(a_{nn} = 0\) then by the positivity (psd) of \(A\), the entries in the last row and last column must vanish (consider the minors \(\begin{pmatrix} a_{jj} & a_{jn} \\ a_{nj} & a_{nn} \end{pmatrix}\) for \(1 \leq j < n\)). But then we are done by the induction hypothesis.

The final case is when \(a_{nn} > 0\). Now consider the following elementary definite integral:

\[
x^\alpha - y^\alpha = \alpha (x - y) \int_0^1 (\lambda x + (1 - \lambda) y)^{\alpha - 1} d\lambda.
\]

Let \(\zeta\) denote the final column of \(A\); applying (9.6) entrywise to \(x\) an entry of \(A\) and \(y\) the corresponding entry of \(B := \zeta^T a_{nn}^{-1} \zeta\) yields:

\[
A^{\alpha \alpha} - B^{\alpha \alpha} = \alpha \int_0^1 (A - B) \circ (\lambda A + (1 - \lambda) B)^{(\alpha - 1)} d\lambda.
\]

By the induction hypothesis, the leading principal \((n - 1) \times (n - 1)\) submatrix of the matrix \((\lambda A + (1 - \lambda) B)^{(\alpha - 1)}\) is positive semidefinite (even though the entire matrix need not be so).

By Lemma 9.5, \(A - B\) is positive semidefinite and has last row and column zero. It follows by the Schur product theorem that the integrand on the right is positive semidefinite. Since \(B^{\alpha \alpha}\) is a rank-one positive semidefinite matrix (this is easy to verify), it follows that \(A^{\alpha \alpha}\) is also positive. This concludes one direction of the proof.

To prove the other half, suppose \(\alpha \notin \mathbb{Z}^\geq 0 \cup [n - 2, \infty)\); now consider \(H_\mu\) where \(\mu = \delta_1 + \epsilon \delta_x\) for \(\epsilon, x > 0\), \(x \neq 1\). Note that \((H_\mu)_{jk} = s_{j+k}(\mu) = 1 + \epsilon x^{j+k}\); and as shown previously, \(H_\mu\) is positive semidefinite of rank 2.

First suppose \(\alpha < 0\). Then consider the leading principal \(2 \times 2\) minor of \(H_\mu^{\alpha \alpha}\), which equals

\[
B := \begin{pmatrix} (1 + \epsilon)^\alpha & (1 + \epsilon x)^\alpha \\ (1 + \epsilon x)^\alpha & (1 + \epsilon x^2)^\alpha \end{pmatrix}.
\]

We claim that \(\det B < 0\), which shows \(H_\mu^{\alpha \alpha}\) is not positive semidefinite. Indeed, note that

\[
(1 + \epsilon)(1 + \epsilon x^2) - (1 + \epsilon x)^2 = \epsilon(x - 1)^2 > 0,
\]
so \( \det B = (1 + \epsilon)^\alpha (1 + \epsilon x)^\alpha - (1 + \epsilon x)^{2\alpha} < 0 \) because \( \alpha < 0 \).

Next suppose that \( \alpha \in (0, n - 2) \setminus \mathbb{N} \). Given \( x > 0 \), for small \( \epsilon \) we know by the binomial theorem that
\[
(1 + \epsilon x)^\alpha = 1 + \sum_{k \geq 1} \binom{\alpha}{k} \epsilon^k x^k, \quad \text{where} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}.
\]
We will produce \( u \in \mathbb{R}^n \) such that \( u^T H_\mu^\circ \alpha u < 0 \); note this shows that \( H_\mu^\circ \alpha \not\in \mathbb{P}_n \).

Starting with the matrix \( H_\mu = 11^T + \epsilon v v^T \) where \( v = (1, x, \ldots, x^{n-1})^T \), we obtain:
\[
H_\mu^\circ \alpha = 11^T + \sum_{k=1}^{\lfloor \alpha \rfloor + 2} \epsilon^k \binom{\alpha}{k} (v^{\circ k})(v^{\circ k})^T + o(\epsilon^{\lfloor \alpha \rfloor + 2}), \tag{9.8}
\]
where \( o(\epsilon^{\lfloor \alpha \rfloor + 2}) \) is a matrix such that the quotient of any entry by \( \epsilon^{\lfloor \alpha \rfloor + 2} \) goes to zero as \( \epsilon \to 0^+ \).

Note that the first term and the sum altogether contain at most \( n \) terms. Since the corresponding vectors \( 1, v, v^{\circ 2}, \ldots, v^{\circ (\lfloor \alpha \rfloor + 2)} \) are linearly independent (by considering the – possibly partial – usual Vandermonde matrix formed by them), there exists a vector \( u \in \mathbb{R}^n \) satisfying:
\[
u T 1 = u^T v = u^T v^{\circ 2} = \cdots = u^T v^{\circ \lfloor \alpha \rfloor + 1} = 0, \quad u^T v^{\circ \lfloor \alpha \rfloor + 2} = 1.
\]
Substituting these into the above computation, we obtain:
\[
u T H_\mu^\circ \alpha u = \epsilon^{\lfloor \alpha \rfloor + 2} \binom{\alpha}{\lfloor \alpha \rfloor + 2} + u^T \cdot o(\epsilon^{\lfloor \alpha \rfloor + 2}) \cdot u.
\]
Since \( \binom{\alpha}{\lfloor \alpha \rfloor + 2} \) is negative if \( \alpha \) is not an integer, it follows that
\[
\lim_{\epsilon \to 0^+} \frac{u^T H_\mu^\circ \alpha u}{\epsilon^{\lfloor \alpha \rfloor + 2}} < 0.
\]
Hence one can choose a small \( \epsilon > 0 \) such that \( u^T H_\mu^\circ \alpha u < 0 \). It follows for this \( \epsilon \) that \( H_\mu^\circ \alpha \) is not positive semidefinite.

\[ \square \]

**Remark 9.9.** As the above proof reveals, the following are equivalent for \( n \geq 2 \) and \( \alpha \in \mathbb{R} \):

1. The entrywise map \( x^\alpha \) preserves positivity on \( \mathbb{P}_n((0, \infty)) \) (or \( \mathbb{P}_n([0, \infty)) \)).
2. \( \alpha \in \mathbb{Z}^\geq 0 \cup [n - 2, \infty) \).
3. The entrywise map \( x^\alpha \) preserves positivity on the (leading principal \( n \times n \) truncations of) Hankel moment matrices of non-negative measures supported on \( \{1, x\} \), for any fixed \( x > 0 \), \( x \neq 1 \).

The use of the Hankel moment matrix ‘counterexample’ \( 11^T + \epsilon v v^T \) for \( v = (1, x, \ldots, x^{n-1})^T \) and small \( \epsilon > 0 \) was not due to FitzGerald and Horn – who used \( v = (1, 2, \ldots, n)^T \) instead – but due to Fallat, Johnson, and Sokal. In fact, the above proof can be made to work if one uses any vector \( v \) with distinct positive real coordinates, and small enough \( \epsilon > 0 \).

As these remarks show, to isolate the entrywise powers preserving positivity on \( \mathbb{P}_n([0, \infty)) \), it suffices to consider a much smaller family – namely, the one-parameter family of truncated moment matrices of the measures \( \delta_1 + \epsilon \delta_x \) – or the one-parameter family \( 1_{n \times n} + \epsilon v v^T \), where \( v = (x_1, \ldots, x_n)^T \) for pairwise distinct \( x_j > 0 \). In fact a stronger result is true. In her 2017 paper in *Linear Algebra Appl.*, Jain was able to eliminate the dependence on \( \epsilon \):

**Theorem 9.10** (Jain). Suppose \( n > 0 \) is an integer, and \( x_1, x_2, \ldots, x_n \) are pairwise distinct positive real numbers. Let \( C := (1 + x_j x_k)^{n}_{j,k=1} \). Then \( C^\circ \alpha \) is positive semidefinite if and only if \( \alpha \in \mathbb{Z}^\geq 0 \cup [n - 2, \infty) \).
In other words, this result identifies a multiparameter family of matrices, each one of which
encodes the positivity preserving powers in the original result of FitzGerald–Horn.

We defer the proof of Theorem 9.10 (in fact, a stronger form shown by Jain in 2020) to Section 15. We then use this stronger variant to prove the corresponding result for entrywise powers preserving other Loewner properties: monotonicity (again shown by Jain in 2020), whence convexity, both with the same multiparameter family of matrices.

The last result in this section applies Theorem 9.3 to classify the entrywise powers that preserve positive definiteness:

Corollary 9.11. Given an integer \( n \geq 2 \) and a scalar \( \alpha \in \mathbb{R} \), the following are equivalent:

1. The entrywise \( \alpha \)th power preserves positive definiteness for \( n \times n \) matrices with positive entries.
2. The entrywise \( \alpha \)th power preserves positive definiteness for \( n \times n \) Hankel matrices with positive entries.
3. \( \alpha \in \mathbb{Z}^{\geq 0} \cup \{n-2, \infty\} \).

Proof. Clearly (1) \( \implies \) (2). Next, if \( \alpha \) is not in \( \mathbb{Z}^{\geq 0} \cup \{n-2, \infty\} \), then given \( 1 \neq x \in (0, \infty) \), there exists \( \epsilon > 0 \) such that \( A^{\alpha} \) has a negative principal minor, where \( A := (1 + e^{\epsilon j^2})_{j,k=0}^{n-1} \) is Hankel. Now perturb \( A \) by \( \delta H_1 \) for small enough \( \delta > 0 \), where \( H_1 := (e^{\epsilon j^2})_{j,k=0}^{n-1} \) is a Hankel ‘principal submatrix’ drawn from the kernel in Lemma 6.9. By Theorem 4.1, \( A + \delta H_1 \) is \( TP \) Hankel for all \( \delta > 0 \), whence positive definite; and for small enough \( \delta > 0 \), its \( \alpha \)th power also has a negative principal minor. This shows the contrapositive to (2) \( \implies \) (3).

Finally, suppose (3) holds. Since the Schur product is a principal submatrix of the Kronecker product, it follows that positive integer powers entrywise preserve positive definiteness. Now suppose \( \alpha \geq n-2 \) and \( A_{n \times n} \) is positive definite. Then all diagonal entries of \( A \) are positive, so there exists \( \epsilon > 0 \) such that \( A - \epsilon \text{Id} \in \mathbb{P}_n([0, \infty)) \). Now we have

\[
A^{\alpha} = (A - \epsilon \text{Id})^{\alpha} + \text{diag}(a_{jj}^{\alpha} - (a_{jj} - \epsilon)^{\alpha})_{j=1}^{n},
\]

and the first term on the right is in \( \mathbb{P}_n \) by Theorem 9.3, so \( A^{\alpha} \) is positive definite. \( \square \)

We conclude by highlighting the power and applicability of the ‘integration trick’ (9.7) of FitzGerald and Horn. First, it in fact applies to general functions, not just to powers. The following observation (by the author and Tao) will be useful below.

Theorem 9.12 (Extension Principle). Let \( 0 < \rho \leq \infty \) and \( I = (0, \rho) \) or \( (-\rho, \rho) \) or its closure. Fix an integer \( n \geq 2 \) and a continuously differentiable function \( h : (0, \rho) \to \mathbb{R} \). If \( h[\cdot] \) preserves positivity on rank-one matrices in \( \mathbb{P}_n(I) \) and \( h'[\cdot] \) preserves positivity on \( \mathbb{P}_{n-1}(I) \), then \( h[\cdot] \) preserves positivity on \( \mathbb{P}_n(I) \).

The proof is exactly as before, but now using the more general integral identity:

\[
h(x) - h(y) = \int_x^y h'(t) \, dt = \int_0^1 (x - y) h'(\lambda x + (1 - \lambda)y) \, d\lambda.
\]

Second, this integration trick is even more powerful, in that it further applies to classify the entrywise powers that preserve other properties of \( \mathbb{P}_n \), including monotonicity and super-additivity. See Section 14 for details on these properties, their power-preservers, and their further application to the distinguished sub-cones \( \mathbb{P}_G \) for non-complete graphs \( G \).
10. Entrywise powers preserving total positivity: I.

The next goal is to study which entrywise power functions preserve total positivity and total non-negativity. The present section is devoted to proving:

**Theorem 10.1.** If $A_{m \times n}$ is $TP_3$, then so is $A^t$ for all $t \geq 1$.

The proof relies on Descartes’ rule of signs (also known as Laguerre’s rule of signs – for the connection, see Section [28.3](#)). Recall that we had shown a ‘weak’ variant of this in Lemma [5.2](#). The next variant is stronger, and relies on the following notion.

**Definition 10.2.** Suppose $F : \mathbb{R} \rightarrow \mathbb{R}$ is infinitely differentiable. Given an integer $k \geq 0$, we say $F$ has a zero of order $k$ at $t_0 \in \mathbb{R}$, if $F(t_0) = F'(t_0) = \cdots = F^{(k-1)}(t_0) = 0$ and $F^{(k)}(t_0) \neq 0$. (Note that a zero of order 0 means that $F(t_0) \neq 0$.)

Descartes’ rule of signs bounds the number of real zeros of generalized Dirichlet polynomials, which are functions of the form

$$F : \mathbb{R} \rightarrow \mathbb{R}, \quad F(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t}, \quad c_j, \alpha_j \in \mathbb{R}.$$  

These functions are so named because changing variables to $x = e^t$ gives

$$f(x) = \sum_{j=1}^{n} c_j x^{\alpha_j} : (0, \infty) \rightarrow \mathbb{R},$$

which are known as generalized polynomials. Another special case of $F(t)$ is when one uses $\alpha_j = -\log(j)$, to obtain $F(t) = \sum_{j=1}^{n} c_j / j^t$; these are called Dirichlet polynomials. The generalized Dirichlet polynomials subsume both of these families of examples.

We can now state

**Theorem 10.3** (Descartes’ rule of signs). Suppose $F(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t}$ as above, with $c_j \in \mathbb{R}$ not all zero, and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ also real. Then the number of real zeros of $F$, counting multiplicities, is at most the number of sign changes, or ‘variations’, in the sequence $c_1, c_2, \ldots, c_n$ (after removing all zero terms).

For instance, the polynomial $x^6 - 8 = (x^2 - 2)(x^4 + 2x^2 + 4)$ has only one sign change, so at most one positive root – which is at $x = e^{2} = 2$.

To prove Theorem 10.3 we require a couple of preliminary lemmas.

**Lemma 10.4** (Generalized Rolle’s theorem). Given an open interval $I$ and a smooth function $F : I \rightarrow \mathbb{R}$, let $Z(F, I)$ denote the number of zeros of $F$ in $I$, counting orders. If $Z(F, I)$ is finite, then we have $Z(F', I) \geq Z(F, I) - 1$.

**Proof.** Suppose $F$ has a zero of order $k_r > 0$ at $x_r, 1 \leq r \leq n$. Then $F'$ has a zero of order $k_r - 1 \geq 0$ at $x_r$. These add up to:

$$\sum_{r=1}^{n} (k_r - 1) = Z(F, I) - n$$

We may also suppose $x_1 < x_2 < \cdots < x_n$. Now by Rolle’s theorem, $F'$ also has at least $n-1$ zeros in the intervals $(x_r, x_{r+1})$ between the points $x_r$. Together, we obtain: $Z(F', I) \geq Z(F, I) - 1$.

**Lemma 10.5.** Let $F, G : I \rightarrow \mathbb{R}$ be smooth and $G \neq 0$ on $I$. If $F$ has a zero of order $k$ at $t_0$ then so does $F \cdot G$.
Proof. This is straightforward: use Leibnitz’s rule to compute \((F \cdot G)^{(j)}(t_0)\) for \(0 \leq j \leq k\).

With these lemmas in hand, we can prove Descartes’ rule of signs.

Proof of Theorem 10.3. The proof again follows Laguerre’s argument (1883), by induction on the number \(s\) of sign changes in the sequence \(c_1, c_2, \ldots, c_n\). (Note that not all \(c_j\) are zero.)

The base case is \(s = 0\), in which case \(F(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t}\) has all nonzero coefficients of the same sign, and hence never vanishes.

For the induction step, we first assume without loss of generality that all \(c_j\) are nonzero. Suppose the last sign change occurs at \(c_k\), i.e., \(c_k c_{k-1} < 0\). Choose and fix \(\alpha \in (\alpha_k, \alpha_{k-1})\), and define \(G(t) := e^{-\alpha t}\). Then,

\[
H(t) := F(t) \cdot G(t) = \sum_{j=1}^{n} c_j e^{(\alpha_j - \alpha) t}
\]

has the same zeros (with orders) as \(F(t)\), by Lemma 10.5. Moreover,

\[
H'(t) = \sum_{j=1}^{n} c_j (\alpha_j - \alpha) e^{(\alpha_j - \alpha) t}
\]

has exactly one less sign change than \(F(t)\), namely, \(s - 1\). It follows by the induction hypothesis that \(Z(H', \mathbb{R}) \leq s - 1\). Hence by Lemma 10.4 \(Z(F, \mathbb{R}) = Z(H, \mathbb{R}) \leq 1 + Z(H', \mathbb{R}) \leq s\), and the proof is complete by induction.

We remark that there are numerous strengthenings of Descartes’ rule of signs in the literature, obtained by Budan [54], Fourier [94], Laguerre [176], Segner [262], Sturm [274], and many others – this was popular even in the 20th century, see e.g. the articles by Curtiss [71] in Ann. of Math. (1918) and by Hurwitz [144] in Math. Ann. (1920). Here we restrict ourselves to mentioning some of these variants without proofs (although we remark that their proofs are quite accessible – see for instance the 2006 survey [148] by Jameson in Math. Gazette).

As above, let \(F(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t}\) with \(\alpha_1 > \alpha_2 > \cdots > \alpha_n\).

(1) Then not only is \(Z(F, \mathbb{R}) \leq s\), but \(s - Z(F, \mathbb{R})\) is an even integer. This was shown by Budan [54] and Fourier [94], and is also attributed to Le Gua. In fact, Budan–Fourier showed a more general result, stated here for polynomials:

**Theorem 10.6** (Budan–Fourier). Suppose \(f\) is a polynomial, and \(-\infty < \alpha < \beta < \infty\). Denoting by \(V(x)\) the number of sign changes in the sequence \((f(x), f'(x), f''(x), \ldots)\),

\[
V(\alpha) - V(\beta) - Z(f, (\alpha, \beta))
\]

is a non-negative integer, which is moreover even if \(\alpha, \beta\) are not zeros of \(f\).

Notice that \(V(\beta) \to 0\) as \(\beta \to \infty\), e.g. by considering \(f\) monic. From this the above assertion for \(Z(f, (0, \infty))\) – or \(Z(F, \mathbb{R})\) (for rational \(\alpha_j\) – follows. Curiously, in his 1934 paper [247] in Math. Z., Schoenberg proved this result using totally non-negative matrices.

(2) Define the partial sums

\[
C_1 := c_1, \quad C_2 := c_1 + c_2, \quad \ldots, \quad C_n := c_1 + c_2 + \cdots + c_n.
\]

Then the number of positive roots of \(F(t)\), counting orders, is at most the number of sign changes in \(C_1, C_2, \ldots, C_n\).
(3) Similarly, the number of negative roots of \( F(t) \), counting orders, is the number of positive roots of \( F(-t) \), hence at most the number of sign changes in the ‘reverse sequence’

\[
D_1 := c_n, \quad D_2 := c_n + c_{n-1}, \quad \ldots, \quad D_n := c_n + c_{n-1} + \cdots + c_1.
\]

Finally, we use Descartes’ rule of signs to show the result stated above: that all powers \( \geq 1 \) preserve total positivity of order 3.

**Proof of Theorem 10.1.** It is easy to check that all entrywise powers \( \alpha \geq 1 \) preserve the \( TP_2 \) property. We now show that the positivity of all \( 3 \times 3 \) matrices \( B \) using the Cauchy–Binet formula, one shows that \( D_1BD_2 \) is also preserved by entrywise applying \( x^t \), \( t \geq 1 \). Without loss of generality, we may assume \( m = n = 3 \) and work with a \( TP \) matrix \( B_{3 \times 3} = (b_{jk})_{j,k=1}^3 \).

The first claim is that we may assume \( B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix} \). Indeed, define the diagonal matrices

\[
D_1 := \begin{pmatrix} 1/b_{11} & 0 & 0 \\ 0 & 1/b_{21} & 0 \\ 0 & 0 & 1/b_{31} \end{pmatrix}, \quad D_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_{11}/b_{12} & 0 \\ 0 & 0 & b_{11}/b_{13} \end{pmatrix}.
\]

Using the Cauchy–Binet formula, one shows that \( B = (b_{jk})_{j,k=1}^3 \) is totally positive if and only if \( D_1BD_2 \) is \( TP \). But check that \( D_1BD_2 \) has only ones in its first row and column, as desired.

The next observation is that a matrix \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix} \) is \( TP_2 \) if and only if \( b, c > a > 1 \) and \( ad > bc \). (This is easily verified, and in turn implies \( d > b, c \).)

Now consider \( A^t \). For \( t \geq 1 \) this is \( TP_2 \) by above, so we only need to consider det \( A^t \) for \( t \geq 1 \). Define the generalized Dirichlet polynomial

\[
F(t) := \det(A^t) = (ad)^t - d^t - (bc)^t + b^t + c^t - a^t, \quad t \in \mathbb{R}.
\]

Notice from the above inequalities involving \( a, b, c, d \) that regardless of whether or not \( d > bc \), and whether or not \( b > c \), the sign sequence remains unchanged when arranging the exponents in \( F \) (namely, \( \log ad, \log d, \log bc, \log b, \ldots \) ) in decreasing order. It follows by Theorem 10.3 that \( F \) has at most three real roots.

As \( t \to \infty, F(t) \to \infty. \) Now one can carry out a Taylor expansion of \( F \) and check that the constant and linear terms vanish, yielding:

\[
F(t) = e^{t \log(ad)} - e^{t \log(d)} - \cdots = t^2(\log(a) \log(d) - \log(b) \log(c)) + o(t^2).
\]

It follows that \( F \) has (at least) a double root at 0. Now claim that \( F \) is indeed positive on \((1, \infty), \) as desired. For if \( F \) is negative on \((1, \infty), \) then since \( F(1) = \det A > 0, \) it follows by continuity that \( F \) has at least two more roots in \((1, \infty), \) which is false. Hence \( F \geq 0 \) on \((1, \infty). \) If \( F(t_0) = 0 \) for some \( t_0 > 1, \) then \( t_0 \) is a global minimum point in \([1, t_0 + 1] \) for \( F, \) whence \( F'(t_0) = 0. \) But then \( F \) has at least two zeros at \( t_0 \in (1, \infty), \) which is false.  \( \Box \)

While Theorem 10.3 sufficed in proving Theorem 10.1, we will need in Section 29.3 below a similar variant for ‘usual’ polynomials – more precisely, for Laurent series with degrees bounded below. This is now shown:
**Theorem 10.7.** Fix an integer $n_0 \geq 0$ and an open interval $I \subset (0, \infty)$. Suppose $F : I \to \mathbb{R}$, sending $t \mapsto \sum_{j=-n_0}^{\infty} c_j t^j$ is a convergent power series with not all $c_j \in \mathbb{R}$ zero. Then the number of zeros of $F$ in $I$, counting multiplicities, is either infinite or at most the number of sign changes in the sequence $c_{-n_0}, c_{1-n_0}, \ldots$ (after removing all zero terms).

*Proof.* If the Maclaurin coefficients $c_j$ have infinitely many sign changes then the result is immediate. Otherwise suppose there are only finitely many sign changes in the $c_j$, say $s$ many. We show the result by induction on $0 \leq s < \infty$, with the result immediate for $s = 0$. For the induction step, suppose the last sign change occurs at $c_k$, i.e., $c_k$ has sign opposite to the immediately preceding nonzero Maclaurin coefficient of $F$.

Now let $R > 0$ denote the radius of convergence of the power series $t^{n_0} F(t)$, so that $F$ is smooth on $(0, R)$ by (repeatedly) using the quotient rule. Consider the function

$$G(u) := u^{1-2k} F(u^2), \quad 0 < u < \sqrt{R}.$$

Then $G$ is smooth on $(0, \sqrt{R})$, and we now proceed as in (Laguerre’s 1883) proof of Theorem [10.3](#) via Rolle’s theorem – now working solely in $(0, \sqrt{R})$. The Laurent series $G'(u)$ has one less sign change than does $F$, so at most $s - 1$ roots (counting multiplicities) in $(0, \sqrt{R})$ by the induction hypothesis. Hence $G$ has at most $s$ roots in $(0, \sqrt{R})$, whence so does $F$ in $(0, R)$, hence in $I$. $\square$
11. ENTRYWISE POWERS PRESERVING TOTAL POSITIVITY: II.

In the previous section, we used Descartes’ rule of signs to show that $x^\alpha$ entrywise preserves the $3 \times 3$ TP matrices, for all $\alpha \geq 1$. Here the goal is twofold: first, to completely classify the entrywise powers that preserve TP/TN for $m \times n$ matrices for each fixed $m, n \geq 1$; and second, to then classify all continuous functions that do the same (at present, only for TN).

**Corollary 11.1.** If $\alpha \geq 1$, then $x^\alpha$ entrywise preserves the $3 \times 3$ TN matrices.

**Proof.** Let $A_{3 \times 3}$ be TN and $\alpha \geq 1$. By Whitney’s density theorem [6.7], there exist $3 \times 3$ TP matrices $B_m$ that entrywise converge to $A$, as $m \to \infty$. Hence $B_m^{\alpha} \to A^{\alpha}$ for $\alpha \geq 1$. Since $B_m^{\alpha}$ is TP by Theorem [10.1] it follows that $A^{\alpha}$ is TN, as claimed. □

The next result classifies all entrywise powers preserving total non-negativity for matrices of any fixed size.

**Theorem 11.2.** Given integers $m, n > 0$, define $d := \min(m, n)$. The following are equivalent for $\alpha \in \mathbb{R}$.

1. $x^\alpha$ preserves (entrywise) the $m \times n$ TN matrices.
2. $x^\alpha$ preserves (entrywise) the $d \times d$ TN matrices.
3. Either $\alpha = 0$ (where we set $0^0 := 1$), or
   (a) For $d = 1, 2$: $\alpha \geq 0$.
   (b) For $d = 3$: $\alpha \geq 1$.
   (c) For $d \geq 4$: $\alpha = 1$.

Thus we see that in contrast to the entrywise preservers of positive semidefiniteness (see Theorem [9.3]), almost no powers preserve the TN matrices – nor the TP matrices, as we show presently.

**Proof.** That (2) $\implies$ (1) is straightforward, as is (1) $\implies$ (2) by padding by zeros – noting that negative powers are not allowed (given zero entries of TN matrices). To show (3) $\implies$ (2), we use Theorem [10.1] as well as that $x^0$ applied to any TN matrix yields the matrix of all ones.

It remains to prove (2) $\implies$ (3). We may rule out negative powers since $(0_{d \times d})^{\alpha}$ is not defined for $d \geq 1$. Similarly, $x^0$ always preserves total non-negativity. This shows (3) for $d = 1, 2$. For $d = 3$, suppose $\alpha \in (0, 1)$ and consider the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{pmatrix}. \quad (11.3)$$

This is a Toeplitz cosine matrix, hence TN (see Example [3.21] or verify directly). Now compute:

$$\det A^{\alpha} = \det \begin{pmatrix} 1 & (\sqrt{2})^{-\alpha} & 0 \\ (\sqrt{2})^{-\alpha} & 1 & (\sqrt{2})^{-\alpha} \\ 0 & (\sqrt{2})^{-\alpha} & 1 \end{pmatrix} = 1 - 2^{1-\alpha},$$

which is negative if $\alpha < 1$. So $A^{\alpha}$ is not TN (not even positive semidefinite, in fact), for $\alpha < 1$, which shows (3) for $d = 3$. 


Next suppose \( d = 4 \) and consider the matrix
\[
N(x) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 \\
0 & 2 & 4 & 6 \\
0 & 3 & 8 & 14
\end{pmatrix} + x, \quad x \geq 0.
\]

One verifies that: all \( 2 \times 2 \) minors are of the form \( ax + bx^2 \), where \( a > 0, b \geq 0; \) all \( 3 \times 3 \) minors are of the form \( cx^2 \), where \( c \geq 0; \) and \( \det N(x) = 0 \). This implies \( N(x) \) is \( TN \) for \( x \geq 0 \). Moreover, for small \( x > 0 \), computations similar to the proof of Theorem 10.1 show that
\[
\det N(x)^{ot} = 2(t^3 - t^4)x^4 + o(x^4),
\]
so given \( t > 1 \), it follows that \( \det N(x)^{ot} < 0 \) for sufficiently small \( x > 0 \). Thus \( N(x)^{ot} \) is not \( TN \), whence \( x^\alpha \) does not preserve \( 4 \times 4 \) \( TN \) matrices for \( \alpha > 1 \). If on the other hand \( \alpha \in (0,1) \), then we work with the \( 4 \times 4 \) \( TN \) matrix
\[
C = \begin{pmatrix}
1 & 1/\sqrt{2} & 0 & 0 \\
1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \\
0 & 1/\sqrt{2} & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
proceeding as in the \( d = 3 \) case. This concludes the proof for \( d = 4 \).

Finally if \( d > 4 \) then we use the \( TN \) matrices \( \begin{pmatrix} N(x) & 0 \\ 0 & 0 \end{pmatrix}_{d \times d} \); for small \( x > 0 \) this rules out the powers \( \alpha > 1 \) as above. Similarly, using the \( TN \) matrix \( \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}_{d \times d} \) rules out the powers in \( (0,1) \). \( \square \)

In turn, Theorem 11.2 helps classify the powers preserving total positivity in each fixed size.

**Corollary 11.4.** Given \( m, n > 0 \), define \( d := \min(m,n) \) as in Theorem 11.2. The following are equivalent for \( \alpha \in \mathbb{R} \):

1. \( x^\alpha \) preserves entrywise the \( m \times n \) \( TP \) matrices.
2. \( x^\alpha \) preserves entrywise the \( d \times d \) \( TP \) matrices.
3. We have:
   - (a) For \( d = 1 \): \( \alpha \in \mathbb{R} \).
   - (b) For \( d = 2 \): \( \alpha > 0 \).
   - (c) For \( d = 3 \): \( \alpha \geq 1 \).
   - (d) For \( d \geq 4 \): \( \alpha = 1 \).

**Proof.** That (2) \( \implies \) (1) is straightforward, as is (1) \( \implies \) (2) (as above) by now using Theorem 7.5. That (3) \( \implies \) (2) was shown in Theorem 10.1 for \( d = 3 \), and is obvious for \( d \neq 3 \). Finally, we show (2) \( \implies \) (3). The \( d = 1 \) case is trivial, while the \( d = 2 \) case follows by considering \( \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \), say. Next, if \( d \geq 3 \) and if \( x^\alpha \) preserves the \( d \times d \) \( TP \) matrices, then \( \alpha > 0 \), by considering any \( TP \) matrix and applying \( x^\alpha \) to any of its \( 2 \times 2 \) minors. Hence \( x^\alpha \) extends continuously to \( x = 0 \); now \( x^\alpha \) preserves the \( d \times d \) \( TN \) matrices by continuity. Theorem 11.2 now finishes the proof. \( \square \)

Next, we tackle the more challenging question of classifying all functions that entrywise preserve total positivity or total non-negativity in fixed dimension \( m \times n \). We will show that (i) every such function must be continuous (barring a one-parameter exceptional family of
TN preservers), and in turn, this implies that (ii) it must be a power function. We first show (ii), beginning with an observation on the (additive) Cauchy functional equation.

**Remark 11.5** (Additive continuous functions). Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Cauchy functional equation $g(x + y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$. Then we claim that $g(x) = cx$ for some $c \in \mathbb{R}$ (and all $x$). Indeed, $g(0 + 0) = g(0) + g(0)$, whence $g(0) = 0$. Next, one shows by induction that $g(n) = ng(1)$ for integers $n > 0$, and hence for all integers $n < 0$ as well. Now one shows that $pg(1) = g(p) = g(q \cdot p/q) = q \cdot g(p/q)$ for integers $p, q$ with $q \neq 0$, from which it follows that $g(p/q) = (p/q)g(1)$ for all rationals $p/q$. Finally, using continuity we conclude that $g(x) = xg(1)$ for all $x \in \mathbb{R}$.

**Proposition 11.6.** Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous and entrywise preserves the $2 \times 2$ TN matrices. Then $f(x) = f(1)x^\alpha$ for some $\alpha \geq 0$.

We recall here that $0^0 := 1$ by convention.

**Proof.** Define the matrices
\[
A(x, y) = \begin{pmatrix} x & xy \\ 1 & y \end{pmatrix}, \quad B(x, y) = \begin{pmatrix} xy & x \\ y & 1 \end{pmatrix}, \quad x, y \geq 0.
\]
Clearly, these matrices are TN, whence by the hypotheses,
\[
\det f[A(x, y)] = f(x)f(y) - f(1)f(xy) \geq 0, \\
\det f[B(x, y)] = f(1)f(xy) - f(x)f(y) \geq 0.
\]
It follows that
\[
f(x)f(y) = f(1)f(xy), \quad \forall x, y \geq 0. \tag{11.7}
\]
There are two cases. First if $f(1) = 0$ then choosing $x = y \geq 0$ in (11.7) gives $f \equiv 0$ on $[0, \infty)$. Else if $f(1) > 0$ then we claim that $f$ is always positive on $(0, \infty)$. Indeed, if $f(x_0) = 0$ for $x_0 > 0$, then set $x = x_0, y = 1/x_0$ in (11.7) to get: $0 = f(1)^2$, which is false.

Now define the functions
\[
g(x) := f(x)/f(1), \quad x > 0, \quad h(y) := \log g(e^y), \quad y \in \mathbb{R}.
\]
Then (11.7) can be successively reformulated as:
\[
g(xy) = g(x)g(y), \quad \forall x, y > 0, \\
h(a + b) = h(a) + h(b), \quad \forall a, b \in \mathbb{R}. \tag{11.8}
\]
Moreover, both $g, h$ are continuous. Since $h$ satisfies the additive Cauchy functional equation, it follows by Remark 11.5 that $h(y) = yh(1)$ for all $y \in \mathbb{R}$. Translating back, we get $g(x) = x^h(1)$ for all $x > 0$. It follows that $f(x) = f(1)x^\alpha$ for $x > 0$, where $\alpha = h(1)$. Finally, since $f$ is also continuous at $0^+$, it follows that $\alpha \geq 0$; and either $\alpha = 0$ and $f \equiv 1$ (so we set $0^0 : = 1$), or $f(0) = 0 < \alpha$. (Note that $\alpha$ cannot be negative, since $f[-]$ preserves TN on the zero matrix, say.)

**Corollary 11.9.** Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous and entrywise preserves the $m \times n$ TN matrices, for some $m, n \geq 2$. Then $f(x) = f(1)x^\alpha$ for some $\alpha \geq 0$, with $f(1) \geq 0$.

**Proof.** Given $m, n \geq 2$, every $2 \times 2$ TN matrix can be embedded as a leading principal submatrix in a $m \times n$ TN matrix, by padding it with (all other) zero entries. Hence the hypotheses imply that $f[-]$ preserves the $2 \times 2$ TN matrices, and we are done by the above Proposition 11.6. \qed

We continue working toward the classification of all entrywise functions preserving $m \times n$ $TP/TN$ matrices. Thus far, we have classified the power functions among these preservers; and we also showed that every continuous map that preserves $m \times n$ $TN$ matrices is a multiple of a power function.

We now show that every function that entrywise preserves the $m \times n$ $TP/TN$ matrices is automatically continuous on $(0, \infty)$ – which allows us to classify all such preservers. The continuity will follow from a variant of a 1929 result by Ostrowski on mid-convex functions on normed linear spaces, and we begin by proving this result.

12.1. Mid-convex functions and continuity.

Definition 12.1. Given a convex subset $U$ of a real vector space, a function $f : U \to \mathbb{R}$ is said to be mid-convex if
\[
f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}, \quad \forall x, y \in U;
\]
and $f$ is convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in U$ and $\lambda \in (0, 1)$.

Notice that convex functions are automatically mid-convex. The converse need not be true in general. However, if a mid-convex function is continuous then it is easy to see that it is also convex. Thus, a natural question for mid-convex functions is to find sufficient conditions under which they are continuous. We now discuss two such conditions, both classical results. The first condition is mild: $f$ is locally bounded, on one neighborhood of one point.

Theorem 12.2. Let $B$ be a normed linear space (over $\mathbb{R}$) and let $U$ be a convex open subset. Suppose $f : U \to \mathbb{R}$ is mid-convex and $f$ is bounded above in an open neighborhood of a single point $x_0 \in U$. Then $f$ is continuous on $U$, and hence convex.

This generalizes to normed linear spaces a special case of a result by Ostrowski, who showed in *Jber. Deut. Math. Ver.* (1929) the same conclusion, but over $B = \mathbb{R}$ and assuming $f$ is bounded above in a measurable subset.

The proof requires the following useful observation:

Lemma 12.3. If $f : U \to \mathbb{R}$ is mid-convex, then $f$ is rationally convex, i.e., $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in U$ and $\lambda \in (0, 1) \cap \mathbb{Q}$.

Proof. Inductively using mid-convexity, it follows that
\[
f \left( \frac{x_1 + x_2 + \cdots + x_{2^n}}{2^n} \right) \leq \frac{f(x_1) + \cdots + f(x_{2^n})}{2^n}, \quad \forall n \in \mathbb{N}, \ x_1, \ldots, x_{2^n} \in U.
\]

Now suppose that $\lambda = \frac{p}{q} \in (0, 1)$, where $p, q > 0$ are integers and $2^{n-1} \leq q < 2^n$ for some $n \in \mathbb{N}$. Let $x_1, \ldots, x_q \in U$, and define $\pi = \frac{1}{q}(x_1 + \cdots + x_q)$. Then,
\[
f \left( \frac{x_1 + \cdots + x_q + (2^n - q)x}{2^n} \right) \leq \frac{f(x_1) + \cdots + f(x_q) + (2^n - q)f(\pi)}{2^n}
\]
\[
\Rightarrow 2^n f(\pi) \leq f(x_1) + \cdots + f(x_q) + (2^n - q)f(\pi)
\]
\[
\Rightarrow q f(\pi) \leq f(x_1) + \cdots + f(x_q)
\]
\[
\Rightarrow f \left( \frac{x_1 + \cdots + x_q}{q} \right) \leq \frac{f(x_1) + \cdots + f(x_q)}{q}
\]
In this inequality, set \( x_1 = \cdots = x_p = x \) and \( x_{p+1} = \cdots = x_q = y \) to complete the proof:
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \]

With Lemma [12.3] in hand, we can prove the theorem above.

**Proof of Theorem 12.2.** We may assume without loss of generality that \( x_0 = 0 \in U \subset B \), and also that \( f(x_0) = f(0) = 0 \).

We claim that \( f \) is continuous at \( 0 \), where \( f \) was assumed to be bounded above in an open neighborhood of \( 0 \). Write this as: \( f(B(0, r)) < M \) for some \( r, M > 0 \), where \( B(x, r) \subset B \) denotes the open ball of radius \( r \) centered at \( x \in B \). Now given \( \epsilon \in (0, 1) \cap \mathbb{Q} \) rational, and \( x \in B(0, \epsilon r) \), compute using Lemma [12.3]
\[
x = \epsilon \left( \frac{x}{\epsilon} \right) + (1 - \epsilon)0 \quad \implies \quad f(x) \leq \epsilon f \left( \frac{x}{\epsilon} \right) + 0 < \epsilon M.
\]

Moreover,\[
0 = \left( \frac{\epsilon}{1 + \epsilon} \right) \left( \frac{-x}{\epsilon} \right) + \frac{x}{1 + \epsilon},
\]
so applying Lemma [12.3] once again, we obtain:
\[
0 \leq \left( \frac{\epsilon}{1 + \epsilon} \right) f \left( \frac{-x}{\epsilon} \right) + \frac{f(x)}{1 + \epsilon} < \frac{\epsilon M}{1 + \epsilon} + \frac{f(x)}{1 + \epsilon} \quad \implies \quad f(x) > -\epsilon M.
\]

Therefore, we have \( x \in B(0, \epsilon r) \implies |f(x)| < \epsilon M \).

Now given \( \epsilon > 0 \), choose \( 0 < \epsilon' < \min(M, \epsilon) \) such that \( \epsilon'/M \) is rational, and set \( \delta := \epsilon' r/M \). Then \( \delta < r \), so \( |f(x)| < \delta M/r = \epsilon' < \epsilon \) whenever \( x \in B(0, \delta) \). Hence \( f \) is continuous at \( x_0 \).

We have shown that if \( f \) is bounded above in some open neighborhood of \( x_0 \), then \( f \) is continuous at \( x_0 \). To finish the proof, we claim that for all \( y \in U \), \( f \) is bounded above on some open neighborhood of \( y \). This would show that \( f \) is continuous on \( U \), which combined with mid-convexity implies convexity.

To show the claim, choose a rational \( \rho > 1 \) such that \( \rho y \in U \) (this is possible as \( U \) is open) and set \( U_y := B(y, (1 - 1/\rho)r) \). Note that \( U_y \subset U \) since for every \( v \in U_y \) there exists \( x \in B(0, r) \) such that
\[
v = y + (1 - 1/\rho)x = \frac{1}{\rho}(\rho y) + \left(1 - \frac{1}{\rho}\right)x.
\]

Thus \( v \) is a convex combination of \( \rho y \in U \) and \( x \in B(0, r) \subset U \). Hence \( U_y \subset U \); in turn,
\[
f(v) \leq \frac{1}{\rho} f(\rho y) + \left(1 - \frac{1}{\rho}\right) f(x) \leq \frac{f(\rho y)}{\rho} + \left(1 - \frac{1}{\rho}\right) M, \quad \forall v \in U_y
\]
by Lemma [12.3]. Since the right-hand side is independent of \( v \in U_y \), the above claim follows. Hence by the first claim, \( f \) is indeed continuous at every point in \( U \).

The second condition, which will be used in a later part in this text, is that \( f \) is Lebesgue measurable. Its sufficiency was proved a decade before Ostrowski’s result, independently by Blumberg in *Trans. Amer. Math. Soc.* (1919) and Sierpiński in *Fund. Math.* (1920). However, the following proof goes via Theorem [12.2].

**Theorem 12.4.** If \( I \subset \mathbb{R} \) is an open interval, and \( f : I \to \mathbb{R} \) is Lebesgue measurable and mid-convex, then \( f \) is continuous, whence convex.

**Proof.** Suppose $f$ is not continuous at a point $x_0 \in I$. Fix $c > 0$ such that $(x_0 - 2c, x_0 + 2c) \subset I$. By Theorem 12.2, $f$ is unbounded on $(x_0 - c, x_0 + c)$. Now let $B_n := \{x \in I : f(x) > n\}$ for $n \geq 1$; note this is Lebesgue measurable. Choose $u_n \in B_n \cap (x_0 - c, x_0 + c)$ and $\lambda \in [0, 1]$; then by mid-convexity,
\[
 n < f(u_n) = f\left[\frac{u_n + \lambda c}{2} + \frac{u_n - \lambda c}{2}\right] \leq \frac{1}{2}(f(u_n + \lambda c) + f(u_n - \lambda c)).
\]
Thus $B_n$ contains at least one of the points $u_n + \lambda c \in I$, i.e., one of $\pm \lambda c$ lies in $B_n - u_n$. We now claim that each $B_n$ has Lebesgue measure $\mu(B_n) \geq c$. Assuming this claim,
\[
 c \leq \lim_{n \to \infty} \mu(B_n) = \mu(\bigcap_{n \geq 1} B_n),
\]
since the $B_n$ are a nested family of subsets. But then $S := \bigcap_{n \geq 1} B_n$ is non-empty, so for any $v \in S$, we have $f(v) > n$ for all $n$, which produces the desired contradiction.

Thus, it remains to show the above claim. Fix $n \geq 1$ and note from above that $M_n := B_n - u_n$ is a Lebesgue measurable set such that for every $\lambda \in [0, 1]$, at least one of $\lambda c, -\lambda c$ lies in $M_n$. Define the measurable sets $A_1 := M_n \cap [-c, 0]$ and $A_2 := M_n \cap [0, c]$, so that $-A_1 \cup A_2 = [0, c]$. This implies:
\[
 c \leq \mu(-A_1) + \mu(A_2) = \mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2) \leq \mu(M_n). \]

**12.2. Functions preserving total non-negativity.** With Theorem 12.2 in hand, it is possible to classify all entrywise functions that preserve total non-negativity or total positivity in a fixed size, or even positive semidefiniteness on $2 \times 2$ matrices. A major portion of the work is carried out by the next result. To state this result, we need the following notion.

**Definition 12.5.** Suppose $I \subset [0, \infty)$ is an interval. A function $f : I \to [0, \infty)$ is multiplicatively mid-convex on $I$ if and only if $f(\sqrt{xy}) \leq \sqrt{f(x)f(y)}$ for all $x, y \in I$.

**Remark 12.6.** A straightforward computation yields that if $f : I \to \mathbb{R}$ is always positive, and $0 \notin I$, then $f$ is multiplicatively mid-convex on $I$ if and only if the auxiliary function $g(y) := \log f(e^y)$ is mid-convex on $\log(I)$.

We now prove the following important result, which is also crucial later.

**Theorem 12.7.** Suppose $I = [0, \infty)$ and $I^+ := I \setminus \{0\}$. A function $f : I \to \mathbb{R}$ satisfies $(f(a) f(b)) f(c)$ is positive semidefinite whenever $a, b, c \in I$ and $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is $TN$, if and only if $f$ is non-negative, non-decreasing, and multiplicatively mid-convex on $I$. In particular,

1. $f'|_{I^+}$ is never zero or always zero.
2. $f'|_{I^+}$ is continuous.

The same results hold if $I = [0, \infty)$ is replaced by $I = (0, \infty)$, $[0, \rho)$, or $(0, \rho)$ for $0 < \rho < \infty$.

This result was essentially proved by H.L. Vasudeva, under some reformulation. In the result, note that $TN$ is the same as ‘positive semidefinite with non-negative entries’, since we are dealing with $2 \times 2$ matrices; thus, the test set of matrices is precisely $\mathbb{P}_2(I)$, and the hypothesis can be rephrased as:
\[
 f[-] : \mathbb{P}_2(I) \to \mathbb{P}_2 = \mathbb{P}_2(\mathbb{R}).
\]

Moreover, all of these matrices are clearly Hankel. This result will therefore also play an important role later, when we classify the entrywise preservers of positive semidefiniteness on low-rank Hankel matrices.
Proof. Let \( I \) be any of the domains mentioned in the theorem. We begin by showing the equivalence. Given a \( TN \) matrix
\[
\begin{pmatrix}
a & b \\
b & c \\
\end{pmatrix}, \quad a, b, c \in I, \quad 0 \leq b \leq \sqrt{ac},
\]
compute via the non-negativity, monotonicity, and multiplicative mid-convexity respectively:
\[
0 \leq f(b) \leq f(\sqrt{ac}) \leq \sqrt{f(a)f(c)}.
\]
It follows that \( \begin{pmatrix} f(a) & f(b) \\ f(b) & f(c) \end{pmatrix} \) is \( TN \) and hence positive semidefinite.

Conversely, if (via the above remarks) \( f[-] : \mathbb{P}_2(I) \to \mathbb{P}_2 \), then apply the entrywise \( f[-] \) to the matrices
\[
\begin{pmatrix}
a & b \\
b & a \\
\end{pmatrix}, \quad \begin{pmatrix}
a & \sqrt{ac} \\
\sqrt{ac} & c \\
\end{pmatrix}, \quad a, b, c \in I,
\]
with \( 0 \leq b \leq a \). From the hypotheses, it successively (and respectively) follows that \( f \) is non-negative, non-decreasing, and multiplicatively mid-convex. This proves the equivalence.

As a brief digression that will be useful later, we remark that the test matrices \( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \) can be replaced by
\[
\begin{pmatrix}
a & b \\
b & a \\
\end{pmatrix}, \quad \begin{pmatrix}
a & \sqrt{ac} \\
\sqrt{ac} & c \\
\end{pmatrix}, \quad a, b, c \in I,
\]
with \( 0 \leq b \leq a \), to conclude as above that \( f \) is non-decreasing on \( I \). Indeed, we obtain \( f(a), f(b) \geq 0 \), and either \( f(b) = 0 \leq f(a) \), or \( 0 < f(b)^2 \leq f(b)f(a) \), leading to the same conclusion.

We now show the two assertions in turn, again on \( I^+ \) for any of the domains \( I \) above; in other words, \( I^+ = (0, \rho) \) for \( 0 < \rho \leq \infty \). For (1), suppose \( f(x) = 0 \) for some \( x \in I^+ \). Since \( f \) is non-negative and non-decreasing on \( I^+ \), it follows that \( f \equiv 0 \) on \((0, x)\). Now claim that \( f(y) = 0 \) if \( y > x \), \( y \in I^+ = (0, \rho) \). Indeed, choose \( n \gg 0 \) such that \( y \sqrt{y/x} < \rho \). Set \( \zeta := \sqrt{y/x} > 1 \), and consider the following rank-one matrices in \( \mathbb{P}_2(I^+) \):
\[
A_1 := \begin{pmatrix} x & x \zeta \\ x \zeta & x \zeta^2 \end{pmatrix}, \quad A_2 := \begin{pmatrix} x \zeta & x \zeta^2 \\ x \zeta^2 & x \zeta^3 \end{pmatrix}, \quad \ldots, \quad A_n := \begin{pmatrix} x \zeta^{n-1} & x \zeta^n \\ x \zeta^n & x \zeta^{n+1} \end{pmatrix}.
\]

The inequalities \( \det f[A_k] \geq 0 \), \( 1 \leq k \leq n \) yield:
\[
0 \leq f(x \zeta^k) \leq \sqrt{f(x \zeta^{k-1})f(x \zeta^{k+1})}, \quad k = 1, 2, \ldots, n.
\]
From this inequality for \( k = 1 \), it follows that \( f(x \zeta) = 0 \). Similarly, these inequalities inductively yield: \( f(x \zeta^k) = 0 \) for all \( 1 \leq k \leq n \). In particular, we have \( f(y) = f(x \zeta^n) = 0 \). This shows that \( f \equiv 0 \) on \( I^+ \), as claimed.

We provide two proofs of (2). If \( f \equiv 0 \) on \( I^+ \), then \( f \) is continuous on \( I^+ \). Otherwise by (1), \( f \) is strictly positive on \( (0, \rho) = I^+ \). Now the ‘classical’ proof uses the above ‘Ostrowski-result’: define the function \( g : \log I^+ := (-\infty, \log \rho) \to \mathbb{R} \) via:
\[
g(y) := \log f(e^y), \quad y < \log \rho.
\]
By the assumptions on \( f \) and the observation in Remark 12.6, \( g \) is mid-convex and non-decreasing on \( (-\infty, \log \rho) \). In particular, \( g \) is bounded above on compact sets. Now apply Theorem 12.2 to deduce that \( g \) is continuous. It follows that \( f \) is continuous on \( (0, \rho) \).
A more recent, shorter proof is by Hiai (2009): given \( f \) as above which is strictly positive and non-decreasing on \((0, \rho)\), fix \( t \in (0, \rho) \) and let \( 0 < \epsilon < \min(t/5, (\rho - t)/4) \). Then \( 0 < t + \epsilon \leq \sqrt{(t + 4\epsilon)(t - \epsilon)} < \rho \), whence

\[
f(t + \epsilon) \leq f \left(\sqrt{(t + 4\epsilon)(t - \epsilon)}\right) \leq \sqrt{f(t + 4\epsilon)f(t - \epsilon)}.
\]

Now letting \( \epsilon \to 0^- \), we have

\[
0 < f(t) \leq f(t^+) \leq f(t^-) \leq f(t), \quad \forall t \in (0, \rho).
\]

Since \( t \in (0, \rho) \) was arbitrary, this shows \( f \) is continuous as claimed. \( \Box \)

**Remark 12.10.** From the proof – see (12.8) or (12.9) – it follows that the assumptions may be further weakened to not work with all symmetric \( 2 \times 2 \) \( TN \) matrices, but with only the rank-one symmetric and the Toeplitz symmetric \( 2 \times 2 \) \( TN \) matrices.

As an application, Theorem 12.7 allows us to complete the classification of all entrywise maps that preserve total non-negativity in each fixed size.

**Theorem 12.11.** Suppose \( m, n \geq 2 \) and \( f : [0, \infty) \to \mathbb{R} \) entrywise preserves the \( m \times n \) \( TN \) matrices. Then either \( f(x) = f(1)x^\alpha \) for \( f(1), \alpha \geq 0 \) and all \( x \geq 0 \) (and these powers were classified in Theorem 11.2), or \( \min(m, n) = 2 \) and \( f(x) = f(1)\sgn(x) \) for \( x \geq 0 \) and \( f(1) > 0 \).

If instead \( \min(m, n) = 1 \), then \( f \) can be any function that maps \([0, \infty)\) into itself.

**Proof.** The result is trivial for \( \min(m, n) = 1 \), so we assume henceforth that \( m, n \geq 2 \). By embedding \( 2 \times 2 \) \( TN \) matrices inside \( m \times n \) \( TN \) matrices, it follows that \( f[-] \) preserves the \( 2 \times 2 \) \( TN \) matrices. In particular, \( f \) is continuous on \([0, \infty)\) by Theorem 12.7 and non-negative and non-decreasing on \([0, \infty)\). Now one can repeat the proof of Proposition 11.6 above, to show that

\[
f(x)f(y) = f(xy)f(1), \quad \forall x, y \geq 0,
\]

and moreover, either \( f \equiv 0 \) on \([0, \infty)\), or \( f(x) = f(1)x^\alpha \) for \( x > 0 \) and some \( \alpha \geq 0 \).

We assume henceforth that \( f \neq 0 \) on \([0, \infty)\), whence \( f(x) = f(1)x^\alpha \) as above – with \( f(1) > 0 \). If now \( f(0) \neq 0 \), then substituting \( x = 0, y \neq 1 \) in (12.12) shows that \( \alpha = 0 \), and now using \( x = y = 0 \) in (12.12) shows \( f(0) = f(1) \), i.e., \( f|_{[0, \infty)} \) is constant (and positive).

Otherwise \( f(0) = 0 \). Now if \( \alpha > 0 \) then \( f(x) = f(1)x^\alpha \) for all \( x \geq 0 \) and \( f \) is continuous on \([0, \infty)\). The final case is where \( f(0) = 0 = \alpha \), but \( f \neq 0 \). Then \( f(0) = 0 \) while \( f(x) = f(1) > 0 \) for all \( x > 0 \). Now if \( \min(m, n) = 2 \) then it is easy to verify that \( f[-] \) preserves \( TN_{m \times n} \). On the other hand, if \( m, n \geq 3 \), then computing \( \det f[A] \) for the matrix

\[
A = \begin{pmatrix}
1 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 1
\end{pmatrix}
\]

shows that \( f[-] \) is not a positivity preserver on \( A \oplus 0_{(m-3) \times (n-3)} \in TN_{m \times n} \). \( \square \)

**12.3. Functions preserving total positivity.** Akin to the above results, we can also classify the entrywise functions preserving total positivity in any fixed size, and they too are essentially power functions.

**Theorem 12.13.** Suppose \( f : (0, \infty) \to \mathbb{R} \) is such that \( f[-] \) preserves the \( m \times n \) \( TP \) matrices for some \( m, n \geq 2 \). Then \( f \) is continuous and \( f(x) = f(1)x^\alpha \), with \( \alpha > 0 \) and \( f(1) > 0 \).

Recall that the powers preserving the \( m \times n \) \( TP \) matrices were classified in Corollary 11.4.

To show the theorem, we make use of the following intermediate lemma, which is also useful later in studying preservers of \( TP \) Hankel kernels. A cruder version of the next result
say that if \( f[-] \) preserves total positivity on \( 2 \times 2 \) matrices, then \( f \) is continuous. More strongly, we have:

**Lemma 12.14.** Suppose \( p \in (1, \infty) \), and \( f : (0, \infty) \to \mathbb{R} \). The following are equivalent:

1. \( f[-] \) preserves total positivity on all symmetric \( 2 \times 2 \) TP matrices.
2. \( f[-] \) preserves positive definiteness on the symmetric TP Hankel matrices
   \[
   \begin{pmatrix}
   a & b \\
   b & c
   \end{pmatrix}, \quad a, c > 0, \sqrt{ac/p} < b < \sqrt{ac}.
   \]
   
3. \( f \) is positive, increasing, and multiplicatively mid-convex on \((0, \infty)\).

In particular, \( f \) is continuous.

**Proof.** Clearly (1) \( \implies \) (2), and that (3) \( \implies \) (1) is left to the reader as it is similar to the proof of Theorem 12.7. We now assume (2) and show (3). The first step is to claim that \( f \) is positive and strictly increasing on \((0, \infty)\). Suppose \( 0 < x < y < \infty \). Choose \( n > \log_p(y/x) \), and define the increasing sequence
\[
x_0 = x, \quad x_1 = x(y/x)^{1/n}, \quad x_2 = x(y/x)^{2/n}, \quad \ldots, \quad x_n = y.
\]
Now the matrix \( \begin{pmatrix} x_{k+1} & x_k \\ x_k & x_k \end{pmatrix} \) is in the given test set, by choice of \( n \), so applying \( f[-] \) and taking determinants, we have
\[
f(x_k), f(x_{k+1}), f(x_k)(f(x_{k+1}) - f(x_k)) > 0, \quad 0 \leq k \leq n - 1.
\]
It follows that \( f \) is positive on \((0, \infty)\), hence also strictly increasing, since \( f(x) = f(x_0) < f(x_1) < \cdots < f(x_n) = f(y) \).

We next show continuity, proceeding indirectly. From above, \( f : (0, \infty) \to (0, \infty) \) has at most countably many discontinuities, and they are all jump discontinuities. Let \( f(x^+) := \lim_{y \to x^+} f(y) \), for \( x > 0 \). Then \( f(x^+) \geq f(x) \forall x \), and \( f(x^+) \) coincides with \( f(x) \) at all points of right continuity and has the same jumps as \( f \). Thus, it suffices to show that \( f(x^+) \) is continuous (since this implies \( f \) is also continuous).

Now given \( 0 < x < y < \infty \), apply \( f[-] \) to the matrices
\[
M(x, y, \epsilon) := \begin{pmatrix} x + \epsilon & \sqrt{xy} + \epsilon \\ \sqrt{xy} + \epsilon & y + \epsilon \end{pmatrix}, \quad x, y, \epsilon > 0,
\]
where \( \epsilon > 0 \) is small enough that \((x + \epsilon)(y + \epsilon) < p(\sqrt{xy} + \epsilon)^2 \). Then an easy verification shows that \( M(x, y, \epsilon) \) is in the given test set. It follows that \( \det [M(x, y, \epsilon)] > 0 \), i.e., \( f(x + \epsilon)f(y + \epsilon) > f(\sqrt{xy} + \epsilon)^2 \). Taking \( \epsilon \to 0^+ \), we obtain:
\[
f(x^+)f(y^+) \geq f(\sqrt{xy} + \epsilon)^2, \quad \forall x, y > 0.
\]
Thus \( f(x^+) \) is positive, non-decreasing and multiplicatively mid-convex on \((0, \infty)\). From the proof of Theorem 12.7(2), we conclude that \( f(x^+) \) is continuous on \((0, \infty)\), whence \( f(x) = f(x^+) \) is also continuous and multiplicatively mid-convex on \((0, \infty)\).

Using this lemma, we now show:

**Proof of Theorem 12.13.** We first show the result for \( m = n = 2 \). By Lemma 12.14, \( f \) is continuous, positive, and strictly increasing on \((0, \infty)\). Now claim that \( f(x) = f(1)x^\alpha \) for all \( x > 0 \) (and some \( \alpha > 0 \)). For this, consider the matrices
\[
A(x, y, \epsilon) := \begin{pmatrix} x & xy \\ 1 - \epsilon & y \end{pmatrix}, \quad B(x, y, \epsilon) := \begin{pmatrix} xy & y \\ x & 1 + \epsilon \end{pmatrix}, \quad \text{where } x, y, \epsilon > 0.
\]
These are both TP matrices, whence so are \( f[A(x, y, \epsilon)] \) and \( f[B(x, y, \epsilon)] \). The positivity of both determinants yields:

\[
f(x)f(y) > f(xy)f(1 - \epsilon), \quad f(xy)f(1 + \epsilon) > f(x)f(y), \quad \forall x, y, \epsilon > 0.
\]

Taking \( \epsilon \to 0^+ \), the continuity of \( f \) and the assumptions imply that \( \frac{f(x)}{f(1)} \) is multiplicative, continuous, positive, non-constant, and strictly increasing on \((0, \infty)\). Hence (e.g. as in the proof of Proposition \[11.6\]), \( f(x) = f(1)x^\alpha \) for all \( x > 0 \), where \( \alpha > 0 \) and \( f(1) > 0 \).

This completes the proof for \( m = n = 2 \). Now suppose more generally that \( m, n \geq 2 \). Recall by a TP completion problem (see Theorem \[7.1\]) that every \( 2 \times 2 \) TP matrix can be completed to an \( m \times n \) TP matrix. It follows from the assumptions that \( f[-] \) must preserve the \( 2 \times 2 \) TP matrices, and we are done. \( \square \)

12.4. **Symmetric TN and TP matrix preservers.** Having classified the preservers of total positivity on all matrices of a fixed size, we turn to TN symmetric matrices:

**Theorem 12.15.** Suppose \( f : [0, \infty) \to \mathbb{R} \) and \( d \geq 1 \). Then \( f[-] \) preserves the symmetric TN \( d \times d \) matrices if and only if \( f \) is a non-negative constant, or:

1. \( (d = 1) \). The function \( f \) is non-negative.
2. \( (d = 2) \). \( f \) is non-negative, non-decreasing, and multiplicatively mid-convex on \([0, \infty)\).
   
   In particular, \( f \) is continuous on \((0, \infty)\).
3. \( (d = 3) \). \( f(x) = cx^\alpha \) for some \( c > 0 \) and \( \alpha \geq 1 \).
4. \( (d = 4) \). \( f(x) = cx^\alpha \) for some \( c > 0 \) and \( \alpha \in \{1\} \cup [2, \infty) \).
5. \( (d = 5) \). \( f(x) = cx \) for some \( c > 0 \).

**Proof.** For \( d = 1 \) the result is immediate. If \( d = 2 \), the result follows from Theorem \[12.7\]. Now suppose \( d = 3 \). One implication follows from Theorem \[12.11\]. Conversely, every symmetric \( 2 \times 2 \) can be padded by a row and column of zeros to remain TN, so by Theorem \[12.7\] \( f \) is continuous, non-negative, and non-decreasing on \((0, \infty)\). We next show that \( f \) is continuous at 0 and that \( f(0) = 0 \) for non-constant \( f \). First, the matrix \( f[x \text{Id}_{3 \times 3}] \) is TN for each \( x > 0 \). If \( f(0) > 0 \), considering various \( 2 \times 2 \) minors yields \( f(x) = f(0) \) for all \( x > 0 \), so \( f \) is constant. The remaining case is \( f(0) = 0 \). Now let \( A_{3 \times 3} \) be as in Equation \[11.3\]. By the hypotheses, \( f[xA] \) is TN for all \( x > 0 \), so

\[
0 \leq \det f[xA] = -f(0^+)^3.
\]

Thus \( f(0^+) = 0 = f(0) \), and \( f \) is continuous.

Next, consider the symmetric TN matrices

\[
A'(x, y) := \begin{pmatrix} x^2 & x & xy \\ x & 1 & y \\ xy & y & y^2 \end{pmatrix}, \quad B'(x, y) := \begin{pmatrix} x^2y & xy & x \\ xy & y & 1 \\ x & 1 & 1/y \end{pmatrix}, \quad x \geq 0, \ y > 0.
\]

Since these contain as non-principal minors the matrices \( \begin{pmatrix} x & xy \\ 1 & y \end{pmatrix} \) and \( \begin{pmatrix} xy & x \\ y & 1 \end{pmatrix} \), we can repeat the proof of Proposition \[11.6\] to conclude that \( f \) is either a constant or \( f(x) = cx^\alpha \) for some \( c > 0, \alpha \geq 0 \). Finally, again using the matrix \( A \) in \[11.3\] and the computations following it, we conclude that \( \alpha \geq 1 \).

The next case is \( d = 4 \). As above, embedding \( 3 \times 3 \) matrices via padding by zeros shows that \( f(x) = cx^\alpha \), with \( c > 0 \) and \( \alpha \geq 1 \). From the proof of Theorem \[9.3\] given \( \alpha \in (1, 2) \) one obtains a \( 4 \times 4 \) Hankel moment matrix (hence this is TN), corresponding to the measure \( \delta_1 + c\delta_x \) for \( 1 \neq x \in (0, \infty) \) and small \( c > 0 \), whose \( \alpha \)th entrywise power is not positive.

semidefinite. This proves one implication; for the reverse, the preceding parts for $d = 2, 3$ imply that the determinants of all proper submatrices of $f[A]$ are non-negative, for every $4 \times 4$ symmetric $TN$ matrix $A$. That $\det f[A] \geq 0$ follows from Theorem 9.3.

The final case is $d \geq 5$. In this case, one implication is trivial, and the reverse implication for $d = 5$ implies the same for all $d > 5$, by padding $5 \times 5$ $TN$ matrices by zeros. Thus, it suffices to classify the non-constant preservers of $5 \times 5$ symmetric $TN$ matrices. By the preceding part, these are of the form $cx^\alpha$ for $\alpha = 1$ or $\alpha \geq 2$, and $c > 0$. Now suppose $\alpha \geq 2$, and consider the family of $5 \times 5$ matrices

$$
T(x) := 1_{5\times5} + x \begin{pmatrix}
2 & 3 & 6 & 14 & 36 \\
3 & 6 & 14 & 36 & 98 \\
6 & 14 & 36 & 98 & 276 \\
14 & 36 & 98 & 284 & 842 \\
36 & 98 & 276 & 842 & 2604
\end{pmatrix}, \quad x > 0. \quad (12.16)
$$

Straightforward computations show that all $k \times k$ minors of $M(x)$ are of the form $ax^{k-1} + bx^k$ for $a, b \geq 0$, for $1 \leq k \leq 4$, and $\det M(x) = 0$. Thus $M(x)$ is $TN$ for all $x > 0$. Let $N(x) := M(x)^{(1)}$ be the truncation of $M(x)$, i.e. with its first row and last column removed. Another computation reveals that for small $x > 0$,

$$
\det N(x)^{\alpha} = 28584(a^3 - a^4)x^4 + O(x^5),
$$

so if $\alpha > 1$, then there exists small $x > 0$ such that $M(x)^{\alpha}$ is not $TN$. \hfill \Box

From this result, it is possible to deduce the classification of $TP$ preservers on symmetric matrices of each fixed size:

Corollary 12.17. Suppose $f : (0, \infty) \rightarrow (0, \infty)$. Then $f[-]$ preserves total positivity on symmetric $TP$ $d \times d$ matrices, if and only if $f$ satisfies:

1. ($d = 1$). The function $f$ is positive.
2. ($d = 2$). $f$ is positive, increasing, and multiplicatively mid-convex on $(0, \infty)$. In particular, $f$ is continuous.
3. ($d = 3$). $f(x) = cx^\alpha$ for some $c > 0$ and $\alpha \geq 1$.
4. ($d = 4$). $f(x) = cx^\alpha$ for some $c > 0$ and $\alpha \in \{1\} \cup [2, \infty)$.
5. ($d = 5$). $f(x) = cx$ for some $c > 0$.

Proof. The equivalence is obvious for $d = 1$, and was shown for $d = 2$ in Lemma 12.14. Now suppose $d \geq 3$ and $A$ is any symmetric $TP$ $2 \times 2$ matrix. By Theorem 7.4, $A$ extends to a symmetric $TP$ $d \times d$ matrix, whence $f[A]$ is $TP$. The $d = 2$ case now implies that $f$ is continuous, increasing, and positive on $(0, \infty)$, and hence extends to a continuous, non-negative, increasing function $\tilde{f} : [0, \infty) \rightarrow [0, \infty)$. By Whitney density for symmetric matrices (Proposition 6.14), $f[-]$ preserves symmetric $TN$ $2 \times 2$ matrices, whence $f$ is of the desired form for each $d \geq 3$ by Theorem 12.15. Conversely, the $d = 3$ case follows from Theorem 10.1, the $d \geq 5$ case is obvious; and for $d = 4$, given $A_{4\times4}$ symmetric $TP$ and $\alpha \geq 2$, note that all $3 \times 3$ submatrices of $A^{\alpha}$ are $TP$ by Theorem 10.1, while $\det A^{\alpha} > 0$ by Corollary 9.11. \hfill \Box

12.5. Totally non-negative Hankel matrices – entrywise preservers. We have seen that if the entrywise map $f[-]$ preserves the $m \times n$ $TP/TN$ matrices for $m, n \geq 4$, then $f$ is either constant on $(0, \infty)$ (and $f(0)$ equals either this constant or zero), or $f(x) = f(1)x$ for all $x$. In contrast, the powers $x^\alpha$ that entrywise preserve positive semidefiniteness on $\mathbb{P}_n((0, \infty))$ (for fixed $n \geq 2$) are $\mathbb{Z}_{\geq0} \cup [n-2, \infty)$. 
12. Entrywise functions preserving total positivity. Mid-convex implies continuous.

The test set of Hankel $TN$ matrices.

This discrepancy is also supported by the fact that $\mathbb{P}_n$ is closed under taking the Schur (or entrywise) product, but already the $3 \times 3$ $TN$ matrices are not. (Hence neither are the $m \times n$ $TN$ or $TP$ matrices for $m, n \geq 3$, by using completions and density arguments.) For example, $A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $B := A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are both $TN$, but $A \circ B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ has determinant $-1$, and hence cannot be $TN$.

Thus, a more refined (albeit technical) question would be to isolate and work with a class of $TN$ matrices that is a closed convex cone, and which is further closed under the Schur product. In fact such a class has already been discussed earlier: the family of Hankel $TN$ matrices (see Corollary 4.2 above). With those results in mind, and for future use, we introduce the following notation.

**Definition 12.18.** Given an integer $n > 0$, let $\mathcal{HTN}_n$ denote the $n \times n$ Hankel $TN$ matrices.

We will study later in this text, the notion of entrywise preservers of $TN$ on $\mathcal{HTN}_n$ for a fixed $n$ and for all $n$. This study turns out to be remarkably similar (and related) to the study of positivity preservers on $\mathbb{P}_n$ – which is perhaps not surprising, given Theorem 4.1.

For now, we work in the setting under current consideration: entrywise power-preservers.

**Theorem 12.19.** For $n \geq 2$ and $\alpha \in \mathbb{R}$, $x^\alpha$ entrywise preserves $TN$ on $\mathcal{HTN}_n$, if and only if $\alpha \in \mathbb{Z} \geq 0 \cup [n - 2, \infty)$.

In other words, $x^\alpha$ preserves total non-negativity on $\mathcal{HTN}_n$ if and only if it preserves positive semidefiniteness on $\mathbb{P}_n([0, \infty))$.

**Proof.** If $\alpha \in \mathbb{Z} \geq 0 \cup [n - 2, \infty)$ then we use Theorem 4.1 together with Theorem 9.3. Conversely, suppose $\alpha \in (0, n - 2) \setminus \mathbb{Z}$. We have previously shown that the ‘moment matrix’ $H := (1 + \epsilon x^{j+k-2})_{j,k=1}^n$ lies in $\mathcal{HTN}_n$ for $x, \epsilon > 0$; but if $x \neq 1$ and $\epsilon > 0$ is small then $H^\alpha \notin \mathbb{P}_n$, as shown in the proof of Theorem 9.3. (Alternately, this holds for all $\epsilon > 0$ by Theorem 9.10.) It follows that $H^\alpha \notin \mathcal{TN}_n$. \qed
13. ENTRYWISE POWERS (AND FUNCTIONS) PRESERVING POSITIVITY: II. MATRICES WITH ZERO PATTERNS.

Having completed the classification of entrywise functions preserving the $TP/TN$ matrices in any fixed size, in this part of the text and the next we restrict ourselves to understanding the entrywise functions preserving positive semidefiniteness – henceforth termed positivity – either in a fixed dimension or in all dimensions. (As mentioned above, there will be minor detours studying the related notion of entrywise preservers of $HTN_n$.)

In this section and the next, we continue to study entrywise powers preserving positivity in a fixed dimension, by refining the test set of positive semidefinite matrices. The plan for the next two sections is as follows:

1. We begin by recalling the test set $\mathbb{P}_G((0, \infty))$ associated to any graph $G$, and discussing some of the modern-day motivations in studying entrywise functions (including powers) that preserve positivity.
2. We then prove some results on general entrywise functions preserving positivity on $\mathbb{P}_G$ for arbitrary non-complete graphs. (The case of complete graphs is the subject of the remainder of the text.) As a consequence, the powers – in fact, the functions – preserving $\mathbb{P}_G((0, \infty))$ for $G$ any tree (or collection of trees) are completely classified.
3. We show how the ‘integration trick’ of FitzGerald–Horn (see the discussion around Equations (9.6) and (9.7)) extends to help classify the entrywise powers preserving other Loewner properties, including monotonicity, and in turn, super-additivity.
4. Using these results, we classify the powers preserving $\mathbb{P}_G$ for $G$ the almost complete graph (i.e., the complete graph minus any one edge).
5. We then state some recent results on powers preserving $\mathbb{P}_G$ for other $G$ (all chordal graphs; cycles), and conclude with some questions for general graphs $G$, which arise naturally from these results.

13.1. Modern-day motivations: graphical models and high-dimensional covariance estimation. As we discuss in the next part, the question of which functions preserve positivity when applied entrywise has a long history, having been studied for the better part of a century in the analysis literature. For now we explain why this question has attracted renewed attention owing to its importance in high-dimensional covariance estimation.

In modern-day scientific applications, one of the most important challenges involves understanding complex multivariate structure and dependencies. Such questions naturally arise in various domains: understanding the interactions of financial instruments, studying markers of climate parameters to understand climate patterns, and modeling gene-gene associations in cancer and cardiovascular disease, to name a few. In such applications, one works with very large random vectors $X \in \mathbb{R}^p$, and a fundamental measure of dependency that is commonly used (given a sample of vectors) is the covariance matrix (or correlation matrix) and its inverse. Unlike traditional regimes, where the sample size $n$ far exceeds the dimension of the problem $p$ (i.e., the number of random variables in the model), these modern applications – among others – involve the reverse situation: $n \ll p$. This is due to the high cost of making, storing, and working with observations, for instance; but moreover, an immediate consequence is that the corresponding covariance matrix built out of the samples $x_1, \ldots, x_n \in \mathbb{R}^p$:

$$\hat{\Sigma} := \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})^T,$$
is highly singular. (Its rank is bounded above by the sample size $n \ll p$.) This makes $\hat{\Sigma}$ a poor estimator of the true underlying covariance matrix $\Sigma$.

A second shortcoming of the sample covariance matrix has to do with zero patterns. In the underlying model, there is often additional domain-specific knowledge which leads to sparsity. In other words, certain pairs of variables are known to be independent, or conditionally independent given other variables. For instance, in probability theory one has the notion of a Markov random field, or graphical model, in which the nodes of a graph represent random variables, and edges the dependency structure between them. Or in the aforementioned climate-related application – specifically, temperature reconstruction – the temperature at one location is assumed to not influence that at another (perhaps faraway) location, at least when conditioned on the neighboring points. Such (conditional) independences are reflected in zero entries in the associated (inverse) covariance matrix. In fact, in the aforementioned applications, several models assume most of the entries ($\sim 90\%$ or more) to be zero.

However, in the observed sample covariance matrix, there is almost always some noise, as a result of which very few entries are zero. This is another of the reasons why the sample covariance is a poor estimator in modern applications.

For such reasons, it is common for statistical practitioners to regularize the sample covariance matrix (or other estimators), in order to improve its properties for a given application. Popular state-of-the-art methods involve inducing sparsity – i.e., zero entries – via convex optimization techniques that impose an $\ell^1$-penalty (since $\ell^0$-penalties are not amenable to such techniques). While these methods induce sparsity and are statistically consistent as $n, p \to \infty$, they are iterative and hence require solving computationally expensive optimization problems. In particular, they are not scalable to ultra high-dimensional data, say for $p \sim 100,000$ or $500,000$, as one often encounters in the aforementioned situations.

A recent promising alternative is to apply entrywise functions on the entries of sample covariance matrices – see e.g. \cite{11, 35, 79, 126, 127, 183, 240, 302} and numerous follow-up papers. For example, the hard and soft thresholding functions set ‘very small’ entries to zero (operating under the assumption that these often come from noise, and do not represent the most important associations). Another popular family of functions used in applications consists of entrywise powers. Indeed, powering up the entries provides an effective way in applications to separate signal from noise.

Note that these ‘entrywise’ operations do not suffer from the same drawback of scalability, since they operate directly on the entries of the matrix, and do not involve optimization-based techniques. The key question now, is to understand when such entrywise operations preserve positive semidefiniteness. Indeed, the regularized matrix that these operations yield, must serve as a proxy for the sample covariance matrix in further statistical analyses, and hence is required to be positive semidefinite.

It is thus crucial to understand when these entrywise operations preserve positivity – and in fixed dimension, since in a given application one knows the dimension of the problem. Note that while the motivation here comes from downstream applications, the heart of the issue is very much a mathematical question involving analysis on the cone $\mathbb{P}_n$.

With these motivations, the current and last few sections deal with entrywise powers preserving positivity in fixed dimension; progress on these questions impacts applied fields. At the same time, the question of when entrywise powers and functions preserve positivity, has been studied in the mathematics literature for almost a century. Thus (looking slightly ahead), in the next part and the last part of this text, we return to the mathematical advances, both classical and recent. This includes proving some of the celebrated characterization
results in this area – by Schoenberg, Rudin, Loewner and Horn, and Vasudeva – using fairly accessible mathematical machinery.

13. Entrywise powers (and functions) preserving positivity: II.
Matrices with zero patterns. 75

In this section and the next, we continue with the theme of entrywise powers and functions preserving positivity in a fixed dimension, now under additional sparsity constraints – i.e., on $\mathbb{P}_G$ for a fixed graph $G$. In this section, we obtain certain necessary conditions on general functions preserving positivity on $\mathbb{P}_G$.

As we will see in the next part, the functions preserving positive semidefiniteness on $\mathbb{P}_n$ for all $n$ (and those preserving $T\mathfrak{N}$ on $HT\mathfrak{N}_n$) for all integers $n \geq 1$ can be classified, and they are precisely the power series with non-negative coefficients:

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \quad \text{with } c_k \geq 0 \forall k.$$

This is a celebrated result of Schoenberg and Rudin. However, the situation is markedly different for entrywise preservers of $\mathbb{P}_n$ for a fixed dimension $n \geq 1$:

- For $n = 1$, clearly any $f : [0, \infty) \to [0, \infty)$ works.
- For $n = 2$, the entrywise preservers of positive semidefiniteness (or of total non-negativity) on $\mathbb{P}_2((0, \infty))$ have been classified by Vasudeva in Ind. J. Pure Appl. Math. (1979): see Theorem 12.7.
- For $n \geq 3$, the problem remains open to date.

Given the open (and challenging!) nature of the problem in fixed dimension, efforts along this direction have tended to work on refinements of the problem: either restricting the class of entrywise functions (to e.g. power functions, or polynomials as we study later), or restricting the class of matrices: to $TP/TN$ matrices, to Toeplitz matrices (done by Rudin), or Hankel $TN$ matrices, or to matrices with rank bounded above (by Schoenberg, Rudin, Loewner and Horn, and subsequent authors), or to matrices with a given sparsity pattern – i.e., $\mathbb{P}_G$ for fixed $G$. It is this last approach that we focus on, in this section and the next.

Given a (finite simple) graph $G = (V,E)$, with $V = [n] = \{1, \ldots, n\}$ for some $n \geq 1$, and a subset $0 \in I \subset \mathbb{R}$, the subset $\mathbb{P}_G(I)$ is defined to be:

$$\mathbb{P}_G(I) := \{A \in \mathbb{P}_n(I) : a_{jk} = 0 \text{ if } j \neq k \text{ and } (j,k) \notin E\}. \quad (13.1)$$

For example, when $G = A_3$ (the path graph on three nodes), $\mathbb{P}_G = \left\{ \begin{pmatrix} a & b & e \\ d & b & 0 \\ e & 0 & c \end{pmatrix} \in \mathbb{P}_3 \right\}$, and when $G = K_n$ (the complete graph on $n$ vertices), we have $\mathbb{P}_G(I) = \mathbb{P}_n(I)$.

We now study the entrywise preservers of $\mathbb{P}_G$ for a graph $G$. To begin, we extend the notion of entrywise functions to $\mathbb{P}_G$, by acting only on the ‘unconstrained’ entries:

**Definition 13.2.** Let $0 \in I \subset \mathbb{R}$. Given a graph $G$ with vertex set $[n]$, and $f : I \to \mathbb{R}$, define $f_G[-] : \mathbb{P}_G(I) \to \mathbb{R}^{n \times n}$ via

$$(f_G[A])_{jk} := \begin{cases} 0, & \text{if } j \neq k, (j,k) \notin E, \\ f(a_{jk}), & \text{otherwise.} \end{cases}$$

Here are some straightforward observations on entrywise preservers of $\mathbb{P}_G([0, \infty))$.

1. When $G$ is the empty graph, i.e., $G = (V, \emptyset)$, the functions $f$ such that $f_G[-]$ preserves $\mathbb{P}_G$ are precisely the functions sending $[0, \infty)$ to itself.
13. Entrywise powers (and functions) preserving positivity: II.

Matrices with zero patterns.

(2) When $G$ is the disjoint union of a positive number of disconnected copies of $K_2$ and isolated nodes, $\mathbb{P}_G$ consists of block diagonal matrices of the form $\oplus_{j=1}^{k} A_j$, where the $A_j$ are either $2 \times 2$ or $1 \times 1$ matrices (blocks), corresponding to copies of $K_2$ or isolated points respectively, and $\oplus$ denotes a block diagonal matrix of the form:

$$
\begin{pmatrix}
A_1 & & \\
& A_2 & \\
& & \ddots \\
& & & A_k
\end{pmatrix}.
$$

(The remaining entries are zero.) By assumption, at least one of the $A_j$ must be a $2 \times 2$ block. For such graphs, we conclude by Theorem 12.7 that $f_G([-) : \mathbb{P}_G([0, \infty)) \rightarrow \mathbb{P}_G((0, \infty))$ if and only if $f$ is non-negative, non-decreasing, multiplicatively mid-convex, and $0 \leq f(0) \leq \lim_{x \rightarrow 0^+} f(x)$.

(3) More generally, if $G$ is a disconnected union of graphs: $G = \bigsqcup_{j \in J} G_j$ then $f_G([-) : \mathbb{P}_G([0, \infty)) \rightarrow \mathbb{P}_G((0, \infty))$ if and only if the entrywise map $f_{G_j}([-) \text{ preserves } \mathbb{P}_{G_j}((0, \infty))$ for all $j$.

In light of these examples, we shall henceforth consider only connected, non-complete graphs $G$, and the functions $f$ such that $f_G([-) \text{ preserves } \mathbb{P}_G([0, \infty))$. We begin with the following necessary conditions:

**Proposition 13.3.** Let $I = [0, \infty)$ and $G$ be a connected, non-complete graph. Suppose $f : I \rightarrow \mathbb{R}$ is such that $f_G([-) : \mathbb{P}_G(I) \rightarrow \mathbb{P}_G(I)$. Then the following statements hold:

1. $f(0) = 0$.
2. $f$ is continuous on $I$.
3. $f$ is super-additive on $I$, i.e., $f(x + y) \geq f(x) + f(y)$ $\forall x, y \geq 0$.

**Remark 13.4.** In particular, $f_G([-) = f[-)$ for (non-)complete graphs $G$. Thus, following the proof of Proposition 13.3 we use $f[-)$ in the sequel.

**Proof.** Clearly $f : I \rightarrow I$. Assume that $G$ has at least 3 nodes, since for connected graphs with two nodes, the proposition is vacuous. A small observation – made by Horn, if not earlier – reveals that there exist three nodes, which we may relabel as $1, 2$, and $3$ without loss of generality, such that $2, 3$ are adjacent to $1$ but not to each other. Since $\mathbb{P}_2(I) \hookrightarrow \mathbb{P}_G(I)$ via

$$
\begin{pmatrix}
a & b \\
b & c
\end{pmatrix} \mapsto \begin{pmatrix} a & b \\
b & c
\end{pmatrix} \oplus 0_{(|V|-2) \times (|V|-2)},
$$

it follows from Theorem 12.7 that $f|_{[0, \infty)}$ is non-negative, non-decreasing, and multiplicatively mid-convex; moreover, $f|_{[0, \infty)}$ is continuous and is identically zero or never zero.

To prove (1), define

$$
B(\alpha, \beta) := \begin{pmatrix} \alpha + \beta & \alpha & \beta \\ \alpha & \alpha & 0 \\ \beta & 0 & \beta \end{pmatrix}, \quad \alpha, \beta \geq 0.
$$

Note that $B(\alpha, \beta) \oplus 0_{(|V|-3) \times (|V|-3)} \in \mathbb{P}_G(I)$. Hence $f_G[B(\alpha, \beta) \oplus 0] \in \mathbb{P}_G(I)$, from which we obtain:

$$
f_G[B(\alpha, \beta)] = \begin{pmatrix} f(\alpha + \beta) & f(\alpha) & f(\beta) \\ f(\alpha) & f(\alpha) & 0 \\ f(\beta) & 0 & f(\beta) \end{pmatrix} \in \mathbb{P}_3(I), \quad \forall \alpha, \beta \geq 0. \quad (13.5)
$$
For \( \alpha = \beta = 0 \), (13.5) yields that \( \det f_G[B(0,0)] = -f(0)^2 \geq 0 \). But since \( f \) is non-negative, it follows that \( f(0) = 0 \), proving (1).

Now if \( f|_{(0,\infty)} \equiv 0 \), the remaining assertions immediately follow. Thus, we assume in the sequel that \( f|_{(0,\infty)} \) is always positive.

To prove (2), let \( \alpha = \beta > 0 \). Then (13.5) gives:

\[
\det f_G[B(\alpha, \alpha)] \geq 0 \implies f(\alpha)^2(f(2\alpha) - 2f(\alpha)) \geq 0 \implies f(2\alpha) - 2f(\alpha) \geq 0.
\]

Taking the limit as \( t \to 0^+ \), we obtain \(-f(0^+) \geq 0 \). Since \( f \) is non-negative, \( f(0^+) = 0 = f(0) \), whence \( f \) is continuous at 0. The continuity of \( f \) on \( I \) now follows from the above discussion.

Finally, to prove (3), let \( \alpha, \beta > 0 \). Invoking (13.5) and again starting with \( \det f_G[B(\alpha, \beta)] \geq 0 \), we obtain

\[
f(\alpha)f(\beta)(f(\alpha + \beta) - f(\alpha) - f(\beta)) \geq 0 \implies f(\alpha + \beta) \geq f(\alpha) + f(\beta).
\]

This shows that \( f \) is super-additive on \((0, \infty)\); since \( f(0) = 0 \), we obtain super-additivity on all of \( I \).

Proposition 13.3 is the key step in classifying all entrywise functions preserving positivity on \( \mathbb{P}_G \) for every tree \( G \). In fact, apart from the case of \( \mathbb{P}_2 = \mathbb{P}_{K_2} \), this is perhaps the only known case (i.e., family of individual graphs) for which a complete classification of the entrywise preservers of \( \mathbb{P}_G \) is available – and proved in the next result.

Recall that a tree is a connected graph in which there is a unique path between any two vertices; equivalently, where the number of edges is one less than the number of nodes; or also where there are no cycle subgraphs. For example, the graph \( A_3 \) considered above (with \( V = \{1, 2, 3\} \) and \( E = \{(1,2), (1,3)\} \)) is a tree.

**Theorem 13.6.** Suppose \( I = [0, \infty) \) and a function \( f : I \to I \). Let \( G \) be a tree on at least 3 vertices. Then the following are equivalent:

1. \( f[-] : \mathbb{P}_G(I) \to \mathbb{P}_G(I) \).
2. \( f[-] : \mathbb{P}_T(I) \to \mathbb{P}_T(I) \) for all trees \( T \).
3. \( f[-] : \mathbb{P}_{A_3}(I) \to \mathbb{P}_{A_3}(I) \).
4. \( f \) is multiplicatively mid-convex and super-additive on \( I \).

**Proof.** Note that \( G \) contains three vertices on which the induced subgraph is \( A_3 \) (consider any induced connected subgraph on three vertices). By padding \( \mathbb{P}_{A_3} \) by zeros to embed inside \( \mathbb{P}_{G|I} \), we obtain (1) \( \implies \) (3). Moreover, that (2) \( \implies \) (1) is clear.

To prove that (3) \( \implies \) (4), note that \( K_2 \hookrightarrow A_3 \). Hence, \( f \) is multiplicatively mid-convex on \((0, \infty)\) by Theorem 12.7. By Proposition 13.3, \( f(0) = 0 \) and \( f \) is super-additive on \( I \). In particular, \( f \) is multiplicative-convex on all of \( I \).

Finally, we show that (4) \( \implies \) (2) by induction on \( n \) for all trees \( T \) with at least \( n \geq 2 \) nodes. For the case \( n = 2 \) by Theorem 12.7, it suffices to show that \( f \) is non-decreasing. Given \( \gamma \geq \alpha \geq 0 \), let \( \beta \geq 0 \) be such that \( \gamma = \alpha + \beta \). Then we have

\[
f(\gamma) = f(\alpha + \beta) \geq f(\alpha) + f(\beta) \geq f(\alpha),
\]

proving the result.

For the induction step, suppose that (2) holds for all trees on \( n \) nodes and let \( G' = (V,E) \) be a tree on \( n + 1 \) nodes. Without loss of generality, let \( V = [n + 1] = \{1, \ldots, n + 1\} \), such that node \( n + 1 \) is adjacent only to node \( n \). (Note: there always exists such a node in every tree.) Let \( G \) be the induced subgraph on the subset \([n]\) of vertices. Then, any \( A \in \mathbb{P}_{G'}(I) \)
can be written as
\[ A = \begin{pmatrix} B & b e_n \\ b e_n^T & c \end{pmatrix}_{(n+1) \times (n+1)}, \] where \( b \in \mathbb{R}, \ c \in I, \ B \in \mathbb{P}_G(I), \)
and \( e_n := ((0, 0, \ldots, 0, 1)_{1\times n})^T \) is a standard basis vector. Since \( f \) is non-negative and super-additive, \( f(0) = f(0 + 0) \geq 2f(0) \), whence \( f(0) = 0. \) If \( f \equiv 0 \), we are done. Thus, we assume that \( f \neq 0 \), whence \( f|_{(0, \infty)} \) is positive by Theorem 12.7.

If \( c = 0 \), then \( b_n c - b^2 \geq 0 \), whence \( b = 0 \) and \( f[A] = \begin{pmatrix} f[B] & 0_{n \times 1} \\ 0_{1 \times n} & 0 \end{pmatrix} \in \mathbb{P}_G \oplus 0_{1 \times 1} \) by the induction hypothesis. Otherwise \( c > 0 \), whence \( f(c) > 0. \) From the properties of Schur complements (Theorem 2.31) we obtain that \( A \) is positive semidefinite (psd) if and only if \( B - \frac{b^2}{c} E_{nn} \) is psd, where \( E_{nn} \) is the elementary \( n \times n \) matrix with \((j, k)\) entry \( \delta_{j,n}\delta_{k,n}; \) and similarly, \( f[A] \) is psd if and only if \( f[B] - \frac{f(b)^2}{f(c)} E_{nn} \) is psd.

By the induction hypothesis, we have that \( f[B] - \frac{b^2}{c} E_{nn} \) is psd. Thus, it suffices to prove that \( f[B] - \frac{f(b)^2}{f(c)} E_{nn} - f[B - \frac{b^2}{c} E_{nn}] \) is psd. Now compute:
\[ f[B] - \frac{f(b)^2}{f(c)} E_{nn} - f[B - \frac{b^2}{c} E_{nn}] = \alpha E_{nn}, \quad \text{where} \quad \alpha = f(b_n) - \frac{f(b)^2}{f(c)} - f(b_n - \frac{b^2}{c}). \]
Therefore, it suffices to show that \( \alpha \geq 0. \) But by super-additivity, we have
\[ \alpha = f(b_n) - \frac{f(b)^2}{f(c)} - f(b_n - \frac{b^2}{c}) \]
\[ = f(b_n) - \frac{b^2}{c} + \frac{b^2}{c} - \frac{f(b)^2}{f(c)} - f(b_n - \frac{b^2}{c}) \]
\[ \geq f(b_n) - \frac{b^2}{c} + f\left(\frac{b^2}{c}\right) - \frac{f(b)^2}{f(c)} - f(b_n - \frac{b^2}{c}) \]
\[ = \frac{f\left(\frac{b^2}{c}\right)}{f(c)} - f\left(\frac{b^2}{c}\right). \]
Moreover, by multiplicative mid-convexity, we obtain that \( f(\frac{b^2}{c})f(c) \geq f(b)^2. \) Hence \( \alpha \geq 0 \) and \( f[A] \) is psd, as desired.

An immediate consequence is the complete classification of entrywise powers preserving positivity on \( \mathbb{P}_T([0, \infty)) \) for \( T \) a tree.

**Corollary 13.7.** \( f(x) = x^\alpha \) preserves \( \mathbb{P}_T([0, \infty)) \) for a tree on at least three nodes, if and only if \( \alpha \geq 1. \)

The proof follows from the observation that \( x^\alpha \) is super-additive on \([0, \infty)\) if and only if \( \alpha \geq 1. \)

Loewner monotonicity and super-additivity.

We next study the set of entrywise powers preserving positivity on matrices with zero patterns. Recall the closed convex cone \( P_G([0,\infty)) \) studied in the previous section, for a (finite simple) graph \( G \). Now define

\[
\mathcal{H}_G := \{ \alpha \geq 0 : A^{\alpha} \in P_G([0,\infty)) \text{ } \forall A \in P_G([0,\infty)) \},
\]

with the convention that \( 0^0 := 1 \). Thus \( \mathcal{H}_G \) is the set of entrywise, or Hadamard, powers that preserve positivity on \( P_G \).

Observe that if \( G \subset H \) are graphs, then \( \mathcal{H}_G \supset \mathcal{H}_H \). In particular, by the FitzGerald–Horn classification in Theorem 9.3,

\[
\mathcal{H}_G \supset \mathcal{H}_{K_n} = \mathbb{Z}^\geq \cup [n-2,\infty),
\]

whenever \( G \) has \( n \) vertices. In particular, there is always a point \( \beta \geq 0 \) beyond which every real power preserves positivity on \( P_G \). We are interested in the smallest such point, which leads us to the next definition (following the FitzGerald–Horn theorem 9.3 in the special case \( G = K_n \)):

**Definition 14.3.** The critical exponent of a graph \( G \) is

\[
\alpha_G := \min\{ \beta \geq 0 : \alpha \in \mathcal{H}_G \text{ } \forall \alpha \geq \beta \}.
\]

**Example 14.4.** We saw earlier that if \( G \) is a tree (but not a disjoint union of copies of \( K_2 \)), then \( \alpha_G = 1 \); and FitzGerald–Horn showed that \( \alpha_{K_n} = n-2 \) for all \( n \geq 2 \).

In this section we are interested in closed-form expressions for \( \alpha_G \) and \( \mathcal{H}_G \). Not only is this a natural mathematical refinement of Theorem 9.3 but as discussed in Section 13.1 this moreover impacts applied fields, providing modern motivation to study the question. Somewhat remarkably, the above examples were the only known cases until very recently.

On a more mathematical note, we are also interested in understanding a combinatorial interpretation of the critical exponent \( \alpha_G \). This is a graph invariant that arises out of positivity; it is natural to ask if it is related to previously known (combinatorial) graph invariants, and more broadly, how it relates to the geometry of the graph.

We explain in this section that there is a uniform answer for a large family of graphs, which includes complete graphs, trees, split graphs, banded graphs, cycles, and other classes; and moreover, there are no known counterexamples to this answer. Before stating the results, we remark that the question of computing \( \mathcal{H}_G, \alpha_G \) for a given graph is easy to formulate, and one can carry out easy numerical simulations by running (software code) over large sets of matrices in \( P_G \) (possibly chosen randomly), to better understand which powers preserve \( P_G \). This naturally leads to accessible research problems for various classes of graphs: say triangle-free graphs, or graphs with small numbers of vertices. For instance, there is a graph on five vertices for which the critical exponent is not known!

Now on to the known results. We begin by computing the critical exponent \( \alpha_G \) – and \( \mathcal{H}_G \), more generally – for a family of graphs that turns out to be crucial in understanding several other families (split, Apollonian, banded, and in fact all chordal graphs):

**Definition 14.5.** The almost complete graph \( K_n^{(1)} \) is the complete graph on \( n \) nodes, with one edge missing.
We will choose a specific labelling of the nodes in \( K_{n}^{(1)} \); note this does not affect the set \( \mathcal{H}_{G} \) or the threshold \( \alpha_{G} \). Specifically, we set the \((1,n)\) and \((n,1)\) entries to be zero, so that 

\[ P_{K_{n}^{(1)}} \text{ consists of matrices of the form } \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \vdots & \ddots \\ 0 & \cdots & \cdots \end{pmatrix} \in \mathbb{P}_{n}. \]

The first goal is to prove:

**Theorem 14.6.** For all \( n \geq 2 \), we have \( \mathcal{H}_{K_{n}}^{(1)} = \mathcal{H}_{K_{n}} = \mathbb{Z}_{>0} \cup [n-2, \infty) \).

### 14.1. Other Loewner properties

In order to prove Theorem 14.6, we need to understand the powers that preserve super-additivity on \( n \times n \) matrices under the positive semidefinite ordering. We now define these notions, as well as a related notion of monotonicity.

**Definition 14.7.** Let \( I \subseteq \mathbb{R} \) and \( n \in \mathbb{N} \). A function \( f : I \to \mathbb{R} \) is said to be

1. **Loewner monotone** on \( \mathbb{P}_{n}(I) \) if we have \( A \geq B \geq 0_{n \times n} \implies f[A] \geq f[B] \).
2. **Loewner super-additive** on \( \mathbb{P}_{n}(I) \) if \( f[A + B] \geq f[A] + f[B] \) for all \( A, B \in \mathbb{P}_{n}(I) \).

In these definitions, we are using the Loewner ordering (or positive semidefinite ordering) on \( n \times n \) matrices: \( A \geq B \) if \( A - B \in \mathbb{P}_{n}(\mathbb{R}) \).

**Remark 14.8.** A few comments to clarify these definitions are in order. First, if \( n = 1 \) then these notions both reduce to their ‘usual’ counterparts for real functions defined on \([0, \infty)\). Second, if \( f(0) \geq 0 \), then Loewner monotonicity implies Loewner positivity. Third, a Loewner monotone function differs from – in fact is the entrywise analogue of – the more commonly studied operator monotone functions, which have the same property but for the functional calculus: \( A \geq B \geq 0 \implies f(A) \geq f(B) \geq 0 \).

Note that if \( n = 1 \) and \( f \) is differentiable, then \( f \) is monotonically increasing if and only if \( f' \) is positive. The following result generalizes this fact to powers acting entrywise on \( \mathbb{P}_{n} \), and classifies the Loewner monotone powers.

**Theorem 14.9 (FitzGerald–Horn).** Given an integer \( n \geq 2 \) and a scalar \( \alpha \in \mathbb{R} \), the power \( x^{\alpha} \) is Loewner monotone on \( \mathbb{P}_{n}([0, \infty)) \) if and only if \( \alpha \in \mathbb{Z}_{>0} \cup [n-1, \infty) \). In particular, the critical exponent for Loewner monotonicity on \( \mathbb{P}_{n} \) is \( n-1 \).

We will see in Section 15 a strengthening of Theorem 14.9 by using individual matrices from a multiparameter family, in the spirit of Jain’s Theorem 9.10 for Loewner positive powers.

**Proof.** The proof strategy is similar to that of Theorem 9.3, use the Schur product theorem for non-negative integer powers, induction on \( n \) for the powers above the critical exponent, and (the same) rank two Hankel moment matrix counterexample for the remaining powers. First, if \( \alpha \in \mathbb{N} \) and \( 0 \leq B \leq A \), then repeated application of the Schur product theorem yields:

\[ 0_{n \times n} \leq B^{\alpha} \leq B^{(\alpha-1)} \circ A \leq B^{(\alpha-2)} \circ A^{2} \leq \cdots \leq A^{\alpha}. \]

Now suppose \( \alpha \geq n-1 \). We prove that \( x^{\alpha} \) is Loewner monotone on \( \mathbb{P}_{n} \) by induction on \( n \); the base case of \( n = 1 \) is clear. For the induction step, if \( \alpha \geq n-1 \), then recall the integration trick (9.7) of FitzGerald and Horn:

\[ A^{\alpha} - B^{\alpha} = \alpha \int_{0}^{1} (A - B) \circ (\lambda A + (1 - \lambda)B)^{\alpha-1} \, d\lambda. \]

Since \( \alpha - 1 \geq n - 2 \), the matrix \( (\lambda A + (1 - \lambda)B)^{\alpha-1} \) is positive semidefinite by Theorem 9.3 and thus \( A^{\alpha} - B^{\alpha} \in \mathbb{P}_{n} \). Therefore \( A^{\alpha} \geq B^{\alpha} \), and we are done by induction.
Loewner monotonicity and super-additivity.

Finally, to show that the threshold \( \alpha = n - 1 \) is sharp, suppose \( \alpha \in (0, n - 1) \setminus \mathbb{N} \) (we leave the case of \( \alpha < 0 \) as an easy exercise). Consider again the Hankel moment matrices

\[
A(\epsilon) := H_\mu \text{ for } \mu = \delta_1 + \epsilon \delta_x, \quad B := A(0) = 1_{n \times n},
\]

where \( x, \epsilon > 0 \), \( x \neq 1 \), and \( H_\mu \) is understood to denote the leading principal \( n \times n \) submatrix of the Hankel moment matrix for \( \mu \). Clearly \( A(\epsilon) \geq B \geq 0_{n \times n} \). As above, let \( v = (1, x, \ldots, x^{n-1})^T \), so that \( A(\epsilon) = 11^T + \epsilon vu^T \). Choose a vector \( u \in \mathbb{R}^n \) that is orthogonal to \( v, v^2, \ldots, v^{\alpha(|\alpha|+1)} \), and \( u^T v^{\alpha(|\alpha|+2)} = 1 \). (Note, this is possible since the vectors \( v, v^2, \ldots, v^{\alpha(|\alpha|+2)} \) are linearly independent, forming the columns of a possibly partial generalized Vandermonde matrix.)

We claim that \( u^T (A(\epsilon)^\alpha - B^\alpha) u < 0 \) for small \( \epsilon > 0 \), which will show that \( x^\alpha \) is not Loewner monotone on \( \mathbb{P}_n([0, \infty)) \). Indeed, compute using the binomial series for \( (1 + x)^\alpha \):

\[
u^T A(\epsilon)^\alpha u - u^T B u = u^T (11^T + \epsilon vu^T)^\alpha u - u^T 11^T u
\]

\[
= u^T \cdot \sum_{k=1}^{\alpha+2} \binom{\alpha}{k} \epsilon^k u^k (v^k)^T u + u^T \cdot o(\epsilon |\alpha|+2) \cdot u
\]

\[
= \left( \frac{\alpha}{\alpha+2} \right) \epsilon^{\alpha+2} + u^T o(\epsilon |\alpha|+2) u
\]

\[
= \epsilon^{\alpha+2} \left( \binom{\alpha}{\alpha+2} + u^T o(1) \cdot u \right),
\]

and this is negative for small \( \epsilon > 0 \). (Here, \( o(\cdot) \) always denotes a matrix, as in (9.8) above.) \( \square \)

Theorem 14.9 is now used to classify the powers preserving Loewner super-additivity. Note that if \( n = 1 \) then \( x^\alpha \) is super-additive on \( \mathbb{P}_n([0, \infty)) = [0, \infty) \), if and only if \( \alpha \geq n = 1 \). The following result generalizes this to all integers \( n \geq 1 \).

**Theorem 14.10** (Guillot, Khare, and Rajaratnam). Given an integer \( n \geq 2 \) and a scalar \( \alpha \in \mathbb{R} \), the power \( x^\alpha \) is Loewner super-additive on \( \mathbb{P}_n([0, \infty)) \) if and only if \( \alpha \in \mathbb{N} \cup [n, \infty) \). Moreover, for each \( \alpha \in (0, n) \setminus \mathbb{N} \) and \( x \in (0, 1) \), for \( \epsilon > 0 \) small enough the matrix

\[
(11^T + \epsilon vu^T)^\alpha - 11^T - (\epsilon vu^T)^\alpha
\]

is not positive semidefinite, where \( v = (1, x, \ldots, x^{n-1})^T \). In particular, the critical exponent for Loewner super-additivity on \( \mathbb{P}_n \) is \( n \).

Thus, once again the same rank two Hankel moment matrices provide the desired counterexamples, for non-integer powers \( \alpha \) below the critical exponent.

**Proof.** As above, we leave the proof of the case \( \alpha < 0 \) to the reader. Next, if \( \alpha = 0 \) then super-additivity fails, since we always get \( -11^T \) from the super-additivity condition, and this is not positive semidefinite.

If \( \alpha > 0 \) is an integer, then by the binomial theorem and the Schur product theorem,

\[
(A + B)^\alpha = \sum_{k=0}^\alpha \binom{\alpha}{k} A^k \circ B^{\alpha(k-k)} \geq A^{\alpha} + B^{\alpha}, \quad \forall A, B \in \mathbb{P}_n.
\]
Next, if $\alpha \geq n$ and $A, B \in \mathbb{P}_n([0, \infty))$, then $x^{\alpha-1}$ preserves Loewner monotonicity on $\mathbb{P}_n$, by Theorem 14.9. Again use the integration trick (9.7) to compute:

$$(A + B)^{\alpha} - A^\alpha = \alpha \int_0^1 B \circ (\lambda(A + B) + (1 - \lambda)A)^{\alpha - 1} \, d\lambda \geq \alpha \int_0^1 B \circ (\lambda B)^{\alpha - 1} \, d\lambda = B^\alpha.$$ 

The final case is if $\alpha \in (0, n) \setminus \mathbb{Z}$. As above, we fix $x > 0$, $x \neq 1$, and define

$$v := (1, x, \ldots, x^{n-1})^T, \quad A(\epsilon) := \epsilon v v^T (\epsilon > 0), \quad B := A(0) = 1_{n \times n}.$$ 

Clearly $A(\epsilon), B \geq 0_{n \times n}$. Now since $\alpha \in (0, n)$, the vectors $v, v^{\alpha_1}, \ldots, v^{\alpha_k}$ are linearly independent (since the matrix with these columns is part of a generalized Vandermonde matrix). Thus, we may choose $u \in \mathbb{R}^n$ that is orthogonal to $v, \ldots, v^{\alpha_1}$ (if $\alpha \in (0, 1)$, this is vacuous) and such that $u^T v^{\alpha} = 1$. Now compute as in the previous proof, using the binomial theorem:

$$(A(\epsilon) + B)^{\alpha} - A(\epsilon)^\alpha - B^\alpha = \sum_{k=1}^{\lfloor \alpha \rfloor} \binom{\alpha}{k} \epsilon^k v^{\alpha_k} (v^{\alpha_k})^T - \epsilon^\alpha v^{\alpha_0} (v^{\alpha_0})^T + o(\epsilon^\alpha);$$

the point here is that the last term shrinks at least as fast as $\epsilon^{\lfloor \alpha \rfloor + 1}$. Hence by choice of $u$,

$$u^T ((A(\epsilon) + B)^{\alpha} - A(\epsilon)^\alpha - B^\alpha) u = -\epsilon^\alpha + u^T \cdot o(\epsilon^\alpha) \cdot u,$$

and this is negative for small $\epsilon > 0$. Hence $x^\alpha$ is not Loewner super-additive even on rank-one matrices in $\mathbb{P}_n((0, 1))$. \hfill $\square$

**Remark 14.11.** The above proof goes through for arbitrary $v = (v_1, \ldots, v_n)^T$, consisting of pairwise distinct positive real scalars.

### 14.2. Entrywise powers preserving $\mathbb{P}_G$.

We now apply the above results to compute the set of entrywise powers preserving positivity on $\mathbb{P}_{K_n^{(1)}}$ (the almost complete graph).

**Proof of Theorem 14.6.** The result is straightforward for $n = 2$, so we assume henceforth that $n \geq 3$. It suffices to show that $\mathcal{H}_{K_n^{(1)}} \subset \mathbb{Z}_{\geq 0} \cup [n - 2, \infty)$, since the reverse inclusion follows from Theorem 9.3 via (14.2). Fix $x > 0$, $x \neq 1$, and define

$$v := (1, x, \ldots, x^{n-3})^T \in \mathbb{R}^{n-2}, \quad A(\epsilon) := \begin{pmatrix} 1 & 1^T & 0 \\ 1 & 1 + \epsilon vv^T & \sqrt{\epsilon} v \\ 0 & \sqrt{\epsilon} v^T & 1 \end{pmatrix}_{n \times n}, \quad \epsilon > 0.$$ 

Note that if $p, q > 0$ are scalars, $a, b \in \mathbb{R}^{n-2}$ are vectors, and $B$ is an $(n - 2) \times (n - 2)$ matrix, then using Schur complements (Theorem 2.31),

$$\begin{pmatrix} p & a^T \\ a & B \\ 0 & b^T \\ 0 & q \end{pmatrix} \in \mathbb{P}_n \iff \begin{pmatrix} p & a^T \\ a & B - q^{-1}bb^T \end{pmatrix} \in \mathbb{P}_{n-1} \quad \iff \quad B - p^{-1}aa^T - q^{-1}bb^T \in \mathbb{P}_{n-2}. \tag{14.12}$$

Applying this to the matrices $A(\epsilon)^{\alpha}$, we obtain:

$$A(\epsilon)^{\alpha} \in \mathbb{P}_n \iff (11^T + \epsilon vv^T)^{\alpha} = (11^T)^{\alpha} - (\epsilon vv^T)^{\alpha} \in \mathbb{P}_{n-2}.$$ 

For small $\epsilon > 0$, Theorem 14.10 now shows that $\alpha \in \mathbb{Z}_{\geq 0} \cup [n - 2, \infty)$, as desired. \hfill $\square$
Loewner monotonicity and super-additivity.

In the remainder of this section, we present what is known about the critical exponents $\alpha_G$ and power-preserver sets $\mathcal{H}_G$ for various graphs. We do not provide proofs below, instead referring the reader to the 2016 paper “Critical exponents of graphs” by Guillot, Khare, and Rajaratnam in J. Combin. Th., Ser. A.

The first family of graphs is that of chordal graphs, and it subsumes not only the complete graphs, trees, and almost complete graphs (for all of which we have computed $\mathcal{H}_G, \alpha_G$ with full proofs above), but also other graphs including split, banded, and Apollonian graphs, which are discussed presently.

**Definition 14.13.** A graph is **chordal** if it has no induced cycle of length $\geq 4$.

Chordal graphs are important in many fields. They are also known as triangulated graphs, decomposable graphs, and rigid circuit graphs. They occur in spectral graph theory, but also in network theory, optimization, and Gaussian graphical models. Chordal graphs play a fundamental role in areas including maximum likelihood estimation in Markov random fields, perfect Gaussian elimination, and the matrix completion problem.

The following is the main result of the aforementioned 2016 paper, and it computes $\mathcal{H}_G$ for every chordal graph.

**Theorem 14.14.** Let $G$ be a chordal graph with $n \geq 2$ nodes and at least one edge. Let $r$ denote the largest integer such that $K_r$ or $K_r^{(1)} \subset G$. Then $\mathcal{H}_G = \mathbb{Z}^{\geq 0} \cup [r - 2, \infty)$.

The point of the theorem is that the study of powers preserving positivity reduces solely to the geometry of the graph, and can be understood combinatorially rather than through matrix analysis (given the theorem). While we do not prove this result here, we remark that the proof crucially uses Theorem 14.6 and the ‘clique-tree decomposition’ of a chordal graph.

As applications of Theorem 14.14, we mention several examples of chordal graphs $G$ and their critical exponents $\alpha_G$: by the preceding theorem, the only powers below $\alpha_G$ that preserve positivity on $\mathbb{P}_G$ are the non-negative integers.

1. The complete and almost complete graph on $n$ vertices are chordal, and have critical exponent $n - 2$.
2. Trees are chordal, and have critical exponent 1.
3. Let $C_n$ denote a cycle graph (for $n \geq 4$), which is clearly not chordal. Any minimal planar triangulation $G$ of $C_n$ is chordal, and one can check that $\alpha_G = 2$ regardless of the size of the original cycle graph or the locations of the additional chords drawn.
4. A banded graph with bandwidth $d > 0$ is a graph with vertex set $[n] = \{1, \ldots, n\}$, and edges $(j, j + x)$ for $x \in \{-d, -d + 1, \ldots, d - 1, d\}$ such that $1 \leq j, j + x \leq n$. One checks (combinatorially) that $\alpha_G = \min(d, n - 2)$ if $n > d$.
5. A split graph consists of a clique $C \subset V$ and an independent (i.e., pairwise disconnected) set $V \setminus C$, whose nodes are connected to various nodes of $C$. Split graphs are an important class of chordal graphs because it can be shown that the proportion of (connected) chordal graphs with $n$ nodes that are split graphs, grows to 1 as $n \to \infty$. Theorem 14.14 implies that for a split graph $G$, $\alpha_G = \max(|C| - 2, \max \deg(V \setminus C))$.
6. Apollonian graphs are constructed as follows: start with a triangle as the first iteration. Given any iteration, which is a subdivision of the original triangle by triangles, choose an interior point of any of these ‘atomic’ triangles, and connect it to the three vertices of the corresponding atomic triangle. This increases the number of atomic
triangles by 2 at each step. If $G$ is an Apollonian graph on $n \geq 3$ nodes, one shows that $\alpha_G = \min(2, n - 2)$.

It is natural to ask what is known for non-chordal graphs. We mention one such result, also shown in the aforementioned 2016 paper.

**Theorem 14.15.** Let $C_n$ denote the cycle graph on $n$ vertices (which is non-chordal for $n \geq 4$). Then $\mathcal{H}_{C_n} = [1, \infty)$ for all $n \geq 4$.

Remarkably, this is the same combinatorial recipe as for chordal graphs (in Theorem 14.14)!

We end with some questions, which can be avenues for further research into this nascent topic.

**Question 14.16.**

(1) Compute the critical exponent (and set of powers preserving positivity) for graphs other than the ones discussed above. In particular, compute $\alpha_G$ for all $G = (V,E)$ with $|V| \leq 5$.

(2) For all graphs $G$ with known critical exponent, the critical exponent turns out to be $r - 2$, where $r$ is the largest integer such that $G$ contains either $K_r$ or $K_r^{(1)}$. Does the same result hold for all graphs?

(3) In fact more is true in all known cases: $\mathcal{H}_G = \mathbb{Z}_{\geq 0} \cup [\alpha_G, \infty)$. Is this true for all graphs?

(4) ‘Taking a step back’: can one show that the critical exponent of a graph is an integer (perhaps without computing it explicitly)?

(5) Does the critical exponent have connections to – or can it be expressed in terms of – other, purely combinatorial graph invariants?

(6) More generally, is it possible to classify the entrywise functions that preserve positivity on $\mathbb{P}_G$, for $G$ a non-complete, non-tree graph? Perhaps the simplest example is a cycle $G = C_n$. 
15. Loewner convexity. Single matrix encoders of entrywise power-preservers of
Loewner properties.

In the preceding section, we classified the entrywise powers that are Loewner monotone
or superadditive on all matrices in \( \mathbb{P}_n([0, \infty)) \). In this section, we will similarly classify the
Loewner convex powers on \( \mathbb{P}_n \) (a notion that is not yet defined above). We first show that
there exist individual matrices which encode the sets of entrywise powers preserving Loewner
positivity, monotonicity, and convexity.

15.1. Matrices encoding Loewner positive powers. We begin with the Loewner positive
powers, and recall Jain’s Theorem [9,10] above. This was strengthened by Jain in her 2020
paper in *Adv. Oper. Th.*:

**Theorem 15.1** (Jain). Suppose \( n \geq 2 \) is an integer, and \( x_1, x_2, \ldots, x_n \) are pairwise distinct
real numbers such that \( 1 + x_j x_k > 0 \) for all \( j, k \). Let \( C := (1 + x_j x_k)^{n_{j,k}} \). Then \( C^{\alpha} \) is
positive semidefinite if and only if \( \alpha \in \mathbb{Z}_{\geq 0} \cup [n - 2, \infty) \).

The remainder of this subsection is devoted to proving Theorem 15.1, beginning with the
following notation.

**Definition 15.2.** Given a real tuple \( x = (x_1, \ldots, x_n) \), define \( A_x := -\infty \) if all \( x_j \leq 0 \), and
\( -1/\max_j x_j \) otherwise. Similarly, define \( B_x := \infty \) if all \( x_j \geq 0 \), and \( -1/\min_j x_j \) otherwise.

Here are a few properties of \( A_x, B_x \); the details are straightforward verifications, left to
the reader.

**Lemma 15.3.**

1. Suppose \( x \in \mathbb{R} \). Then \( 1 + yx > 0 \) for a real scalar \( y \), if and only if \( \text{sgn}(x)y \in (-1/|x|, \infty) \), where we set \( 1/|x| := \infty \) if \( x = 0 \).
2. Given real scalars \( x_1, \ldots, x_n, y \), we have \( 1 + yx_j > 0 \) for all \( j \) if and only if \( y \in (A_x, B_x) \).
3. \( A_{-x} = -B_x \) and \( A_x < 0 < B_x \) for all \( x \in \mathbb{R}^n \).

**Proof.** The first part follows by using \( x = \text{sgn}(x)|x| \); the second follows by intersecting
the solution-intervals for each \( x_j \). The final assertion is a consequence of the first two. \( \square \)

We now show an intermediate result that resembles Descartes’ rule of signs (Lemma 5.2),
except that it holds for powers of \( 1 + ux \) rather than \( \exp(ux) \):

**Proposition 15.4.** Fix a real number \( r \), an integer \( n \geq 1 \), and two real tuples \( c = (c_1, \ldots, c_n) \neq 0 \) and \( x = (x_1, \ldots, x_n) \) with pairwise distinct \( x_j \). Then the function

\[
\varphi_{x,c,r} : (A_x, B_x) \to \mathbb{R}, \quad u \mapsto \sum_{j=1}^n c_j (1 + ux_j)^r
\]

is either identically zero or has at most \( n - 1 \) zeros, counting multiplicities.

**Proof.** If \( r = 0 \) then the result is straightforward, so assume henceforth that \( r \neq 0 \). Let
\( s = S^r(c) \) denote the number of sign changes in the nonzero tuple \( c = (c_1, \ldots, c_n) \). Now
claim more strongly that the number of zeros is at most \( s \). The proof is by induction on
\( n \geq 1 \) and then on \( s \in [0, n - 1] \). The base case of \( n = 1 \) is clear; and for any \( n \), the base
case of \( s = 0 \) is also immediate. Thus, suppose \( n \geq 2 \) and \( s \geq 1 \), and suppose the result
holds for all tuples \( c \) of length \( < n \), as well as for all tuples \( c \in \mathbb{R}^n \setminus \{0\} \) with at most \( s - 1 \)
sign changes. Because of this, we may also assume that all $c_j$ are nonzero. Let $s$ denote the number of sign changes in $c$.

We begin by relabelling the $x_j$ if required, to lie in increasing order:

$$x_1 < \cdots < x_n.$$  

Now suppose there does not exist $0 < k \leq n$ such that $c_{k-1}c_k < 0$ (there is a sign change here in the tuple $c$, which is also relabelled corresponding to the $x_j$ if required) and $x_k > 0$. Then $x_k \leq 0$, so that $x_{k-1} < 0$. Now work with the tuples $-x$ and $c' := (c_n, \ldots, c_1)$, i.e.,

$$-x_n < -x_{n-1} < \cdots < -x_1, \quad \varphi_{-x,c',r}(v) := \sum_{j=1}^{n} c_j(1 - ux_j)^{r}.$$  

Here $v = -u \in (-B_X, -A_X) = (A_{-X}, B_{-X})$ by Lemma 15.3, so the result for $\varphi_{-x,c',r}(v)$ would prove that for $\varphi_{x,c,r}(u)$. Using this workaround if needed, it follows that there exists $1 \leq k \leq n$ such that $c_{k-1}c_k < 0$ and $x_k > 0$. In particular, there exists $v > 0$ such that

$$1 - vx_n < \cdots < 1 - vx_k < 0 < 1 - vx_{k-1} < \cdots < 1 - vx_1.$$  

Define

$$\psi : (A_X, B_X) \to \mathbb{R}, \quad \psi(u) := \sum_{j=1}^{n} c_j(1 - vx_j)(1 + ux_j)^{r-1}.$$  

By choice of $v$, the sequence $(c_1(1 - vx_1), \ldots, c_n(1 - vx_n))$ has precisely $s - 1$ sign changes, so $\psi$ has at most $s - 1$ zeros. Now for $u \in (A_X, B_X)$, we compute:

$$\psi(u) = \sum_{j=1}^{n} c_j(1 + ux_j - (u + v)x_j)(1 + ux_j)^{r-1}$$

$$= \varphi_{x,c,r}(u) - (u + v) \sum_{j=1}^{n} c_j x_j (1 + ux_j)^{r-1} = -(u + v)^{r+1} \frac{h'(u)}{r},$$

where $h(u) := (u + v)^{-r} \varphi_{x,c,r}(u)$ and $r \neq 0$. Since $x_k > 0$, we obtain from above:

$$u \in (A_X, B_X) \implies u + v > A_X + v = v - x_n^{-1} > v - x_k^{-1} > 0.$$  

Thus $u + v$ is well-defined and positive on $(A_X, B_X)$, whence $h : (A_X, B_X) \to \mathbb{R}$ is well-defined as well. From above, $\psi$ has at most $s - 1$ zeros on $(A_X, B_X)$, whence so does $h'$. But then by Rolle’s theorem, $h$ has at most $s$ zeros on $(A_X, B_X)$, and hence, so does $\varphi_{x,c,r}$. \hfill $\square$

A second intermediate result involves a homotopy argument that will be crucial in proving Theorem 15.1.

**Proposition 15.5.** Let $n \geq 2$ be an integer and fix real scalars

$$x_1 < \cdots < x_n, \quad 0 < y_1 < \cdots < y_n$$  

such that $1 + x_j x_k > 0$ for all $j, k$. Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, the ‘linear homotopies’ (between $x_j$ and $\epsilon y_j$)

$$x_j^{(\epsilon)}(t) := x_j + t(\epsilon y_j - x_j), \quad t \in [0, 1]$$  

all satisfy

$$1 + x_j^{(\epsilon)}(t) x_k^{(\epsilon)}(t) > 0, \quad \forall j, k = 1, \ldots, n, \quad t \in [0, 1].$$
Notice that this is not immediate, nor even true for ‘arbitrary’ \( \epsilon \). For instance, suppose \( n = 2, \epsilon = 1 \), and \((y_1, y_2) = (1, 2)\). If say \((x_1, x_2) = (-199, 0)\) then an easy ‘completion of squares’ shows that the assertion is false at most times in the homotopy:

\[
1 + x_1^{(1)}(t)x_2^{(1)}(t) \leq 0, \quad \forall t \in \left[ \frac{398}{800} - \frac{1}{20}\sqrt{\frac{398^2}{40^2} - 1}, \frac{398}{800} + \frac{1}{20}\sqrt{\frac{398^2}{40^2} - 1} \right] \supset [0.0026, 0.9924].
\]

Similarly, if say \((x_1, x_2) = (-8.5, 0.1)\), then

\[
1 + x_1^{(1)}(t)x_2^{(1)}(t) \leq 0, \quad \forall t \in \left[ \frac{8 - \sqrt{61}}{19}, \frac{8 + \sqrt{61}}{19} \right] \supset [0.01, 0.8321].
\]

Thus, the \( \epsilon \) in the statement is crucial for the result to hold.

**Proof of Proposition 15.5.** We begin with three observations; in all of them, \( x_j(t) = x_j^{(\epsilon)}(t) \) for some fixed \( \epsilon > 0 \), and all \( j, t \). First: we have \( x_1(t) < \cdots < x_n(t) \) for all \( t \in [0, 1] \).

Second: if \( x_1 \geq 0 \), then it is clear that \( x_j(t) \geq 0 \) for all \( j \in \{1, n\} \) and all \( t \in [0, 1] \), and the result is immediate. Thus, we suppose henceforth that \( x_1 < 0 \).

Third: if there exist integers \( j < k \) and a time \( t \in [0, 1] \) such that \( 1 + x_j(t)x_k(t) \leq 0 \), then \( x_j(t) < 0 < x_k(t) \), whence \( x_1(t) < 0 < x_n(t) \). One then verifies easily that \( 1 + x_1(t)x_n(t) \leq 1 + x_j(t)x_k(t) \leq 0 \).

Thus, for every choice of \( x_j, y_j \) as above, with \( x_1 < 0 \), it suffices to produce \( \epsilon_0 > 0 \) such that \( 1 + x_1^{(\epsilon)}(t)x_n^{(\epsilon)}(t) > 0 \) for all \( t \in (0, 1) \) and all \( 0 < \epsilon \leq \epsilon_0 \). There are two cases, depending on the sign of \( x_n \):

**Case 1:** \( x_n \geq 0 \). Then \( x_n < 1/|x_1| \). Now claim that \( \epsilon_0 := 1/(|x_1|y_n) \) works; to see this, compute using the known inequalities on \( x_j, y_j \):

\[
1 + x_1^{(\epsilon)}(t)x_n^{(\epsilon)}(t) = 1 + (\epsilon y_1 + (1-t)x_1)(\epsilon y_n + (1-t)x_n) \\
> 1 + (1-t)x_1(\epsilon y_n + (1-t)x_n) > 1 + (1-t)x_1(\epsilon y_n + (1-t)/|x_1|),
\]

where the (final) two inequalities are strict because \( t \in (0, 1) \). Continuing, this last expression equals

\[
=1 - (1-t)^2 + t(1-t)\epsilon y_n x_1 \geq t(2-t - (1-t)\epsilon_0 y_n |x_1|) = t > 0.
\]

**Case 2:** \( x_n < 0 \). For \( \epsilon \) close to 0, define

\[
f(\epsilon) := 1 - \frac{\epsilon^2(x_n y_1 - x_1 y_n)^2}{4(\epsilon y_n - x_n)(\epsilon y_1 - x_1)}.\]

This function is continuous in \( \epsilon \) and \( f(0) > 0 \). Hence there exists \( \epsilon_0 > 0 \) such that \( f(\epsilon) > 0 \) for all \( 0 \leq \epsilon \leq \epsilon_0 \).

We show that \( \epsilon_0 \) satisfies the desired properties. Let \( 0 < \epsilon \leq \epsilon_0 \), and set

\[
t_j^{(\epsilon)} := -x_j/(\epsilon y_j - x_j), \quad 1 \leq j \leq n.
\]

Note that \( x_j^{(\epsilon)}(t) \) is negative, zero, or positive when \( t < t_j^{(\epsilon)}, t = t_j^{(\epsilon)}, t > t_j^{(\epsilon)} \) respectively. Also note by the observations above that

\[
0 < t_n^{(\epsilon)} < t_{n-1}^{(\epsilon)} < \cdots < t_1^{(\epsilon)} < 1.
\]

Now if \( 0 \leq t \leq t_1^{(\epsilon)} \) or \( t_n^{(\epsilon)} \leq t \leq 1 \), then \( x_1^{(\epsilon)}(t), x_n^{(\epsilon)}(t) \) are both non-positive or non-negative respectively. Hence \( 1 + x_1^{(\epsilon)}(t)x_n^{(\epsilon)}(t) \geq 1 \), as desired.

Next, suppose \( t \in (t_1^{(e)}, t_1^{(e)}) \). Observe that
\[
x_j^{(e)}(t) = t e y_j + (1 - t)x_j = (t - t_1^{(e)})(e y_j - x_j), \quad \forall j \in [1, n], \ t \in [0, 1].
\]

Now using the AM–GM inequality, the proof is complete:
\[
1 + x_1^{(e)}(t)x_n^{(e)}(t) = 1 + (t - t_1^{(e)})(t - t_n^{(e)})(e y_1 - x_1)(e y_n - x_n)
\geq 1 - \frac{1}{4}(t_1^{(e)} - t_n^{(e)})^2(e y_1 - x_1)(e y_n - x_n) = f(\epsilon) > 0. \quad \square
\]

With all of the above results at hand, we can proceed:

Proof of Theorem 15.1. For ease of exposition, we break up the proof into steps.

**Step 1:** The first observation is slightly more general than applies here: Suppose \( y_1, \ldots, y_n \) are distinct real numbers such that \( 1 + y_k x_j > 0 \) for all \( 1 \leq j, k \leq n \). Let \( S := (1 + y_k x_j) \) and let \( r \) be real. If \( r \in \{0, 1, \ldots, n - 2\} \) then \( S^{or} \) has rank \( r + 1 \), else \( S^{or} \) is non-singular.

Indeed, for \( r \in \{0, \ldots, n - 2\} \), we have
\[
S^{or} = (W_y^{(r)})^T D W_x^{(r)}, \quad \text{where} \quad W_x^{(r)} := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
: & : & \cdots & : \\
x_1^r & x_2^r & \cdots & x_n^r
\end{pmatrix},
\]
and with \( D_{(r+1) \times (r+1)} \) a diagonal matrix with entries \((0), (1), \ldots, (r)\). Now \( D \) is non-singular and \( W_y^{(r)}, W_x^{(r)} \) are submatrices of Vandermonde matrices, hence of full rank – so \( S^{or} \) has rank \( r + 1 \).

Now suppose \( r \neq 0, 1, \ldots, n - 2 \) and \( S^{or} c^T = 0 \) for some tuple \( c = (c_1, \ldots, c_n) \neq 0 \). Rewrite \( S^{or} c^T = 0 \) to obtain:
\[
\varphi_{x,c,r}(y_k) = \sum_{j=1}^n c_j (1 + y_k x_j)^r = 0, \quad k = 1, \ldots, n.
\]

Since \( 1 + y_k x_j > 0 \) for all \( j, k \), we have \( y_k \in (A_x, B_x) \) by Lemma 15.3(2). So \( \varphi_{x,c,r} \equiv 0 \) on \((A_x, B_x)\), by Proposition 15.4 whence \( \varphi_{x,c,r}(0) = 0 \) for \( l = 0, 1, \ldots, n - 1 \) by Lemma 15.3(3). Reformulating this,
\[
\sum_{j=1}^n (c_j (r - 1) \cdots (r - l + 1)) x_j^l = 0, \quad \forall l = 0, 1, \ldots, n - 1.
\]

i.e., \( W_x^{(n-1)} D_{c^T} = 0 \), where \( D \) is a diagonal matrix with diagonal entries \( 1, r, r(r-1), \ldots, r(r-1) \cdots (r - n + 2) \). Now \( W_x^{(n-1)} \) is a non-singular (Vandermonde) matrix, as is \( D \) by choice of \( r \). Thus the tuple \( c \) is zero, i.e. \( S^{or} \) is non-singular.

**Step 2:** We now turn to the proof of the theorem. First if \( \alpha \in \mathbb{Z}^\geq 0 \cup \{n - 2, \infty\} \), then \( C^{o\alpha} \) is positive semidefinite by Theorem 9.3. Next, if \( \alpha < 0 \) then the leading \( 2 \times 2 \) principal minor of \( C(x)^{o\alpha} \) is easily seen to be negative. Finally, suppose \( \alpha \in (0, n - 2) \setminus \mathbb{Z} \). Given a real vector \( y \in \mathbb{R}^n \), define \( C(y) := l_{n \times n} + y y^T \). Now apply the previous step, fixing \( r = \alpha \) and and all \( y_j = x_j \). Thus \( \det C(x)^{o\alpha} \neq 0 \) for every \( x \) with all \( 1 + x_j x_k > 0 \).

Now recall the proof of Theorem 9.3 and the subsequent remarks. Thus if \( x_0 < \min_j x_j \), then there exists \( \epsilon_1 > 0 \) such that \( C(\sqrt{\epsilon}(x - x_01))^1 \) has a negative eigenvalue for all \( 0 < \epsilon < \epsilon_1 \). Now consider the linear homotopy

\[
x(t) := (1 - t + t\sqrt{\epsilon_2})x - t\sqrt{\epsilon_2}x_01, \quad t \in [0, 1]
\]

which goes from \( x \) to \( \sqrt{\epsilon_2}(x - x_01) \) as \( t \) goes from 0 to 1. Here we choose \( \epsilon_2 \in (0, \epsilon_1) \) such that \( \epsilon_0 := \sqrt{\epsilon_2} \) satisfies the conclusions of Proposition 15.5 for \( x_j \) as above and \( y_j := x_j - x_0 \) (suitably relabelled to be in increasing order if desired).

Again applying the previous step (for the same fixed \( r = \alpha \)), \( \det C(x(t))^{1/\alpha} \neq 0 \) for all \( t \in [0, 1] \). We also know that \( C(x(1))^{1/\alpha} \) has a negative eigenvalue, whence \( \lambda_{\min}(C(x(1))) < 0 \). Now claim by the ‘continuity of eigenvalues’ that \( \lambda_{\min}(C(x(t))) < 0 \) for all \( t \in [0, 1] \), and in particular, at \( t = 0 \). This is shown in the next step, and completes the proof.

**Step 3:** The claim in the preceding paragraph follows from the following more general fact: Suppose \( C : [0, 1] \to \mathbb{P}_n(C) \) is a continuous Hermitian matrix-valued function such that each \( C(t) \) is non-singular. If \( C(1) \) has a negative eigenvalue, then so does \( C(t) \) for all \( t \in [0, 1] \).

It remains to show this statement, and we use a simpler approach (than the full power of ‘continuity of roots’ in Proposition 8.7 above) to do so. Let \( X := \{ t \in [0, 1] : \lambda_{\min}(C(t)) \geq 0 \} \). Since the cone of positive semidefinite matrices is closed, it follows that \( X \) is closed in \([0, 1]\). Now the claim follows from the sub-claim that \( X^c := [0, 1] \setminus X \) is also closed: since \([0, 1]\) is connected and \( 1 \in X^c \), it follows that \( X^c = [0, 1] \) and so \( 0 \in X^c \) as desired.

To show the sub-claim, let \( \|C(t)\| := \left( \sum_{j,k=1}^n |c_{jk}|^2 \right)^{1/2} \). It is clear using the Cauchy–Schwarz inequality that all eigenvalues of \( C(t) \) lie in \([-\|C(t)\|, \|C(t)\|] \). Now given a sequence \( t_n \in X^c \) that converges to \( t_0 \in [0, 1] \), all entries of \( \{C(t_n) : n \geq 1\} \) lie in a compact set, whence so do the corresponding ‘minimum eigenvalues’ \( \lambda_{\min}(C(t_n)) \). Pick a subsequence \( n_k \) such that \( \lambda(C(t_{n_k})) \to \lambda_0 \). Now \( \lambda_0 \leq 0 \). Also pick a unit length eigenvector \( v_0 \) of \( C(t_n) \) corresponding to the eigenvalue \( \lambda_{\min}(C(t_n)) \); as the unit complex sphere is compact, there is a further subsubsequence \( n_{ki} \) such that \( v_{n_{ki}} \to v_0 \) as \( l \to \infty \), with \( v_0 \) also of unit norm.

With these choices at hand, write the equation \( C(t_{n_{ki}})v_{n_{ki}} = \lambda_{\min}(C(t_{n_{ki}}))v_{n_{ki}} \) and let \( l \to \infty \). Then \( C(t_0)v_0 = \lambda_0v_0 \), with \( \lambda_0 \leq 0 \). It follows from the hypotheses that \( \lambda_{\min}(C(t_0)) < 0 \), and the proof is complete.

**15.2. Matrices encoding Loewner monotone powers.** We now turn to Loewner monotonicity (recall Theorem 14.9). The next result – again by Jain in 2020 – shows that akin to the previous subsection, there exist individual matrices that encode the Loewner monotone powers:

**Corollary 15.6.** Suppose \( n \geq 1 \) and \( x_1, \ldots, x_n \) are distinct nonzero real numbers such that \( 1 + x_jx_k > 0 \) for all \( j, k \). Let \( x := (x_1, \ldots, x_n)^T \) and \( \alpha \in \mathbb{R} \). Then \( (1_{n \times n} + xx^T)^{\alpha} \geq 1_{n \times n} \) if and only if \( \alpha \in \mathbb{Z}^{\geq 0} \cup [n - 1, \infty) \), if and only if \( x^\alpha \) is Loewner monotone on \( \mathbb{P}_n((0, \infty)) \).

Notice that here we cannot take \( x_j = 0 \) for any \( j \); if for instance \( x_n = 0 \) and we call the matrix \( X \), then the monotonicity of \( X^\alpha \) over \( 1_{n \times n} \) is actually equivalent to the positivity of \( X^\alpha \), and so the result fails to hold.

**Proof.** If \( \alpha \in \mathbb{Z}^{\geq 0} \cup [n - 1, \infty) \), then Theorem 14.9 implies \( x^\alpha \) is Loewner monotone on \( \mathbb{P}_n((0, \infty)) \), whence on \( X := 1_{n \times n} + xx^T \geq 1_{n \times n} \). Conversely, suppose \( x^\alpha \) is Loewner
monotone on $X \geq 1_{n \times n}$. Let $x' := (x^T, 0)^T \in \mathbb{R}^{n+1}$; then
\[ \tilde{X} := 1_{(n+1) \times (n+1)} + x'(x')^T = \begin{pmatrix} X & 1 \\ 1^T & 1 \end{pmatrix} \]
satisfies the hypotheses of Theorem 15.1. Now by the theory of Schur complements (Theorem 2.31), $X^\alpha \geq 1_{n \times n}$ if and only if $X^\infty \in \mathbb{P}_{n+1}$. But this is if and only if $\alpha \in \mathbb{Z}^{\geq 0} \cup [n-1, \infty)$, by Theorem 15.1. \hfill \square

15.3. Loewner convex powers, and individual matrices encoding them. Finally, we turn to the entrywise powers preserving Loewner convexity.

Definition 15.7. Let $I \subset \mathbb{R}$ and $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is said to be Loewner convex on a subset $V \subset \mathbb{P}_n(I)$ if $f[\lambda A + (1 - \lambda)B] \leq \lambda f[A] + (1 - \lambda) f[B]$, whenever $A \geq B \geq 0_{n \times n}$ lie in $V$, and $\lambda \in [0, 1]$.

The final theorem in this section classifies the Loewner convex powers, in the spirit of Theorems 9.3, 14.9, and 14.10. It shows in particular that there is a critical exponent for convexity as well. It also shows the encoding of these powers by individual matrices, in the spirit of Corollary 15.6.

Theorem 15.8 (Loewner convex entrywise powers). Fix an integer $n \geq 1$ and a scalar $\alpha \in \mathbb{R}$. The following are equivalent:

1. The entrywise power $x^\alpha$ is Loewner convex on $\mathbb{P}_n([0, \infty))$.
2. Fix distinct nonzero real scalars $x_1, \ldots, x_n$ such that $1 + x_jx_k > 0$ for all $j, k$. Then $x^\alpha$ is Loewner convex on $A := (1 + x_jx_k)^{n}_{j,k=1} \geq B = 1_{n \times n} \geq 0$.
3. $\alpha \in \mathbb{Z}^{\geq 0} \cup [n, \infty)$.

In particular, the critical exponent for Loewner convexity on $\mathbb{P}_n$ is $n$.

Thus, there are rank 2 Hankel $TN$ matrices (with $x_j = x_0^j$ for $x_0 \in (0, \infty) \setminus \{1\}$), which encode the Loewner convex powers.

To prove this result, we require a preliminary result connecting Loewner convex functions with Loewner monotone ones. We also prove a parallel result connecting monotone maps to positive ones.

Proposition 15.9. Suppose $n \geq 1$ and $A \geq B \geq 0_{n \times n}$ are positive semidefinite matrices with real entries such that $A - B = uu^T$, with $u$ having all nonzero entries. Fix any open interval $I$ containing the entries of $A, B$, and suppose $f : I \to \mathbb{R}$ is differentiable.

1. Then both notions of the ‘interval’ $[B, A]$ agree, i.e.,
\[ \{C : B \leq C \leq A\} = \{\lambda A + (1 - \lambda)B : \lambda \in [0, 1]\}. \]
2. If $f[-]$ is Loewner monotone on the interval $[B, A]$, then $f'[-]$ is Loewner positive on $[B, A]$. The converse holds for arbitrary matrices $0 \leq B \leq A$.
3. If $f[-]$ is Loewner convex on the interval $[B, A]$, then $f'[-]$ is Loewner monotone on $[B, A]$. The converse holds for arbitrary matrices $0 \leq B \leq A$.

Proof.

1. That the left hand side contains the right is straightforward. Conversely, if $B \leq C \leq A$ then $0 \leq C - B \leq A - B$, which has rank one. Write $A - B = uu^T$; now if $u^Tv = 0$ then $\|\sqrt{C - B} \cdot v\|^2 = v^T(C - B)v = 0$, whence $(C - B)v = 0$. This inclusion of kernels shows that $\ker(C - B)$ has codimension at most one. If $C \neq B$, then $\ker(C - B) = \ker u^T$ and $C - B$ has column space spanned by $u$, by the orthogonality...
of eigenspaces of Hermitian matrices for different eigenvalues. Thus $C - B = \lambda uu^T$ for some $\lambda \in (0, 1]$. But then,

$$C = B + \lambda(A - B) = \lambda A + (1 - \lambda)B,$$

as desired.

(2) Suppose $f'$ is Loewner positive on $[B, A]$ (for any $0 \leq B \leq A$). We show that $f$ is monotone on this interval by using the integration trick (9.7) (see also Theorem 9.12). Indeed,

$$f[A] - f[B] = \int_0^1 (A - B) \circ f'\left[\lambda A + (1 - \lambda)B\right] \, d\lambda.$$

By assumption and the Schur product theorem, the integrand is positive semidefinite, whence so is the left-hand side, as desired. The same argument applies to show that $f[A_\lambda] \geq f[A_\mu]$, where $A_\lambda := \lambda A + (1 - \lambda)B$ and $0 \leq \mu \leq \lambda \leq 1$.

Conversely, suppose $f[A_\lambda] \geq f[A_\mu]$ for all $0 \leq \mu \leq \lambda \leq 1$. Now given $\lambda \in (0, 1)$, let $0 < h \leq 1 - \lambda$; then $f[A_{\lambda+h}] \geq f[A_\lambda]$, so

$$0 \leq \lim_{h \to 0^+} \frac{1}{h} \left(f[A_{\lambda+h}] - f[A_\lambda]\right)$$

$$= \lim_{h \to 0^+} \frac{1}{h} \left(f[\lambda A + (1 - \lambda)B + h(A - B)] - f[\lambda A + (1 - \lambda)B]\right)$$

$$= f'[A_\lambda] \circ (A - B).$$

By the assumptions, $(A - B)^{\mu(\lambda)}$ is also a rank-one positive semidefinite matrix with all nonzero entries, so taking the Schur product, we have $f'[A_\lambda] \geq 0$ for all $\lambda \in (0, 1)$, as desired.

(3) Suppose $f'$ is Loewner monotone on $[B, A]$ (for any $0 \leq B \leq A$). As above, we use the integration trick to show that $f$ is convex, beginning with:

$$f[(A + B)/2] - f[B] = \frac{1}{2} \int_0^1 (A - B) \circ f'\left[\lambda \frac{A + B}{2} + (1 - \lambda)B\right] \, d\lambda, \quad \text{(15.10)}$$

$$\frac{f[A] + f[B]}{2} - f[B] = \frac{f[A] - f[B]}{2} = \frac{1}{2} \int_0^1 (A - B) \circ f'\left[\lambda A + (1 - \lambda)B\right] \, d\lambda.$$

Now by the hypotheses on $f'$ and the Schur product theorem, it follows that

$$(A - B) \circ f'[\lambda A + (1 - \lambda)B] \geq (A - B) \circ f'\left[\lambda \frac{A + B}{2} + (1 - \lambda)B\right].$$

This, combined with (15.10), yields $f[(A + B)/2] \leq (f[A] + f[B])/2$. One now proves by induction – first on $N$ and then on $j$ – that

$$f\left[\frac{j}{2^N} A + \left(1 - \frac{j}{2^N}\right) B\right] \leq \frac{j}{2^N} f[A] + \left(1 - \frac{j}{2^N}\right) f[B], \quad \forall N \geq 1, \ 1 \leq j \leq 2^N.$$

Now given any $\lambda \in [0, 1]$, approximate $\lambda$ by a sequence of dyadic rationals $j/2^N$ as above, and use the continuity of $f[-]$ and the preceding inequality to conclude that $f[-]$ is Loewner convex on $\{B, A\}$. The same argument can be adapted to show that $f[-]$ is Loewner convex on $\{A_\lambda, A_\mu\}$ as in the preceding part.
Conversely, since we have $f[\lambda A + (1 - \lambda)B] \leq \lambda f[A] + (1 - \lambda)f[B]$ for $0 \leq \lambda \leq 1$, it follows for $\lambda \in (0, 1)$ that
\[
\frac{f[B + \lambda(A - B)] - f[B]}{\lambda} \leq f[A] - f[B],
\]
\[
\frac{f[A + (1 - \lambda)(B - A)] - f[A]}{1 - \lambda} \leq f[B] - f[A].
\]
Letting $\lambda \to 0^+$ and $\lambda \to 1^-$, respectively, yields:
\[
(A - B) \circ f'[B] \leq f[A] - f[B], \quad (B - A) \circ f'[A] \leq f[B] - f[A].
\]
Adding these yields: $(A - B) \circ (f'[A] - f'[B]) \geq 0$. Finally, as above $A - B$ has all entries nonzero, so has a rank-one Schur-inverse; taking the Schur product with this yields $f'[A] \geq f'[B]$. As above, the same argument can be adapted to show that $f'[-]$ is Loewner monotone on $\{A_\lambda, A_\mu\}$.

Finally, we have:

**Proof of Theorem 15.8** Clearly (1) $\Rightarrow$ (2). Now setting $f(x) := x^\alpha$, both (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) follow via Proposition 15.9 (3) and Corollary 15.6.

The above results on individual (pairs of) matrices encoding the entrywise powers preserving Loewner positivity, monotonicity, and convexity naturally lead to the following question.

**Question 15.11.** Given an integer $n \geq 1$, do there exist matrices $A, B \in \mathbb{P}_n(\mathbb{R})$ such that $(A + B)^{\alpha} \geq A^\alpha + B^\alpha$ if and only if $\alpha \in \mathbb{Z}_{\geq 0} \cup [n, \infty)$? In other words, for each $n \geq 1$, is the set of Loewner super-additive entrywise powers on $\mathbb{P}_n(\mathbb{R})$ (see Theorem 14.10) also encoded by a single pair of matrices?

We provide a partial solution here, for the matrices studied above. Suppose $u = (u_1, \ldots, u_n)^T \in (0, \infty)^n$ has pairwise distinct coordinates. Let $A := 1_{n \times n}$, $B := uu^T$. By computations similar to the proof of Theorem 14.6 it follows that
\[
(A + B)^{\alpha} \geq A^\alpha + B^\alpha \iff \begin{pmatrix} 1 + uu^T & u & 1_{n \times 1} \\ u^T & 1 & 0 \\ 1_{1 \times n} & 0 & 1 \end{pmatrix}^{\alpha} \in \mathbb{P}_{n+2}. \tag{15.12}
\]

When $\alpha = 1$, denote the matrix on the right in (15.12) by $M(u)$; this is easily seen to have rank three. Now considering any diagonal entry of the inequality on the left in (15.12), we obtain $\alpha \geq 1$. By Theorem 14.10 and Remark 14.11, it suffices to now assume $\alpha \in (1, n) \setminus \mathbb{Z}$. But if $\alpha \in (1, n - 1)$, then Corollary 15.6 yields the desired result, by considering the leading principal $(n + 1) \times (n + 1)$ submatrices in the preceding inequality on the left in (15.12).

Thus, it remains to show that for $\alpha \in (n - 1, n)$, the matrix $M(u)^{\alpha} \notin \mathbb{P}_n$ for all $u$ with pairwise distinct, positive coordinates. In fact, we claim that it suffices to show for $\alpha \in (n - 1, n)$ that $M(u)^{\alpha}$ is non-singular for all $u$ as above. Indeed, this would imply by a different homotopy argument that $M(\sqrt{\epsilon}u)^{\alpha}$ is non-singular for all $\epsilon > 0$; but for small enough $\epsilon > 0$ the proof of Theorem 14.10 shows that $M(\sqrt{\epsilon}u)^{\alpha}$ has a negative eigenvalue, whence the same holds for all $\epsilon > 0$ by the continuity of eigenvalues (or see Step 3 in the proof of Theorem 15.1).

In light of this discussion, we end this section with a question closely related to the preceding question above.

**Question 15.13.** Suppose $n \geq 2$ and $\alpha \in (n - 1, n)$. Is $M(u)^{\alpha} \in \mathbb{P}_{n+2}$, where $u \in (0, \infty)^n$ has pairwise distinct coordinates, and $M(u)$ is as in (15.12)?
The entrywise calculus was initiated by Schur in 1911, when he defined the map $f[A]$ (which he called $f^0(A)$), in the same paper [258] where he proved the Schur product theorem. Schur also proved the first result involving entrywise maps; see also page cxii of the survey [176].

Theorem 9.3 and Lemma 9.5, which help classify the Loewner positive powers, are by FitzGerald and Horn [89]. The use of the rank-two Hankel matrices in the proof, as well as the powers preserving positive definiteness in Corollary 9.11, are by Fallat–Johnson–Sokal [82]. The ‘individual’ matrices encoding Loewner positive powers were constructed in Theorem 9.10 by Jain [146]; the ‘extension principle’ in Theorem 9.12 is by Khare–Tao [163]. Also note the related papers by Bhatia–Elsner [34], Hiai [129], and Guillot–Khare–Rajaratnam [113], which study ‘two-sided’ entrywise powers: $R \to R$, and which of these are Loewner positive on $\mathbb{P}_n(\mathbb{R})$ for fixed $n \geq 1$.

The historical account of Descartes’ rule of signs in Theorem 10.3 is taken in part from Jameson’s article [148]; once again the proof of this result – via Rolle’s theorem – can be attributed to Laguerre [176]. The proof provided of Theorem 10.1 – which classifies the powers preserving totally positive $3 \times 3$ matrices – is by Fallat–Johnson–Sokal [82]. The Cauchy functional equation (see Remark 11.5) has been studied in numerous papers; we mention Banach [12] and Sierpiński [265], both papers appearing in the same volume of Fund. Math. in 1920. The results in Section 11 prior to Remark 11.5 are shown by Fallat–Johnson–Sokal [82], or essentially follow from there. The results following Remark 11.5 are by Belton–Guillot–Khare–Putinar [23].

Section 12.1 on the continuity of bounded mid-convex functions is taken from the book of Roberts–Varberg [237]; the first main Theorem 12.2 there closely resembles a result by Ostrowski [213], while the second Theorem 12.4 was proved independently by Blumberg [37] and Sierpiński [265]. Theorem 12.7, classifying the Loewner positive maps on $\mathbb{P}_2((0, \infty))$ and $\mathbb{P}_2([0, \infty))$, is essentially by Vasudeva [278]; see also [20, 116] for the versions that led to the present formulation. The short argument for mid-convexity implying continuity, at the end of that proof, is due to Hiai [129]. The remainder of Section 12 classifying all entrywise maps preserving $TN$ and $TP$ in each fixed dimension (for all matrices and for all symmetric matrices) is taken from Belton–Guillot–Khare–Putinar [23]. The one exception is Theorem 12.19, which classifies the powers preserving $TN$ on Hankel $n \times n$ matrices, and it is from [20].

Section 13 on the entrywise functions preserving positivity on $\mathbb{P}_G$ for $G$ a non-complete graph (and specifically, a tree) follows Guillot–Khare–Rajaratnam [114]. Section 14 classifying the Loewner positive powers on $\mathbb{P}_G$ for $G$ a chordal graph – and hence computing the critical exponent of $G$ for Loewner positivity – is due to Guillot–Khare–Rajaratnam [115] (see also the summary in [117]). The two exceptions are Theorem 14.9 by FitzGerald–Horn [89] and Theorem 14.10 by Guillot–Khare–Rajaratnam [113], which classify the Loewner monotone and super-additive powers on $\mathbb{P}_n((0, \infty))$, respectively. Also see [151] for a survey of critical exponents in the matrix theory literature.

Theorem 15.1 and Corollary 15.6 about individual matrices encoding the Loewner positive and monotone powers respectively, are by Jain [147]. The arguments proving these results are taken from [147] (some of these are variants of results in her earlier works) and from Khare [162] – specifically, the homotopy argument in Proposition 15.5 which differs from Jain’s similar assertion in [147] and avoids SSR (strictly sign regular) matrices. Finally, the classification of the Loewner convex powers on $\mathbb{P}_n$ (i.e., the equivalence $1 \iff 3$ in Theorem 15.8) was shown by Hiai [129], via the intermediate Proposition 15.9; see also...
Guillot–Khare–Rajaratnam [113] for a rank-constrained version of Theorem 15.8. The further equivalence to Theorem 15.8(2), which obtains individual matrix-encoders of the Loewner convex powers, is taken from [162].
Part 3:
Entrywise functions preserving positivity in all dimensions
Part 3: Entrywise functions preserving positivity in all dimensions


In this part, we take a step back and explore the foundational results on entrywise preservers of positive semidefiniteness – as well as the rich history that motivated these results.

16.1. History of the problem. In the forthcoming sections, we will answer the question: “Which functions, when applied entrywise, preserve positivity (positive semidefiniteness)?”

(Henceforth we use the word ‘positivity’ to denote ‘positive semidefiniteness’.) This question has been the focus of a concerted effort and significant research activity over the past century. It began with the Schur product theorem (1911) and the following consequence:

Lemma 16.1 (Pólya–Szegö, 1925). Suppose a power series \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) is convergent on \( I \subset \mathbb{R} \) and \( c_k \geq 0 \) for all \( k \geq 0 \). Then \( f[-] : \mathbb{P}_n(I) \to \mathbb{P}_n(\mathbb{R}) \) for all \( n \geq 1 \).

Proof. By induction and the Schur product theorem 3.10, \( f(x) = x^k \) preserves positivity on \( \mathbb{P}_n(\mathbb{R}) \) for all integers \( k \geq 0 \) and \( n \geq 1 \), and hence sends \( \mathbb{P}_n(I) \) to \( \mathbb{P}_n(\mathbb{R}) \). From this the lemma follows, using that \( \mathbb{P}_n(\mathbb{R}) \) is a closed convex cone. \( \square \)

With Lemma 16.1 in hand, Pólya and Szegő asked if there exist any other functions that preserve positivity on \( \mathbb{P}_n \) for all \( n \geq 1 \). A negative answer would essentially constitute the converse result to the Schur product theorem; and indeed, this was shown by Schur’s student Schoenberg (who is well-known for inventing splines), for continuous functions:

Theorem 16.2 (Schoenberg, 1942). Suppose \( I = [-1,1] \) and \( f : I \to \mathbb{R} \) is continuous. The following are equivalent:

1. The entrywise map \( f[-] \) preserves positivity on \( \mathbb{P}_n(I) \) for all \( n \geq 1 \).
2. The function \( f \) equals a convergent power series \( \sum_{k=0}^{\infty} c_k x^k \) for all \( x \in I \), with the Maclaurin coefficients \( c_k \geq 0 \) for all \( k \geq 0 \).

Schoenberg’s 1942 paper (in Duke Math. J.) is well-known in the analysis literature. In a sense, his Theorem 16.2 is the (harder) converse to the Schur product theorem, i.e. Lemma 16.1, which is the implication (2) \( \Rightarrow \) (1). Some of these points were discussed in Section 13.2.

Schoenberg’s theorem can also be stated for \( I = (-1,1) \). In this setting, the continuity hypothesis was subsequently removed from assertion (1) by Rudin, who moreover showed that in order to prove assertion (2) in Theorem 16.2, one does not need to work with the full test set \( \bigcup_{n \geq 1} \mathbb{P}_n(I) \). Instead, it is possible to work with only the low-rank Toeplitz matrices:

Theorem 16.3 (Rudin, 1959). Suppose \( I = (-1,1) \) and \( f : I \to \mathbb{R} \). Then the assertions in Schoenberg’s theorem 16.2 are equivalent on \( I \), and further equivalent to:

3. \( f[-] \) preserves positivity on the Toeplitz matrices in \( \mathbb{P}_n(I) \) of rank \( \leq 3 \), for all \( n \geq 1 \).

Schoenberg’s theorem also has a ‘one-sided’ variant, over the semi-axis \( I = (0,\infty) \):

Theorem 16.4 (Vasudeva, 1979). Suppose \( I = (0,\infty) \) and \( f : I \to \mathbb{R} \). Then the two assertions of Schoenberg’s theorem 16.2 are equivalent on \( I \) as well.

Our goal in this part of the text is to prove stronger versions of the theorems of Schoenberg and Vasudeva. Specifically, we will (i) remove the continuity hypothesis, and (ii) work with severely reduced test sets in each dimension, consisting of only the Hankel matrices of rank at most 3. For instance, we will show Theorem 16.3, but with the word ‘Toeplitz’ replaced by
‘Hankel’. Similarly, we will show a strengthening of Theorem 16.4 using totally non-negative Hankel matrices of rank at most 2. These results are stated and proved in the coming sections.

16.2. Digression: the complex case. In the aforementioned 1959 paper in Duke Math. J., Rudin made some observations about the complex case a la Pólya–Szegő, and presented a conjecture. First observe that the Schur product theorem holds for complex Hermitian matrices as well, with the same proof coming from the spectral theorem:

“If $A, B$ are $n \times n$ complex (Hermitian) positive semidefinite matrices, then so is $A \circ B$.”

As a consequence, every monomial $z \mapsto z^m$ preserves positivity on $\mathbb{P}_n(\mathbb{C})$ for all integers $m \geq 0$ and $n \geq 1$. (Here $\mathbb{P}_n(\mathbb{C})$ comprises the complex Hermitian matrices $A_{n \times n}$ such that $u^*Au \geq 0$ for all $u \in \mathbb{C}^n$.) But more is true: the (entrywise) conjugation map also preserves positivity on $\mathbb{P}_n(\mathbb{C})$ for all $n \geq 1$. Now using the Schur product theorem, the functions

$$z \mapsto z^m(z)^k, \quad m, k \geq 0$$

each preserve positivity on $\mathbb{P}_n(\mathbb{C})$, for all $n \geq 1$. Since $\mathbb{P}_n(\mathbb{C})$ is easily seen to be a closed convex cone as well, Rudin observed that if a series

$$f(z) = \sum_{m,k \geq 0} c_{m,k} z^m(z)^k, \quad \text{with } c_{m,k} \geq 0,$$

is convergent on a disc $D(0, \rho) \subset \mathbb{C}$, then $f[\cdot]$ entrywise preserves positivity on $\mathbb{P}_n(D(0, \rho))$ for $n \geq 1$. Rudin conjectured that there are no other preservers. This was proved soon after:

**Theorem 16.5** (Herz, Ann. Inst. Fourier, 1963). Suppose $I = D(0,1) \subset \mathbb{C}$ and $f : I \to \mathbb{C}$. The following are equivalent:

1. The entrywise map $f[\cdot]$ preserves positivity on $\mathbb{P}_n(I)$ for all $n \geq 1$.
2. $f$ is of the form $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m(z)^k$ on $I$, with $c_{m,k} \geq 0$ for all $m, k \geq 0$.

For completeness, we also point out [214] for a non-commutative variant of the Schur product and Schoenberg’s theorem.

As a final remark, the Schoenberg–Rudin / Vasudeva / Herz results are reminiscent of an earlier, famous result, by Loewner in the parallel (and well-studied) setting of the matrix functional calculus. Namely, given a complex Hermitian matrix $A$ with eigenvalues in a real interval $(a, b)$, a function $f : (a, b) \to \mathbb{R}$ acts on $A$ as follows: let $A = UDU^*$ be a spectral decomposition of $A$; then $f(A) := U f(D) U^*$, where $f(D)$ is the diagonal matrix with diagonal entries $f(d_{jj})$. Now Loewner showed in Math. Z. (1934) even before Schoenberg:

**Theorem 16.6** (Loewner). Let $-\infty \leq a < b \leq \infty$, and $f : (a, b) \to \mathbb{R}$. The following are equivalent:

1. $f$ is matrix monotone: if $A \leq B$ are square matrices with eigenvalues in $(a, b)$, then $f(A) \leq f(B)$.
2. There exist real constants $p \geq 0$ and $q$, and a finite measure $\mu$ on $\mathbb{R} \setminus (a, b)$, such that

$$f(x) = q + px + \int_{\mathbb{R} \setminus (a,b)} \frac{1 + xy}{y - x} \, d\mu(y).$$

3. There exists a function $\tilde{f}$ that is analytic on $(\mathbb{C} \setminus \mathbb{R}) \setminus (a, b)$, such that (a) $f \equiv \tilde{f}_{\mid (a,b)}$ and (b) if $\Re z > 0$ then $\Im f(z) > 0$.

Notice similar to the preceding results, the emergence of analyticity from the dimension-free preservation of a matrix property. This is also the case with a prior result of Rudin with Helson, Kahane, and Katznelson in Acta Math. in 1958, which directly motivated Rudin’s 1959 paper discussed above.
16.3. Origins of positive matrices: Menger, Fréchet, Schoenberg, and metric geometry. In this subsection and the next two, we study some of the historical origins of positive (semi)definite matrices. This class of matrices of course arises as Hessians of twice-differentiable functions at local minima; however, the branch of early 20th century mathematics that led to the development of positivity preservers is metric geometry. More precisely, the notion of a metric space – emerging from the works of Fréchet and Hausdorff – and isometric embeddings of such structures into Euclidean and Hilbert spaces, spheres, hyperbolic and homogeneous spaces, were studied by Schoenberg, Bochner, and von Neumann among others; and it is this work that led to the study of matrix positivity and its preservation.

Definition 16.7. A metric space is a set \( X \) together with a metric \( d : X \times X \to \mathbb{R} \), satisfying:

1. Positivity: \( d(x, y) \geq 0 \) for all \( x, y \in X \), with equality if and only if \( x = y \).
2. Symmetry: \( d(x, y) = d(y, x) \) for all \( x, y \in X \).
3. Triangle inequality: \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).

In this section, we will state and prove three results by Schoenberg, which explain his motivations in studying positivity and its preservers, and serve to illustrate the (by now well-explored) connection between metric geometry and matrix positivity. We begin with a sample result on metric embeddings, shown by Fréchet in Math. Ann. in 1910: If \( (X, d) \) is a metric space with \( |X| = n + 1 \), then \( (X, d) \) isometrically embeds into \( (\mathbb{R}^n, \ell_\infty) \).

Such results led to exploring which metric spaces isometrically embed into Euclidean spaces. Specifically, in Menger’s 1931 paper in Amer. J. Math., and Fréchet’s 1935 paper in Ann. of Math., the authors explored the following question: Given integers \( n, r \geq 1 \), characterize the tuples of \( \binom{n+1}{2} \) positive real numbers that can denote the distances between the vertices of an \( (n+1) \)-simplex in \( \mathbb{R}^r \) but not in \( \mathbb{R}^{r-1} \). In other words, given a finite metric space \( X \), what is the smallest \( r \), if any, such that \( X \) isometrically embeds into \( \mathbb{R}^r \)?

In his 1935 paper in Ann. of Math., Schoenberg gave an alternate characterization of all such ‘admissible’ tuples of distances. This characterization used... matrix positivity!

Theorem 16.8 (Schoenberg, 1935). Fix integers \( n, r \geq 1 \), and a finite set \( X = \{x_0, \ldots, x_n\} \) together with a metric \( d \) on \( X \). Then \( (X, d) \) isometrically embeds into \( \mathbb{R}^r \) (with the Euclidean distance/norm) but not into \( \mathbb{R}^{r-1} \) if and only if the \( n \times n \) matrix

\[
A := (d(x_0, x_j)^2 + d(x_0, x_k)^2 - d(x_j, x_k)^2)_{j,k=1}^n
\]

is positive semidefinite of rank \( r \).

Proof. If \( (X, d) \) isometrically embeds into \( (\mathbb{R}^r, \| \cdot \|) \), then

\[
d(x_0, x_j)^2 + d(x_0, x_k)^2 - d(x_j, x_k)^2 = \|x_0 - x_j\|^2 + \|x_0 - x_k\|^2 - \|x_0 - x_j\| - \|x_0 - x_k\|\]
\[= 2\langle x_0 - x_j, x_0 - x_k \rangle.
\]

But then the matrix \( A \) in (16.9) is the Gram matrix of a set of vectors in \( \mathbb{R}^r \), and hence is positive semidefinite. Here and below in this section, we use Theorem 2.4 and Proposition 2.14 (and their proofs) without further reference. Thus \( A = B^T B \), where the columns of \( B \) are \( x_0 - x_j \in \mathbb{R}^r \). But then \( A \) has rank at most the rank of \( B \), whence at most \( r \). Since \( (X, d) \) does not embed in \( \mathbb{R}^{r-1} \), by the same argument \( A \) has rank precisely \( r \).

Conversely, suppose the matrix \( A \) in (16.9) is positive semidefinite of rank \( r \). First consider the case when \( r = n \), i.e. \( A \) is positive definite. By Theorem 2.4, \( \frac{1}{2} A = B^T B \) for a square invertible matrix \( B \). Thus left-multiplication by \( B \) sends the \( r \)-simplex with vertices \( 0, e_1, \ldots, e_r \) to an \( r \)-simplex, where \( e_j \) comprise the standard basis of \( \mathbb{R}^r \).
Now claim that the assignment \( x_0 \mapsto 0, \ x_j \mapsto Be_j \) for \( 1 \leq j \leq r \), is an isometry : \( X \to \mathbb{R}^r \) whose image, being the vertex set of an \( r \)-simplex, necessarily cannot embed inside \( \mathbb{R}^{r-1} \). Indeed, computing distances gives:

\[
\begin{align*}
    d(Be_j,0)^2 &= \|B^T e_j\|^2 = \frac{1}{2} e_j^T A e_j = \frac{1}{2} a_{jj} = d(x_0, x_j)^2, \\
    d(Be_j,Be_k)^2 &= \|Be_j - Be_k\|^2 = \frac{a_{jj} + a_{kk} - a_{jk}}{2} \\
    &= d(x_0, x_j)^2 + d(x_0, x_k)^2 - (d(x_0, x_j)^2 + d(x_0, x_k)^2 - d(x_j, x_k)^2) \\
    &= d(x_j, x_k)^2.
\end{align*}
\]

Since all distances here are non-negative, taking square roots of the above equations proves the claim (and with it, the theorem for \( r = n \)).

Next suppose \( r < n \). Then \( A = B^T PB \) for some invertible matrix \( B \), where \( P \) is the projection operator \( \begin{pmatrix} I_{r \times r} & 0 \\ 0 & 0_{(n-r) \times (n-r)} \end{pmatrix} \). Let \( \Delta := \{PB e_j, \ 1 \leq j \leq n\} \sqcup \{0\} \) denote the projection under \( P \) of the vertices of an \((n + 1)\) simplex. Repeating the above proof shows that the map : \( x_0 \mapsto 0, \ x_j \mapsto PBe_j \) for \( 1 \leq j \leq r \), is an isometry from \( X \) onto \( \Delta \). By construction, \( \Delta \) lies in the image of the projection \( P \), whence in a copy of \( \mathbb{R}^r \). But being the image under \( P \) of the vertex set of an \( n \)-simplex, \( \Delta \) cannot lie in a copy of \( \mathbb{R}^{r-1} \) (else so would its span, which is all of \( P(\mathbb{R}^n) \cong \mathbb{R}^r \)). \( \square \)

We end this part with an observation. A (real symmetric square) matrix \( A_{(n+1)\times(n+1)} \) is said to be **conditionally positive semidefinite** if \( u^T A u \geq 0 \) whenever \( \sum_j u_j = 0 \). Such matrices are also studied in the literature (though not as much as positive semidefinite matrices). The following lemma reformulates Theorem \[16.8\] into the conditional positivity of a related matrix:

**Lemma 16.11.** Let \( X = \{x_0, \ldots, x_n\} \) be a finite set equipped with a metric \( d \). Then the matrix \( A_{n \times n} \) as in \[16.9\] is positive semidefinite if and only if the \((n + 1) \times (n + 1)\) matrix

\[
A' := (-d(x_j, x_k)^2)_{j,k=0}^n \tag{16.12}
\]

is conditionally positive semidefinite.

In particular, Schoenberg’s papers in the 1930s feature both positive semidefinite matrices (Theorem \[16.8\] above) and conditionally positive semidefinite matrices (Theorem \[16.15\]). Certainly the former class of matrices were a popular and recurrent theme in the analysis literature, with contributions from Carathéodory, Hausdorff, Hermite, Nevanlinna, Pick, Schur, and many others.

**Proof of Lemma \[16.11\]**. Let \( u_1, \ldots, u_n \in \mathbb{R} \) be arbitrary, and set \( u_0 := -(u_1 + \cdots + u_n) \). Defining \( u := (u_1, \ldots, u_n)^T \) and \( u' := (u_0, \ldots, u_n)^T \), we compute using that the diagonal entries of \( A' \) are zero:

\[
(u')^T A' u' = \sum_{k=1}^n \left( \sum_{j=1}^n u_j \right) d(x_0, x_k)^2 u_k + \sum_{j=1}^n u_j d(x_j, x_0)^2 \left( \sum_{k=1}^n u_k \right) - \sum_{j,k=1}^n u_j d(x_j, x_k)^2 u_k \\
= \sum_{j,k=1}^n u_j u_k \left( d(x_0, x_k)^2 + d(x_j, x_0)^2 - d(x_j, x_k)^2 \right) \\
= u^T A u, \quad \text{for all } u \in \mathbb{R}^n.
\]

This proves the result. \( \square \)
16.4. Origins of positivity preservers: Schoenberg, Bochner, and positive definite functions. We continue with our historical journey, this time into the origins of the entrywise calculus on positive matrices. As Theorem 16.8 and Lemma 16.11 show, applying entrywise the function \(-x^2\) to any distance matrix \((d(x_j, x_k))_{j,k=0}^n\) from Euclidean space yields a conditionally positive semidefinite matrix \(A'\).

It is natural to want to remove the word ‘conditionally’ from the above result. Namely: which entrywise maps send distance matrices to positive semidefinite (psd) matrices? These are precisely the positive definite functions:

**Definition 16.13.** Given a metric space \((X,d)\), a function \(f: [0, \infty) \to \mathbb{R}\) is positive definite on \(X\) if for any finite set of points \(x_1, x_2, \ldots x_n \in X\), the matrix \(f((d(x_j, x_k))_{j,k=1}^n)\) is psd.

Note here that positive definite functions are not positivity preservers, since no distance matrix is positive semidefinite unless all \(x_j\) are equal (in which case we get the zero matrix). On a different note, given any metric space \((X,d)\), the positive definite functions on \(X\) form a closed convex cone, by Lemma 3.1.

In arriving at Theorem 16.2 Schoenberg was motivated by metric geometry – as we just studied – as well as the study of positive definite functions. This latter was also of interest to other mathematicians in that era: Bochner, Pólya, and von Neumann, to name a few. In fact, positive definite functions are what originally led to the development of the entrywise calculus. Note that Bochner – and previously Carathéodory, Herglotz, Mathias, and others – studied functions on groups \(G\) that were positive definite in the ‘more standard’ sense – namely, where in the above definition \(f: G \to \mathbb{C}\), and one substitutes \(d(x_j, x_j)\) by \(x_j^{-1}x_k\).


We now present – from this paper – another characterization by Schoenberg of metric space embeddings into a Euclidean space \(\mathbb{R}^r\), this time via positive definite functions. This requires a preliminary observation involving the positive definiteness of an even kernel:

**Lemma 16.14.** Given \(\sigma > 0\), the Gaussian kernel \(T_{G_{\sigma}}(x,y) := \exp(-\sigma \|x - y\|^2)\) – in other words, the function \(\exp(-\sigma x^2)\) – is positive definite on \(\mathbb{R}^r\) for all \(r \geq 1\).

**Proof.** Observe that the case of \(\mathbb{R}^r\) for general \(r\) follows from the \(r = 1\) case, via the Schur product theorem. In turn, the \(r = 1\) case is a consequence of Pólya’s Lemma 6.8 above. □

The following result of Schoenberg in *Trans. Amer. Math. Soc.* (1938) relates metric space embeddings with this positive definiteness of the Gaussian kernel:

**Theorem 16.15** (Schoenberg, 1938). A finite metric space \((X,d)\) with \(X = \{x_0, \ldots, x_n\}\) embeds isometrically into \(\mathbb{R}^r\) for some \(r > 0\) (which turns out to be at most \(n\)), if and only if the \((n+1) \times (n+1)\) matrix with \((j,k)\) entry

\[\exp(-\lambda^2 d(x_j, x_k)^2), \quad 0 \leq j, k \leq n\]

is positive semidefinite, along any sequence of nonzero scalars \(\lambda\) converging to 0 (equivalently, for all \(\lambda \in \mathbb{R}\)).

**Proof.** Clearly if \((X,d)\) embeds isometrically into \(\mathbb{R}^r\), then identifying the \(x_j\) with their images in \(\mathbb{R}^r\), it follows by Lemma 16.14 that the matrix with \((j,k)\) entry

\[\exp(-\lambda^2 \|x_j - x_k\|^2) = \kappa(\lambda x_j, \lambda x_k)\]

is positive semidefinite for all \(\lambda \in \mathbb{R}\).
Conversely, let \( \lambda_m \to 0 \) with all \( \lambda_m \) nonzero. From the positivity of the exponentiated distance matrices for \( \lambda_m \), it follows for any vector \( u := (u_0, \ldots, u_n)^T \in \mathbb{R}^{n+1} \) that
\[
\sum_{j,k=0}^{n} u_j u_k \exp(-\lambda_m^2 d(x_j, x_k)^2) \geq 0.
\]
Expanding into Taylor series and interchanging the infinite sum with the two finite sums,
\[
\sum_{l=0}^{\infty} (-\lambda_m^2)^l \sum_{j,k=0}^{n} u_j u_k d(x_j, x_k)^{2l} \geq 0, \quad \forall m \geq 1.
\]
Suppose we restrict to the vectors \( u' \) satisfying: \( \sum_{j=0}^{n} u_j = 0 \). Then the \( l = 0 \) term vanishes. Now dividing throughout by \( \lambda_m^2 \) and taking \( m \to \infty \), the “leading term” in \( \lambda_m \) must be non-negative. It follows that if \( A' := (-d(x_j, x_k)^2)_{j,k=0}^{n} \), then \( (u')^T A' u' \geq 0 \) whenever \( \sum_j u_j = 0 \).

By Lemma 16.11 and Theorem 16.8, \((X, d)\) embeds isometrically into \( \mathbb{R}^r \), where \( r \leq n \) denotes the rank of the matrix \( A_{n \times n} \) as in [16.9]. \( \square \)

16.5. **Schoenberg: from spheres to correlation matrices, to positivity preservers.**

The previous result, Theorem 16.15, says that Euclidean spaces \( \mathbb{R}^r \) – or their direct limit / union \( \mathbb{R}^{\infty} \) (which should more accurately be denoted \( \mathbb{R}^{\infty} \)), or even its closure \( \ell^2 \) of square-summable real sequences (which Schoenberg and others called *Hilbert space*) – can be characterized by the property that the maps
\[
\exp(-\lambda^2 x^2), \quad \lambda \in (0, \rho)
\]
are all positive definite on each (finite) metric subspace. As we saw, such a characterization holds for each \( \rho > 0 \).

Given this characterization, it is natural to seek out similar characterizations of distinguished submanifolds \( M \) in \( \mathbb{R}^r \) or \( \mathbb{R}^{\infty} \) or \( \ell^2 \). In fact in the aforementioned 1935 Ann. of Math. paper, Schoenberg showed the first such classification result, for \( M = S^{r-1} \) a unit sphere – as well as for the Hilbert sphere \( S^{\infty} \).

Note here that the unit sphere \( S^{r-1} := \{ x \in \mathbb{R}^r : \|x\|^2 = 1 \} \), while the Hilbert sphere \( S^{\infty} \subset \ell^2 \) is the subset of all square-summable sequences with unit \( \ell^2 \)-norm. (This is the closure of the set of all real sequences with finitely many nonzero coordinates and unit \( \ell^2 \)-norm – which is the unit sphere \( \bigcup_{r \geq 1} S^{r-1} \) in \( \bigcup_{r \geq 1} \mathbb{R}^r \).)

One defines a rotationally invariant metric on \( S^{\infty} \) (whence on each \( S^{r-1} \)) as follows. The distance between \( x \) and \(-x\) is \( \pi \), and given points \( x \neq \pm y \) in \( S^{\infty} \), there exists a unique plane passing through \( x \) and \( y \), and the origin. This plane intersects the sphere \( S^{\infty} \) in a unit circle \( S^1 \):
\[
\{ \alpha x + \beta y : \alpha, \beta \in \mathbb{R}, \; 1 = \|\alpha x + \beta y\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta \langle x, y \rangle \},
\]
and we let \( d(x, y) \) denote the angle – i.e. arclength – between \( x \) and \( y \):
\[
d(x, y) := \angle(\langle x, y \rangle) = \arccos(\langle x, y \rangle) \in [0, \pi].
\]

Now we come to Schoenberg’s characterization of spheres. He showed that in contrast to the family (16.16) of positive definite functions for Euclidean spaces, for spheres it suffices to consider a single function! This function is the cosine:

\footnote{A related result on positive definite functions on / Hilbert space embeddings of a topological space \( X \) is by Kolmogorov around 1940. He showed that a continuous function \( K : X \times X \to \mathbb{C} \) is positive definite, if and only if there exists a Hilbert space \( \mathcal{H} \) and a norm-continuous map \( \varphi : X \to \mathcal{H} \) such that \( K(x_1, x_2) = \langle \varphi(x_1), \varphi(x_2) \rangle \) for all \( x_1, x_2 \in X \).}

Proposition 16.17 (Schoenberg, 1935). Let \((X,d)\) be a finite metric space with \(X = \{x_1, \ldots, x_n\}\). Fix an integer \(r \geq 2\). Then \(X\) isometrically embeds into \(S^{r-1}\) but not \(S^{r-2}\), if and only if \(d(x_j, x_k) \leq \pi\) for all \(1 \leq j, k \leq n\) and the matrix \((\cos d(x_j, x_k))_{j,k=1}^n\) is positive semidefinite of rank \(r\).

In particular, \(X\) embeds isometrically into the Hilbert sphere \(S^\infty\) if and only if (a) \(\text{diam}(X) \leq \pi\) and (b) \(\cos(\cdot)\) is positive definite on \(X\).

Thus matrix positivity is also intimately connected with spherical embeddings, which may not be surprising given Theorem 16.8.

Proof. If there exists an embedding as claimed, then letting \(x_0 := 0 \in \mathbb{R}^r\) and identifying \(x_j\) with its image in \(S^{r-1}\), we compute as in (16.10) for \(j, k > 0\):

\[
d'(x_0, x_j)^2 + d'(x_0, x_k)^2 - d'(x_j, x_k)^2 = 2(x_j, x_k) = 2\cos d(x_j, x_k),
\]

where \(d'\) denotes Euclidean distance in \(\mathbb{R}^r\). Hence by Schoenberg’s theorem 16.8, the matrix with these entries must be positive semidefinite of rank \(r\). Moreover, the spherical distance between \(x_j, x_k\) (for \(j, k > 0\)) is at most \(\pi\), as desired.

Conversely, since \(A := (\cos (x_j, x_k))_{j,k=1}^n\) is positive, it is a Gram matrix of rank \(r\), whence \(A = B^T B\) for some \(B_{r \times n}\) of rank \(r\) by Theorem 2.4. Let \(y_j \in \mathbb{R}^r\) denote the columns of \(B\); then clearly \(y_j \in S^{r-1}\) \(\forall j\); moreover,

\[
\cos \angle(y_j, y_k) = \langle y_j, y_k \rangle = a_{jk} = \cos d(x_j, x_k), \quad \forall 1 \leq j, k \leq n.
\]

Since \(d(x_j, x_k)\) lies in \([0, \pi]\) by assumption, as does \(\cos \angle(y_j, y_k)\), we obtain an isometry \(\varphi : X \to S^{r-1}\), sending \(x_j \mapsto y_j\) for all \(j > 0\). Finally, \(\text{im}(\varphi)\) is not contained in \(S^{r-2}\), for otherwise \(A\) would have rank at most \(r - 1\). This shows the result for \(S^{r-1}\); the case of \(S^\infty\) is similar. 

The proof of Proposition 16.17 shows that \(\cos(\cdot)\) is a positive definite function on unit spheres of all dimensions.

Note that Proposition 16.17 and the preceding two theorems by Schoenberg in the 1930s

(i) characterize metric space embeddings into Euclidean spaces via matrix positivity;

(ii) characterize metric space embeddings into Euclidean spaces via the positive definite functions \(\exp(-\lambda^2(\cdot)^2)\) on \(\mathbb{R}^r\) or \(\mathbb{R}^\infty\) (so this involves positive matrices); and

(iii) characterize metric space embeddings into spheres \(S^{r-1}\) or \(S^\infty\) via a different positive definite function \(\cos(\cdot)\) on \(S^\infty\).

Around the same time (in the 1930s), S. Bochner had classified all of the positive definite functions on \(\mathbb{R}\). This result was extended in 1940 simultaneously by Weil, Povzner, and Raikov to classify the positive definite functions on any locally compact abelian group. Amidst this backdrop, in his loc. cit. 1942 paper Schoenberg was interested in understanding the positive definite functions on a unit sphere \(S^{r-1} \subset \mathbb{R}^r\), where \(r \geq 2\).

To present Schoenberg’s result, first consider the \(r = 2\) case. As mentioned above, distance (i.e., angle) matrices are not positive semidefinite; but if one applies the cosine function entrywise, then we obtain the matrix with \((j,k)\) entry \(\cos(\theta_j - \theta_k)\), and this is positive semidefinite by Lemma 2.16. But now \(f[-]\) preserves positivity on a set of Toeplitz matrices (among others), by Lemma 2.16 and the subsequent discussion. For general dimension \(r \geq 2\), we have \(\cos(d(x_j, x_k)) = \langle x_j, x_k \rangle\) (see also the proof of Proposition 16.17), whence \(\cos([d(x_j, x_k)])_{j,k}\) always yields Gram matrices. Hence \(f[-]\) would once again preserve positivity on a set of positive matrices. It was this class of functions that Schoenberg characterized:

Theorem 16.18 (Schoenberg, 1942). Suppose \(f : [-1, 1] \to \mathbb{R}\) is continuous, and \(r \geq 2\) is an integer. Then the following are equivalent:


(1) \((f \circ \cos)\) is positive definite on \(S^{r-1}\).

(2) The function \(f(x) = \sum_{k=0}^{\infty} c_k C_k^{(\alpha^2)}(x)\), where \(c_k \geq 0, \forall k\), and \(\{C_k^{(\alpha^2)}(x) : k \geq 0\}\)

comprise the first Chebyshev or Gegenbauer family of orthogonal polynomials.

Remark 16.19. Theorem [16.18] has an interesting reformulation in terms of entrywise positivity preservers on correlation matrices. Recall that on the unit sphere \(S^{r-1}\), applying \(\cos[-\] entrywise to a distance matrix of points \(x_j\) yields precisely the Gram matrix with entries \(\langle x_j, x_k \rangle\), which is positive of rank at most \(r\). Moreover, as the vectors \(x_j\) lie on the unit sphere, the diagonal entries are all 1 and hence we obtain a correlation matrix. Putting these facts together, \(f \circ \cos\) is positive definite on \(S^{r-1}\) if and only if \(f[-]\) preserves positivity on all correlation matrices of arbitrary size but rank at most \(r\). Thus Schoenberg’s works in 1935 and 1942 already contained connections to entrywise preservers of correlation matrices, which brings us around to the modern-day motivations that arise from precisely this question (now arising in high-dimensional covariance estimation, and discussed in Section [13.1] above).

Remark 16.20. Schoenberg’s work has been followed by numerous papers trying to understand positive definite functions on locally compact groups, spheres, two-point homogeneous metric spaces, and products of these. See e.g. [14, 15, 16, 29, 30, 43, 57, 112, 198, 199, 297, 298, 303] for a selection of works. The connection to spheres has led to work in statistics on spatio-temporal covariance functions on spheres, modeling the earth as a sphere [106, 227, 292] and in Duke Math. J. 2014, 2018 [63, 64]; and Musin in Ann. of Math. 2008, 2014 and 2018 [63, 64].

Returning to the above discussion, if instead we let \(r = \infty\), then a corresponding result would classify positivity preservers on all correlation matrices (without rank constraints) by Remark 16.19. And indeed, Schoenberg achieves this goal in the same paper:

Theorem 16.22 (Schoenberg, 1942). Suppose \(f : [-1, 1] \rightarrow \mathbb{R}\) is continuous. Then \(f \circ \cos\) is positive definite on \(S^\infty\) if and only if there exist scalars \(c_k \geq 0\) such that

\[ f(\cos \theta) = \sum_{k \geq 0} c_k \cos^k \theta, \quad \theta \in [0, \pi]. \]

Notice here that \(\cos^k \theta\) is positive definite on \(S^\infty\) for all integers \(k \geq 0\), by Proposition 16.17 and the Schur product theorem. Hence so is \(\sum_{k \geq 0} c_k \cos^k \theta\) if all \(c_k \geq 0\).

Freed from the sphere context, the preceding theorem says that a continuous function \(f : [-1, 1] \rightarrow \mathbb{R}\) preserves positivity when applied entrywise to all correlation matrices, if and only if \(f(x) = \sum_{k \geq 0} c_k x^k\) on \([-1, 1]\) with all \(c_k \geq 0\). This finally explains how and why Schoenberg reached at his celebrated converse to the Schur product theorem – namely, Theorem 16.22 on entrywise positivity preservers.

16.6. Digression on ultraspherical polynomials. Before proceeding further, we describe the orthogonal polynomials \(C_k^{(\alpha)}(x)\) for \(k \geq 0\), where \(\alpha = \alpha(r) = (r - 2)/2\). Given \(r \geq 2\), note that \(\alpha(r)\) ranges over the non-negative half-integers. Though not used below, here are several different (equivalent) definitions of the polynomials \(C_k^{(\alpha)}\) for general real \(\alpha \geq 0\).
First, if \( \alpha = 0 \) then \( C_k^{(0)}(x) := T_k(x) \), the Chebyshev polynomials of the first kind:

\[
T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \ldots, \quad T_k(\cos(\theta)) = \cos(k\theta) \, \forall k \geq 0.
\]

A second way to compute the polynomials \( C_k^{(0)}(x) \) is through their generating function:

\[
\frac{1 - xt}{1 - 2xt + t^2} = \sum_{k=0}^{\infty} C_k^{(0)}(x) t^k.
\]

For higher \( \alpha \): setting \( \alpha = \frac{1}{2} \) yields the family of Legendre polynomials. If \( \alpha = 1 \), we obtain the Chebyshev polynomials of the second kind. For general \( \alpha > 0 \), the functions \((C_k^{(\alpha)}(x))_{k \geq 0}\)

are the Gegenbauer/ultraspherical polynomials, defined via their generating function:

\[
(1 - 2xt + t^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^{(\alpha)}(x) t^k.
\]

For all \( \alpha \geq 0 \), the polynomials \((C_k^{(\alpha)}(x))_{k \geq 0}\) form a complete orthogonal set in the Hilbert space \( L^2([-1, 1], w_\alpha) \), where \( w_\alpha \) is the weight function

\[
w_\alpha(x) := (1 - x^2)^{\alpha - \frac{1}{2}}, \quad x \in (-1, 1).
\]

Thus, another definition of \( C_k^{(\alpha)}(x) \) is that it is a polynomial of degree \( k \), with \( C_0^{(\alpha)}(x) = 1 \), and such that the \( C_k^{(\alpha)} \) are orthogonal with respect to the bilinear form

\[
\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)w_\alpha(x) \, dx, \quad f, g \in L^2([-1, 1], w_\alpha),
\]

and satisfy:

\[
\langle C_k^{(\alpha)}, C_k^{(\alpha)} \rangle = \frac{\pi 2^{1-2\alpha} \Gamma(k + 2\alpha)}{k!(\alpha)!(\Gamma(\alpha))^2}.
\]

Yet another definition is that the Gegenbauer polynomials \( C_k^{(\alpha)}(x) \) for \( \alpha > 0 \) satisfy the differential equation

\[
(1 - x^2)y'' - (2\alpha + 1)xy' + k(k + 2\alpha)y = 0.
\]

We also have a direct formula

\[
C_k^{(\alpha)}(x) := \sum_{j=0}^{[k/2]} (-1)^j \frac{\Gamma(k - j + \alpha)}{\Gamma(\alpha)j!(k - 2j)!} (2x)^{k-2j},
\]

as well as a recursion:

\[
C_0^{(\alpha)}(x) := 1, \quad C_1^{(\alpha)}(x) := 2\alpha x,
\]

\[
C_k^{(\alpha)}(x) := \frac{1}{k} \left( 2x(k + \alpha - 1)C_{k-1}^{(\alpha)}(x) - (k + 2\alpha - 2)C_{k-2}^{(\alpha)}(x) \right) \, \forall k \geq 2.
\]

16.7. Sketch of proof of Theorem 16.18. Schoenberg’s theorem 16.18 has subsequently been studied by many authors, and in a variety of settings over the years. This includes classifying the positive definite functions on different kinds of spaces: locally compact groups, spheres, and products of these. We next give a proof-sketch of this result. In what follows, we use without reference the observation that (akin to Lemma 3.1), the set of functions \( f \) such that \( f \circ \cos \) is positive definite on \( S^{n-1} \), also forms a closed convex cone, which is moreover closed under taking entrywise products.
We first outline why (2) \(\implies\) (1) in Theorem 16.18. By the above observation, it suffices to show that \(C_k^{(\alpha)} \circ \cos\) is positive definite on \(S^{r-1}\). The proof is by induction on \(r\). For the base case \(r = 2\), let \(\theta_1, \theta_2, \ldots, \theta_n \in S^1 = [0, 2\pi]\). Up to sign, their distance matrix has \((i, j)\) entry \(d(\theta_i, \theta_j) = \theta_i - \theta_j\) (or a suitable translate modulo \(2\pi\)). Now by Lemma 2.16, the matrix 
\[(\cos(k(\theta_i - \theta_j)))_{i,j=1}^n\] is positive semidefinite. But this is precisely the matrix obtained by applying \(C_k^{(0)} \circ \cos\) to the distance matrix above. This proves one implication for \(d = 2\). The induction step (for general \(r \geq 2\)) follows from addition formulas for \(C_k^{(\alpha)}\).

For the converse implication, set \(\alpha := (r - 2)/2\) and note that \(f \in L^2([-1, 1], w_\alpha)\). Hence \(f\) has a series expansion 
\[\sum_{k=0}^{\infty} c_k C_k^{(\alpha)}(x),\] with \(c_k \in \mathbb{R}\). Now recover the \(c_k\) via:

\[c_k = \int_{-1}^{1} f(x) C_k^{(\alpha)}(x) w_\alpha(x) \, dx,\]

since the \(C_k^{(\alpha)}\) form an orthonormal family. Note that \(C_k^{(\alpha)}\) and \(f\) are both positive definite (upon pre-composing with the cosine function), whence so is their product by the Schur product theorem. A result of W.H. Young now shows that \(c_k \geq 0\) for all \(k \geq 0\). \(\square\)

16.8. **Entrywise preservers in fixed dimension.** We conclude by discussing a natural mathematical refinement of Schoenberg’s theorem:

"Which functions entrywise preserve positivity in fixed dimension?"

This turns out to be a challenging, yet important question from the point of view of applications (see Section 13.1 for more on this.) In particular, note that there exist functions which preserve positivity on \(\mathbb{P}_n\) but not on \(\mathbb{P}_{n+1}\): the power functions \(x^\alpha\) with \(\alpha \in (n-3, n-2)\) for \(n \geq 3\), by Theorem 9.3. By Vasudeva’s theorem 16.4 it follows that these ‘non-integer’ power functions cannot be absolutely monotonic.

Surprisingly, though Schoenberg’s theorem is classical and provides a complete description in the dimension-free case, not much is known about the fixed-dimension case: namely, the classification of functions \(f : I \to \mathbb{R}\) such that \(f([-]) : \mathbb{P}_n(I) \to \mathbb{P}_n(\mathbb{R})\) for a fixed integer \(n \geq 1\).

- If \(n = 1\), then clearly, any function \(f : [0, \infty) \to [0, \infty)\) works.
- For \(n = 2\) and \(I = (0, \infty)\), these are precisely the functions \(f : (0, \infty) \to \mathbb{R}\) that are non-negative, non-decreasing, and multiplicatively mid-convex. This was shown by Vasudeva (see Theorem 12.7), and it implies similar results for \(I = [0, \infty)\) and \(I = \mathbb{R}\).
- For every integer \(n \geq 3\), the question is open to date.

Given the scarcity of results in this direction, a promising line of attack has been to study refinements of the problem. These can involve restricting the test set of matrices in fixed dimension (say under rank or sparsity constraints) or the test set of functions (say to only the entrywise powers) as was studied in the previous sections; or to use both restrictions. See Section 13.2 for more on this discussion, as well as the final part of the text, where we study polynomial preservers in a fixed dimension.

To conclude: while the general problem in fixed dimension \(n \geq 3\) is open to date, there is a known result: a necessary condition satisfied by positivity preservers on \(\mathbb{P}_n\), shown by R.A. Horn in his 1969 paper in *Trans. Amer. Math. Soc.* and attributed to his advisor, Loewner. The result is above fifty years old; yet even today, it remains essentially the only known result for general preservers \(f\) on \(\mathbb{P}_n\). In the next two sections, we will state and prove this result – in fact, a stronger version. We will then show (stronger versions of) Vasudeva’s and Schoenberg’s theorems, via a different approach than the one by Schoenberg, Rudin, or others: we crucially use the fixed-dimension theory, via the result of Horn and Loewner.

As mentioned in the previous section, the goal in this part is to prove a stronger form of Schoenberg’s theorem in the spirit of Rudin’s theorem, but replacing the word ‘Toeplitz’ by ‘Hankel’. In order to do so, we will first prove a stronger version of Vasudeva’s theorem in which the test set is once again reduced to only low-rank Hankel matrices.

In turn, our proof of this version of Vasudeva’s theorem relies on a fixed-dimension result, alluded to at the end of the previous section. Namely, we state and prove a stronger version of a 1969 theorem of Horn (attributed by him to Loewner), in this section and the next.

Theorem 17.1 (Horn–Loewner theorem, stronger version). Let \( I = (0, \infty) \), and fix \( u_0 \in (0, 1) \) and an integer \( n \geq 1 \). Define \( u := (1, u_0, \ldots, u_0^{n-1})^T \). Suppose \( f : I \to \mathbb{R} \) is such that \( f[-] \) preserves positivity on the set \( \{ a_1^{n \times n} + b u u^T : a, b > 0 \} \) as well as on the rank-one matrices in \( \mathbb{P}_2(I) \) and the Toeplitz matrices in \( \mathbb{P}_2(I) \). Then:

1. \( f \in C^{n-3}(I) \) and \( f, f', \ldots, f^{(n-3)} \) are non-negative on \( I \). Moreover, \( f^{(n-3)} \) is convex and non-decreasing on \( I \).
2. If moreover \( f \in C^{n-1}(I) \), then \( f^{(n-2)}, f^{(n-1)} \) are also non-negative on \( I \).

Notice that the test set consists entirely of Hankel matrices of rank at most 2 – and these are moreover totally non-negative by Corollary 17.3, since they arise as the truncated moment matrices of the measures \( a_0 + b_0 u \). This fits in with our future use of this result to prove stronger versions of Vasudeva’s and Schoenberg’s theorems, with similarly reduced test sets of low-rank Hankel matrices.

Remark 17.2. In the original result by Horn (and Loewner), \( f \) was assumed to be continuous and to preserve positivity on all of \( \mathbb{P}_N((0, \infty)) \). In Theorem 17.1, we have removed the continuity hypothesis, in the spirit of Rudin’s work, and also greatly reduced the test set.

Remark 17.3. We also observe that Theorem 17.1 is ‘best possible’, in that the number of nonzero derivatives that must be positive is sharp. For example, let \( n \geq 2 \), \( I = (0, \infty) \), and \( f : I \to \mathbb{R} \) be given by: \( f(x) := x^\alpha \), where \( \alpha \in (n-2, n-1) \). Using Theorem 9.3, \( f[-] \) preserves positivity on the test sets \( \{ a_1^{n \times n} + b u u^T : a, b > 0 \} \) and \( \mathbb{P}_2(I) \). Moreover, \( f \in C^{n-1}(I) \) and \( f, f', \ldots, f^{(n-1)} \) are strictly positive on \( I \). However, \( f^{(n)} \) is negative on \( I \).

This low-rank Hankel example (and more generally, Theorem 9.3) also shows that there exist (power) functions that preserve positivity on \( \mathbb{P}_n \) but not on \( \mathbb{P}_{n+1} \). In the final part, we will show that this also holds for polynomial preservers in the fixed dimension case.

We now proceed toward the proof of Theorem 17.1 for general functions. A major piece of the proof is the following preliminary calculation, which will essentially prove the result for smooth functions.

Proposition 17.4. Fix an integer \( n > 0 \) and define \( N := \binom{n}{2} \). Suppose \( a \in \mathbb{R} \) and let a function \( f : (a - \epsilon, a + \epsilon) \to \mathbb{R} \) be \( N \)-times differentiable for some fixed \( \epsilon > 0 \). Now fix vectors \( u, v \in \mathbb{R}^n \), and define \( \Delta : (-\epsilon', \epsilon') \to \mathbb{R} \) via:

\[
\Delta(t) := \det f[1_{n \times n} + t u v^T],
\]

for a sufficiently small \( \epsilon' \in (0, \epsilon) \). Then \( \Delta(0) = \Delta'(0) = \cdots = \Delta^{(N-1)}(0) = 0 \), and

\[
\Delta^{(N)}(0) = \begin{pmatrix} N \\ 0, 1, \ldots, n - 1 \end{pmatrix} V(u)V(v) \prod_{k=0}^{n-1} f^{(k)}(a),
\]
where the first factor on the right is the multinomial coefficient, and given a vector \( \mathbf{u} = (u_1, \ldots, u_n)^T \), we define its Vandermonde determinant to be 1 if \( n = 1 \), and

\[
V(\mathbf{u}) := \prod_{1 \leq j < k \leq n} (u_k - u_j) = \det \begin{pmatrix}
1 & u_1 & \cdots & u_1^{n-1} \\
1 & u_2 & \cdots & u_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & u_n & \cdots & u_n^{n-1}
\end{pmatrix}, \quad \text{if } n > 1. \tag{17.5}
\]

While the result seemingly involves (higher) derivatives, it is in fact a completely algebraic phenomenon, valid over any ground ring. For the interested reader, we isolate this phenomenon in Proposition 17.8 below; its proof is more or less the same as the one now provided for Proposition 17.4. To gain some feel for the computations, the reader may wish to work out the \( N = 3 \) case first.

Proof. Let \( \mathbf{w}_k \) denote the \( k \)th column of \( a \mathbf{1}_{n \times n} + t \mathbf{u} \mathbf{v}^T \); thus \( \mathbf{w}_k \) has \( j \)th entry \( a + tu_jv_k \). To differentiate \( \Delta(t) \), we will use the multilinearity of the determinant and the Laplace expansion of \( \Delta(t) \) into a linear combination of \( n! \) ‘monomials’, each of which is a product of \( n \) terms \( f(\cdot) \). By the product rule, taking the derivative yields \( n \) terms from each monomial, and we may rearrange all of these terms into \( n \) ‘clusters’ of terms (grouping by the column which gets differentiated), and regroup back using the Laplace expansion to obtain:

\[
\Delta'(t) = \sum_{k=1}^{n} \det(f[\mathbf{w}_1] | \cdots | f[\mathbf{w}_{k-1}] | v_k \mathbf{u} \circ f'[\mathbf{w}_k] | f[\mathbf{w}_{k+1}] | \cdots | f[\mathbf{w}_n]).
\]

Now apply the derivative repeatedly, using this principle. By the Chain Rule, for \( m \geq 0 \) the derivative \( \Delta^{(m)}(t) \) – evaluated at \( t = 0 \) – is an integer linear combination of terms of the form

\[
\det(v_1^{m_1}u_1^{m_1} \circ f^{(m_1)}[\mathbf{a}1] | \cdots | v_n^{m_n}u_n^{m_n} \circ f^{(m_n)}[\mathbf{a}1]) = \det(f^{(m_1)}(a)v_1^{m_1}u_1^{m_1} | \cdots | f^{(m_n)}(a)v_n^{m_n}u_n^{m_n}), \quad m_1 + \cdots + m_n = m, \tag{17.6}
\]

where \( \mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n \) and all \( m_j \geq 0 \). Notice that if any \( m_j = m_k \) for \( j \neq k \) then the corresponding determinant 17.6 vanishes. Thus, the lowest degree derivative \( \Delta^{(m)}(0) \) whose expansion contains a non-vanishing determinant is when \( m = 0 + 1 + \cdots + (n-1) = N \). This proves the first part of the result.

To show the second part, consider \( \Delta^{(N)}(0) \). Once again, the only determinant terms that do not vanish in its expansion correspond to applying \( 0, 1, \ldots, n-1 \) derivatives to the columns in some order. We first compute the integer multiplicity of each such determinant, noting by symmetry that these multiplicities are all equal. As we are applying \( N \) derivatives to \( \Delta \) (before evaluating at 0), the derivative applied to get \( f' \) in some column can be any of \( \binom{N}{k} \); now the two derivatives applied to get \( f'' \) in a (different) column can be chosen in \( \binom{N-2}{k} \) ways; and so on. Thus, the multiplicity is precisely

\[
\binom{N}{1} \binom{N-1}{2} \binom{N-3}{3} \cdots \binom{2n-3}{n-2} = \prod_{k=0}^{n-1} \binom{N-(k)}{k} = \frac{N!}{\prod_{k=0}^{n-1} k!} = \binom{N}{0,1,\ldots,n-1}.
\]

We next compute the sum of all determinant terms. Each term corresponds to a unique permutation of the columns \( \sigma \in S_n \), with say \( \sigma_k - 1 \) the order of the derivative applied to
the $k$th column $f[w_k]$. Using (17.6), the determinant corresponding to $\sigma$ equals
\[
\prod_{k=0}^{n-1} f^{(k)}(a)v_k^{\sigma k-1} \cdot (-1)^{\sigma} \cdot \det(u^{\sigma 0} | u^{\sigma 1} | \cdots | u^{\sigma(n-1)})
\]
\[
= V(u) \prod_{k=0}^{n-1} f^{(k)}(a) \cdot (-1)^{\sigma} \prod_{k=0}^{n-1} v_k^{\sigma k-1}.
\]
Summing this term over all $\sigma \in S_n$ yields precisely:
\[
V(u) \prod_{k=0}^{n-1} f^{(k)}(a) \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{k=0}^{n-1} v_k^{\sigma k-1} = V(u) \prod_{k=0}^{n-1} f^{(k)}(a) \cdot V(v).
\]
Now multiplying by the (common) integer multiplicity computed above, the proof is complete. \qed

We next present the promised algebraic formulation of Proposition 17.4. For this, some notation is required. Fix a commutative (unital) ring $R$ and an $R$-algebra $S$. The first step is to formalize the notion of the derivative, on a sub-class of $S$-valued functions. This involves more structure than the more common notion of a derivation, so we give it a different name.

**Definition 17.7.** Given a commutative ring $R$, a commutative $R$-algebra $S$ (with $R \subset S$), and an $R$-module $X$, a **differential calculus** is a pair $(A, \partial)$, where $A$ is an $R$-subalgebra of functions $X \to S$ (under pointwise addition and multiplication and $R$-action) which contains the constant functions, and $\partial : A \to A$ satisfies the following properties:

1. $\partial$ is $R$-linear:
   \[
   \partial \sum_j r_j f_j = \sum_j r_j \partial f_j, \quad \forall r_j \in R, \; f_j \in A, \; \forall j.
   \]

2. $\partial$ is a derivation (product rule):
   \[
   \partial(fg) = f \cdot (\partial g) + (\partial f) \cdot g, \quad \forall f, g \in A.
   \]

3. $\partial$ satisfies a variant of the ‘Chain Rule’ for composing with linear functions. Namely, if $x' \in X, r \in R$, and $f \in A$, then the function $g : X \to S, g(x) := f(x' + rx)$ also lies in $A$, and moreover,
   \[
   (\partial g)(x) = r \cdot (\partial f)(x' + rx).
   \]

With this definition in hand, we can now state the desired algebraic generalization of Proposition 17.4; the proof is essentially the same.

**Proposition 17.8.** Suppose $R, S, X$ are as in Definition 17.7 with an associated differential calculus $(A, \partial)$. Now fix an integer $n > 0$, two vectors $u, v \in R^n$, a vector $a \in X$, and a function $f \in A$. Define $N \in \mathbb{N}$ and $\Delta : X \to R$ via:
\[
N := \binom{n}{2}, \quad \Delta(t) := \det f[a1_{n \times n} + tuv^T], \quad t \in X.
\]
Then $\Delta(0_X) = (\partial \Delta)(0_X) = \cdots = (\partial^{N-1} \Delta)(0_X) = 0_R$, and
\[
(\partial^N \Delta)(0_X) = \binom{N}{0, 1, \ldots, n-1} V(u)V(v) \prod_{k=0}^{n-1} (\partial^k f)(a).
\]
Notice that the algebra $A$ is supposed to remind the reader of ‘smooth functions’, and is used here for ease of exposition. One can instead work with an appropriate algebraic notion of ‘$N$-times differentiable functions’ in order to “truly” generalize Proposition 17.4; we leave the details to the interested reader.

**Remark 17.9.** Note that Proposition 17.4 is slightly more general than the original argument of Horn and Loewner, which involved the special case $u = v$. As the above proof (as well as the preceding proposition 17.8) shows, the argument is essentially ‘algebraic’, and hence holds for arbitrary $u, v$.

Finally, we use Proposition 17.4 to prove the Horn–Loewner theorem 17.1 for smooth functions. The remainder of the proof will be discussed in the next section.

**Proof of Theorem 17.1 for smooth functions.** Suppose $f$ is smooth – or more generally, $C^N$ where $N = \binom{n}{2}$. Then the result is shown by induction on $n$. For $n = 1$ the result says that $f$ is non-negative if it preserves positivity on the given test set, which is obvious. For the induction step, we know that $f, f', \ldots, f^{(n-2)} \geq 0$ on $I$, since the given test set of $(n-1) \times (n-1)$ matrices can be embedded into the test set of $n \times n$ matrices. Now define $f_\epsilon(x) := f(x) + \epsilon x^n$ for each $\epsilon > 0$, and note by the Schur product theorem 3.10 (or Lemma 16.1) that $f_\epsilon$ also satisfies the hypotheses.

Given $a, t > 0$ and the vector $u$ as in the theorem, define $\Delta(t) := \det f_\epsilon[a1_{n \times n} + tuu^T]$ as in Proposition 17.4 (but replacing $f, v$ by $f_\epsilon, u$ respectively). Then $\Delta(t) \geq 0$ for $t > 0$ by assumption, whence

$$0 \leq \lim_{t \to 0^+} \frac{\Delta(t)}{t^N}, \quad \text{where } N = \binom{n}{2}.$$ 

On the other hand, by Proposition 17.4 and L’Hôpital’s rule,

$$\lim_{t \to 0^+} \frac{\Delta(t)}{t^N} = \frac{\Delta^{(N)}(0)}{N!} = \frac{1}{N!} \binom{N}{0, 1, \ldots, n-1} V(u)^2 \prod_{k=0}^{n-1} f_\epsilon^{(k)}(a) = V(u)^2 \prod_{k=0}^{n-1} \frac{f_\epsilon^{(k)}(a)}{k!}.$$ 

Thus, the right-hand side here is non-negative. Since $u$ has distinct coordinates, we can cancel all positive factors to conclude that

$$\prod_{k=0}^{n-1} f_\epsilon^{(k)}(a) \geq 0, \quad \forall \epsilon, a > 0.$$ 

But $f_\epsilon^{(k)}(a) = f^{(k)}(a) + \epsilon n(n-1) \cdots (n-k+1)a^{n-k}$, and this is positive for $k = 0, \ldots, n-2$ by the induction hypothesis. Hence,

$$f_\epsilon^{(n-1)}(a) = f^{(n-1)}(a) + \epsilon n! \geq 0, \quad \forall \epsilon, a > 0.$$ 

It follows that $f^{(n-1)}(a) \geq 0$, whence $f^{(n-1)}$ is non-negative on $(0, \infty)$, as desired.

We conclude by mentioning that the Horn–Loewner theorem, as well as Proposition 17.4 and its algebraic avatar in Proposition 17.8 afford generalizations; the latter results reveal a surprising and novel application to Schur polynomials and to symmetric function identities. For more details, the reader is referred to the 2018 preprint *Smooth entrywise positivity preservers, a Horn–Loewner master theorem, and Schur polynomial identities* by this author.

We continue with the proof of the Horn–Loewner theorem \[17.1\] This is in three steps:

1. Theorem \[17.1\] holds for smooth functions. This was proved in the previous section.
2. If Theorem \[17.1\] holds for smooth functions, then it holds for continuous functions. Here, we need to assume \( n \geq 3 \).
3. If \( f \) satisfies the hypotheses in Theorem \[17.1\] then it is continuous. This follows from Vasudeva’s \( 2 \times 2 \) result – see (the proof of) Theorem \[12.7\] and Remark \[12.10\].

To carry out the second step – as well as a similar step in proving Schoenberg’s theorem below – we will use a standard tool in analysis called \textit{mollifiers}.

18.1. An introduction to (one-variable) mollifiers. In this subsection, we examine some basic properties of mollifiers of one variable; the theory extends to \([\mathbb{R}^n] \) for all \( n > 1 \), but that is not required in what follows.

First recall that one can construct smooth functions \( g : \mathbb{R} \to \mathbb{R} \) such that \( g \) and all its derivatives vanish on \((-\infty, 0)\): for instance, \( g(x) = \exp(-1/x) \cdot 1(x > 0) \). Indeed, one shows that \( g^{(n)}(x) = p_n(1/x)g(x) \) for some polynomial \( p_n \); hence \( g^{(n)}(x) \to 0 \) as \( x \to 0 \). Hence:

\begin{lemma} \[18.1\] Given scalars \(-1 < a < b < 0\), there exists a smooth function \( \phi \) that vanishes outside \([a, b]\), is positive on \((a, b)\), and is a probability distribution on \(\mathbb{R}\).
\end{lemma}

Of course, the assumption \([a, b] \subset (-1, 0)\) is completely unused in the proof of the lemma, but is included for easy reference since we will require it in what follows.

\begin{proof}
The function \( \varphi(x) := g(x-a)g(b-x) \) is non-negative, smooth, and supported precisely on \((a, b)\). In particular, \( \int_{\mathbb{R}} \varphi > 0 \), so the normalization \( \phi := \varphi/\int_{\mathbb{R}} \varphi \) has the desired properties.
\end{proof}

We now introduce mollifiers.

\begin{definition} \[18.2\] A \textit{mollifier} is a one-parameter family of functions (in fact probability distributions) \[ \{ \phi_\delta(x) := \frac{1}{\delta} \phi\left(\frac{x}{\delta}\right) : \delta > 0 \} \]
with real domain and range, corresponding to any function \( \phi \) satisfying Lemma \[18.1\].

A continuous, real-valued function \( f \) (with suitable domain inside \(\mathbb{R}\)) is said to be \textit{mollified} byconvolving with the family \( \phi_\delta \). In this case, we define \( f_\delta(x) := \int_{\mathbb{R}} f(t)\phi_\delta(x-t) \, dt \), where one extends \( f \) outside its domain by zero. (This is called \textit{convolution}: \( f_\delta = f * \phi_\delta \).)
\end{definition}

\begin{remark} \[18.3\] Mollifiers, or Friedrichs mollifiers, were used by Horn and Loewner in the late 1960s, as well as previously by Rudin in his 1959 proof of Schoenberg’s theorem; they were a relatively modern tool at the time, having been introduced by Friedrichs in his seminal 1944 paper on PDEs in \textit{Trans. Amer. Math. Soc.}, as well as slightly earlier by Sobolev in his famous 1938 paper in \textit{Mat. Sbornik} (which contained the proof of the Sobolev embedding theorem).

Returning to the definition of a mollifier, notice by the change of variables \( u = x - t \) and Lemma \[18.1\] that \[ f_\delta(x) = \frac{1}{\delta} \int_{\mathbb{R}} f(x-u)\phi\left(\frac{u}{\delta}\right) \, du = \int_{-\delta}^{0} f(x-u)\phi_\delta(u) \, du. \] (18.4)\]
In particular, \( f_\delta \) is a ‘weighted average’ of the image set \( f([x, x+\delta]) \), since \( \phi \) is a probability distribution. Now it is not hard to see that \( f_\delta \) is continuous, and converges to \( f \) pointwise as \( \delta \to 0^+ \). In fact, more is true:

**Proposition 18.5.** If \( I \subset \mathbb{R} \) is a right-open interval and \( f : I \to \mathbb{R} \) is continuous, then for all \( \delta > 0 \), the mollified functions \( f_\delta \) are smooth on \( \mathbb{R} \) (where we extend \( f \) outside \( I \) by zero), and converge uniformly to \( f \) on compact subsets of \( I \) as \( \delta \to 0^+ \).

To prove this result, we show two lemmas in somewhat greater generality. First, some notation: a (Lebesgue measurable) function \( f : \mathbb{R} \to \mathbb{R} \) is said to be *locally \( L^1 \)* if it is \( L^1 \) on each compact subset of \( \mathbb{R} \).

**Lemma 18.6.** If \( f : \mathbb{R} \to \mathbb{R} \) is locally \( L^1 \), and \( \psi : \mathbb{R} \to \mathbb{R} \) is continuous with compact support, then \( f \ast \psi : \mathbb{R} \to \mathbb{R} \) is also continuous.

**Proof.** Suppose \( x_n \to x \) in \( \mathbb{R} \); without loss of generality \( |x_n - x| < 1 \) for all \( n > 0 \). Also choose \( r, M > 0 \) such that \( \psi \) is supported on \([-r, r]\) and \( M = \|\psi\|_{L^\infty(\mathbb{R})} = \max_{\mathbb{R}} |\psi(x)| \). Then for each \( t \in I \), we have:

\[
f(t)\psi(x_n - t) \to f(t)\psi(x - t), \quad |f(t)\psi(x_n - t)| \leq M|f(t)| \cdot 1(|x - t| \leq r + 1)
\]

(the second inequality follows by considering separately the cases \(|x - t| \leq r + 1\) and \(|x - t| > r + 1\)). Since the right-hand side is integrable, Lebesgue’s dominated convergence theorem applies:

\[
\lim_{n \to \infty} (f \ast \psi)(x_n) = \lim_{n \to \infty} \int_{\mathbb{R}} f(t)\psi(x_n - t) \, dt = \int_{\mathbb{R}} \lim_{n \to \infty} f(t)\psi(x_n - t) \, dt
\]

\[
= \int_{\mathbb{R}} f(t)\psi(x - t) \, dt = (f \ast \psi)(x),
\]

whence \( f \ast \psi \) is continuous on \( \mathbb{R} \). \( \square \)

**Lemma 18.7.** If \( f : \mathbb{R} \to \mathbb{R} \) is locally \( L^1 \), and \( \psi : \mathbb{R} \to \mathbb{R} \) is \( C^1 \) with compact support, then \( f \ast \psi : \mathbb{R} \to \mathbb{R} \) is also \( C^1 \), and \( (f \ast \psi)' = f \ast \psi' \) on \( \mathbb{R} \).

**Proof.** We compute:

\[
(f \ast \psi)'(x) = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} f(y)\psi(x + h - y) \, dy - \frac{1}{h} \int_{\mathbb{R}} f(y)\psi(x - y) \, dy
\]

\[
= \lim_{h \to 0} \int_{\mathbb{R}} f(y) \left( \frac{\psi(x + h - y) - \psi(x - y)}{h} \right) \, dy
\]

where for each \( y \in \mathbb{R} \), \( c(h, y) \in [0, h] \) is chosen using the Mean Value Theorem. While \( c(h, y) \to 0 \) as \( h \to 0 \), the problem is that \( y \) is not fixed inside the integral. Thus, to proceed, we argue as follows: suppose \( \psi \) is supported inside \([-r, r]\) as above, whence so is \( \psi' \). Choose any sequence \( h_n \to 0 \). Now **claim** that the last integral above, evaluated at \( h_n \), converges to \( (f \ast \psi')(x) \) as \( n \to \infty \) – whence so does the limit of the last integral above. Indeed, we may first assume all \( h_n \in (-1, 1) \); and then check for each \( n \) that the above integral equals

\[
\int_{\mathbb{R}} f(y)\psi'(x - y + c(h_n, y)) \, dy = \int_{x-(r+1)}^{x+(r+1)} f(y)\psi'(x - y + c(h_n, y)) \, dy
\]
by choice of \( r \). Now the integrand on the right-hand side is bounded above by \( M_1 |f(y)| \) in absolute value, where \( M_1 := \| \psi' \|_{L^{\infty}(\mathbb{R})} \). Hence by the dominated convergence theorem,

\[
\lim_{n \to \infty} \int_{x-(r+1)}^{x+(r+1)} f(y)\psi'(x - y + c(h_n, y)) \, dy = \int_{x-(r+1)}^{x+(r+1)} f(y)\psi'(x - y) \, dy = (f * \psi')(x).
\]

where the first equality also uses that \( \psi \) is \( C^1 \), and the second is by choice of \( r \). Since this happens for every sequence \( h_n \to 0 \), it follows that \( (f * \psi')(x) = (f * \psi')(x) \). Moreover, \( f * \psi' \) is continuous by Lemma \[18.6\] since \( \psi \) is \( C^1 \). This shows \( f \ast \psi \) is \( C^1 \) as claimed. \( \square \)

Finally, we show the claimed properties of mollified functions.

**Proof of Proposition \[18.5\].** Extending \( f \) by zero outside \( I \), it follows that \( f \) is locally \( L^1 \) on \( \mathbb{R} \). Repeatedly applying Lemma \[18.7\] to \( \psi = \phi_\delta, \phi'_\delta, \phi''_\delta, \ldots \), we conclude that \( f_\delta \in C^\infty(\mathbb{R}) \).

To prove local uniform convergence, let \( K \) be a compact subset of \( I \) and \( \epsilon > 0 \). Denote \( b := \sup K \) and \( a := \inf K \). Since \( I \) is right-open, there is a number \( l > 0 \) such that \( J := [a, b + l] \subset I \). Since \( f \) is uniformly continuous on \( J \), given \( \epsilon > 0 \) there exists \( \delta \in (0, l) \) such that \( |x - y| < \delta \Rightarrow x, y \in J \Rightarrow |f(x) - f(y)| < \epsilon \).

Now claim that if \( 0 < \xi < \delta \) then \( \|f_\xi - f\|_{L^\infty(K)} \leq \epsilon \); note this proves the uniform convergence of the family \( f_\delta \) to \( f \) on \( K \). To show the claim, compute using\[18.4\] for \( x \in K \):

\[
|f_\xi(x) - f(x)| = \left| \int_{-\delta}^{0} (f(x-u) - f(x))\phi_\xi(u) \, du \right| \\
\leq \int_{-\delta}^{0} |f(x-u) - f(x)|\phi_\xi(u) \, du \leq \epsilon \int_{-\delta}^{0} \phi_\xi(u) \, du = \epsilon.
\]

This is true for all \( x \in K \) by the choice of \( \xi < \delta \), and hence proves the claim. \( \square \)

### 18.2. Completing the proof of the Horn–Loewner theorem.

With mollifiers in hand, we finish the proof of Theorem \[17.1\] As mentioned above, the proof can be divided into three steps, and two of them are already worked out. It remains to show that if \( n \geq 3 \) and if the result holds for smooth functions, then it holds for continuous functions.

Thus, suppose \( I = (0, \infty) \) and \( f : I \to \mathbb{R} \) is continuous. Define the mollified functions \( f_\delta, \delta > 0 \) as above; thus \( f_\delta \) are smooth. Moreover, given \( a, b > 0 \), by \[18.4\] the function \( f_\delta \) satisfies:

\[
f_\delta[a1_{n \times n} + buu^T] = \int_{-\delta}^{0} \phi_\delta(y) \cdot f([a + |y|]1_{n \times n} + buu^T) \, dy,
\]

and this is positive semidefinite by the assumptions for \( f \). Thus \( f_\delta[-] \) preserves positivity on the given test set in \( \mathbb{P}_n(I) \); a similar argument shows that \( f_\delta[-] \) preserves positivity on \( \mathbb{P}_2(I) \). Hence by the proof in the previous section, \( f_\delta, f_\delta^2, \ldots, f_\delta^{(n-1)} \) are non-negative on \( I \).

Observe that the theorem amounts to deducing a similar statement for \( f \); however, as \( f \) is a priori known only to be continuous, we can only deduce non-negativity for a ‘discrete’ version of the derivatives – namely, divided differences:

**Definition 18.9.** Suppose \( I \) is a real interval and a function \( f : I \to \mathbb{R} \). Given \( h > 0 \) and an integer \( k \geq 0 \), the \( k \)th order forward differences with step size \( h > 0 \) are defined as follows:

\[
(\Delta_h^0 f)(x) := f(x), \quad (\Delta_h^k f)(x) := (\Delta_h^{k-1} f)(x+h) - (\Delta_h^{k-1} f)(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x+jh),
\]

\( \square \)
whenever $k > 0$ and $x, x + kh \in I$. Similarly, the $k$th order divided differences with step size $h > 0$ are

$$ (D^k_h f)(x) := \frac{1}{h^k}(\Delta^k_h f)(x), \quad \forall k \geq 0, \ x, x + kh \in I. $$

The key point is that if a function is differentiable to some order, and its derivatives of that order are non-negative on an open interval, then using the mean-value theorem for divided differences, one shows the corresponding divided differences are also non-negative, whence so are the corresponding forward differences. Remarkably, the converse also holds, including differentiability! This is a classical result by Boas and Widder:

**Theorem 18.10.** Suppose $I \subset \mathbb{R}$ is an open interval, bounded or not, and $f : I \to \mathbb{R}$.

1. (Cauchy’s mean-value theorem for divided differences: special case.) If $f$ is $k$-times differentiable in $I$ for some integer $k > 0$, and $x, x + kh \in I$ for $h > 0$, then there exists $y \in (x, x + kh)$ such that $(D^k_h f)(x) = f^{(k)}(y)$.

2. (Boas–Widder, Duke Math. J., 1940.) Suppose $k \geq 2$ is an integer, and $f : I \to \mathbb{R}$ is continuous and has all forward differences of order $k$ non-negative on $I$:

$$ (\Delta^k f)(x) \geq 0, \quad \text{whenever } h > 0 \text{ and } x, x + kh \in I. $$

Then on all of $I$, the function $f^{(k-2)}$ exists, is continuous and convex, and has non-decreasing left and right hand derivatives.

We make a few remarks on Boas and Widder’s result. First, for $k = 2$ the result seems similar to Ostrowski’s theorem 12.2, except for the local boundedness being strengthened to continuity. Second, note that while $f^{(k-1)}$ is non-decreasing by the theorem, one can not claim here that the lower-order derivatives $f, \ldots, f^{(k-2)}$ are non-decreasing on $I$. Indeed, a counterexample for such an assertion for $f^{(0)}$, where $0 \leq l \leq k - 2$, is $f(x) = -x^{l+1}$ on $I \subset \mathbb{R}$. Finally, we refer the reader to Section 23.1 for additional related observations and results.

**Proof.** The second part will be proved in detail in Section 23. For the first, consider the Newton form of the Lagrange interpolation polynomial $P(X)$ for $f(X)$ at $X = x, x+h, \ldots, x+kh$. The highest term of $P(X)$ is

$$ (D^k_h f)(x) \cdot (X - x_{k-1}) \cdots (X - x_1)(X - x_0), \quad \text{where } x_j = x_0 + jh \ \forall j \geq 0. $$

Writing $g(X) := f(X) - P(X)$ to be the remainder function, note that $g$ vanishes at $x, x + h, \ldots, x + kh$. By successively applying Rolle’s theorem to $g, g', \ldots, g^{(k-1)}$, it follows that $g^{(k)}$ has a root in $(x, x + kh)$, say $y$. But then,

$$ 0 = g^{(k)}(y) = f^{(k)}(y) - (D^k_h f)(x)k!, $$

which concludes the proof.

Returning to our proof of the stronger Horn–Loewner theorem 17.1, since $f_0, f_0', \ldots, f_0^{(n-1)} \geq 0$ on $I$, by the above theorem the divided differences of $f_0$ up to order $n-1$ are non-negative on $I$, whence the same holds for the forward differences of $f_0$. Applying Proposition 18.5 the forward differences of $f$ of orders $k = 0, \ldots, n-1$ are also non-negative on $I$. Finally, invoke the Boas–Widder theorem for $k = 2, \ldots, n-1$ to conclude the proof of the (stronger) Horn–Loewner theorem – noting for ‘low orders’ that $f$ is non-negative and non-decreasing on $I$ by using forward differences of orders $k = 0, 1$ respectively, whence $f, f' \geq 0$ on $I$ as well.
19. The stronger Vasudeva and Schoenberg theorems. Bernstein’s theorem.
Moment-sequence transforms.

19. The stronger Vasudeva and Schoenberg theorems. Bernstein’s theorem.
Moment-sequence transforms.

19.1. The theorems of Vasudeva and Bernstein. Having shown (the stronger form of) the Horn–Loewner theorem [17.1] we use it to prove the following strengthening of Vasudeva’s theorem [16.4]. In it, recall from Definition [12.18] that \( \mathfrak{HTN}_n \) denotes the set of \( n \times n \) Hankel totally non-negative matrices. (These are automatically positive semidefinite.)

**Theorem 19.1** (Vasudeva’s theorem, stronger version – also see Remark 19.18). Suppose \( I = (0, \infty) \) and \( f : I \to \mathbb{R} \). The following are equivalent:

1. The entrywise map \( f[-] \) preserves positivity on \( \mathbb{P}_n(I) \) for all \( n \geq 1 \).
2. The entrywise map \( f[-] \) preserves positivity on all matrices in \( \mathfrak{HTN}_n \) with positive entries and rank at most 2, for all \( n \geq 1 \).
3. The function \( f \) equals a convergent power series \( \sum_{k=0}^{\infty} c_k x^k \) for all \( x \in I \), with the Maclaurin coefficients \( c_k \geq 0 \) for all \( k \geq 0 \).

To show the theorem, we require the following well-known classical result by Bernstein:

**Definition 19.2.** If \( I \subset \mathbb{R} \) is open, we say that \( f : I \to \mathbb{R} \) is absolutely monotonic if \( f \) is smooth on \( I \) and \( f^{(k)} \geq 0 \) on \( I \) for all \( k \geq 0 \).

**Theorem 19.3** (Bernstein). Suppose \( -\infty < a < b \leq \infty \). If \( f : [a, b) \to \mathbb{R} \) is continuous at \( a \) and absolutely monotonic on \( (a, b) \), then \( f \) can be extended analytically to the complex disc \( D(a, b - a) \).

With Bernstein’s theorem in hand, the ‘stronger Vasudeva theorem’ follows easily:

**Proof of Theorem 19.1** By the Schur product theorem or Lemma 16.1 (3) \( \implies \) (1); and clearly (1) \( \implies \) (2). Now suppose (2) holds. By the stronger Horn–Loewner theorem [17.1] \( f^{(k)} \geq 0 \) on \( I \) for all \( k \geq 0 \), i.e., \( f \) is absolutely monotonic on \( I \). In particular, \( f \) is non-negative and non-decreasing on \( I = (0, \infty) \), so it can be continuously extended to the origin via: \( f(0) := \lim_{x \to 0^+} f(x) \geq 0 \). Now apply Bernstein’s theorem with \( a = 0 \) and \( b = \infty \) to deduce that \( f \) agrees on \( [0, \infty) \) with an entire function \( \sum_{k=0}^{\infty} c_k x^k \). Moreover, since \( f^{(k)} \geq 0 \) on \( I \) for all \( k \), it follows that \( f^{(k)}(0) \geq 0 \), i.e., \( c_k \geq 0 \ \forall k \geq 0 \). Restricting to \( I \), we obtain (3), as desired. \( \square \)

On a related note, recall Theorems 9.3 and 12.19 which showed that when studying entrywise powers preserving the two closed convex cones \( \mathbb{P}_n([0, \infty)) \) and \( \mathfrak{HTN}_n \), the answers were identical. This is perhaps not surprising, given Theorem 4.1. In this vein, we observe that such an equality of preserver sets also holds when classifying the entrywise maps preserving Hankel \( TN \) matrices with positive entries:

**Corollary 19.4.** With \( I = (0, \infty) \) and \( f : I \to \mathbb{R} \), the three assertions in Theorem 19.1 are further equivalent to:

4. The entrywise map \( f[-] \) preserves total non-negativity on all matrices in \( \mathfrak{HTN}_n \) with positive entries, for all \( n \geq 1 \).
5. The entrywise map \( f[-] \) preserves total non-negativity on the matrices in \( \mathfrak{HTN}_n \) with positive entries and rank at most 2, for all \( n \geq 1 \).

**Proof.** Clearly (4) \( \implies \) (5) \( \implies \) (2), where (1)–(3) are as in Theorem 19.1. That (3) \( \implies \) (4) follows from Lemma 16.1 and Theorem 4.1. \( \square \)
Remark 19.5. Here is one situation where the two sets of preservers – of positivity on $\mathbb{P}_n$ for all $n$, and of total non-negativity on $\mathcal{HTN}_n$ for all $n$ – differ: if we also allow zero entries, as opposed to only positive entries as in the preceding corollary and Theorem 19.1. In this case, one shows that the preservers of $\mathcal{HTN}_n$ for all $n$ are the functions $f$ such that $f_{(0,\infty)}$ is absolutely monotonic, whence a power series with non-negative Maclaurin coefficients; and such that $0 \leq f(0) \leq \lim_{x \to 0^+} f(x)$, since the only Hankel $TN$ matrices with a zero entry arise as truncated moment matrices of measures $a\delta_0$. On the other hand, by considering the rank-two Hankel positive semidefinite matrix $A := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and considering the inequality

$$\lim_{x \to 0^+} \det f[xA] \geq 0,$$

it follows that $f$ is continuous at $0^+$. (Note, $A$ is not $TN$.) In particular, the entrywise preservers of positivity on $\bigcup_{n \geq 1} \mathbb{P}_n([0,\infty))$ are precisely the functions $\sum_{k \geq 0} c_k x^k$, with all $c_k \geq 0$.

Remark 19.6. As a reminder, we recall that if one instead tries to classify the entrywise preservers of total non-negativity on all (possibly symmetric) $TN$ matrices, then one obtains only the constant or linear functions $f(x) = c, cx$ for $c, x \geq 0$. See Theorem 12.11 above.

To complete the proof of the stronger Vasudeva theorem 19.1 as well as its corollary above, it remains to show Bernstein’s theorem.

Proof of Bernstein’s theorem 19.3. First we claim that $f^{(k)}(a^+)$ exists and equals $\lim_{x \to a^+} f^{(k)}(x)$ for all $k \geq 0$. The latter limit here exists because $f^{(k+1)} \geq 0$ on $(a, b)$, so $f^{(k)}(x)$ is non-negative and non-decreasing on $(a, b)$.

It suffices to show the claim for $k = 1$. But here we compute:

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0^+} f'(a + c(h)),$$

where $c(h) \in [0, h]$ exists and goes to zero as $h \to 0^+$, by the Mean Value Theorem. The claim now follows from the previous paragraph. In particular, $f^{(k)}$ exists and is continuous, non-negative, and non-decreasing on $[a, b]$.

Applying Taylor’s theorem, we have

$$f(x) = f(a^+) + f'(a^+)(x - a) + \cdots + f^{(n)}(a^+)\frac{(x - a)^n}{n!} + R_n(x),$$

where $R_n$ is the Taylor remainder:

$$R_n(x) = \int_a^x \frac{(x - t)^n}{n!} f^{(n+1)}(t) \, dt. \quad (19.7)$$

By the assumption on $f$, we see that $R_n(x) \geq 0$. Changing variables to $t = a + y(x - a)$, the limits for $y$ change to $0, 1$, and we have:

$$R_n(x) = \frac{(x - a)^{n+1}}{n!} \int_0^1 (1 - y)^n f^{(n+1)}(a + y(x - a)) \, dy.$$

Since $f^{(n+2)} \geq 0$ on $[a, b]$, if $a \leq x \leq c$ for some $c < b$, then uniformly in $[a, c]$ we have:

$$0 \leq f^{(n+1)}(a + y(x - a)) \leq f^{(n+1)}(a + y(c - a)).$$

Therefore using Taylor’s remainder formula
19. The stronger Vasudeva and Schoenberg theorems. Bernstein’s theorem.
Moment-sequence transforms.

once again, we obtain:

\[ 0 \leq R_n(x) \leq \frac{(x-a)^{n+1}}{n!} \int_0^1 (1 - y)^n f^{(n+1)}(a + y(c-a)) \, dy \]

\[ = R_n(c) \frac{(x-a)^{n+1}}{(c-a)^{n+1}} \]

\[ = \frac{(x-a)^{n+1}}{(c-a)^{n+1}} \left( f(c) - \sum_{k=0}^{n} f^{(k)}(a^+) \frac{(c-a)^k}{k!} \right) \]

\[ \leq f(c) \frac{(x-a)^{n+1}}{(c-a)^{n+1}}. \]

From this it follows that \( \lim_{n \to \infty} R_n(x) = 0 \) for all \( x \in [a,c) \). Since this holds for all \( c \in (a,b) \), the Taylor series of \( f \) converges to \( f \) on \( [a,b) \). In other words,

\[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a^+)}{k!}(x-a)^k, \quad x \in [a,b]. \]

Now if \( z \in D(a,b-a) \), then clearly \( a+|z-a| < a+(b-a) = b \). Choosing any \( c \in (a+|z-a|,b) \), we check that the Taylor series converges (absolutely) at \( z \):

\[ \left| \sum_{k=0}^{\infty} \frac{f^{(k)}(a^+)}{k!}(z-a)^k \right| \leq \sum_{k=0}^{\infty} \frac{f^{(k)}(a^+)}{k!}|z-a|^k \leq \sum_{k=0}^{\infty} \frac{f^{(k)}(a^+)}{k!}|c-a|^k = f(c) < \infty. \]

This completes the proof of Bernstein’s theorem – and with it, the stronger form of Vasudeva’s theorem.

\[ \square \]

\textbf{Remark 19.8.} We mention for completeness that Bernstein’s theorem was extended by Bernstein in 1926, to show that even if \( f^{(k)} \geq 0 \) in \( (a,b) \) only for \( k \geq 0 \) even, then \( f \) is necessarily analytic in \( (a,b) \). (In fact Bernstein worked only with divided differences – see Theorem \[37.10\] below.) This was further extended by Boas in \textit{Duke Math. J.}, as follows.

\textbf{Theorem 19.9 (Boas, 1941).} Let \( \{n_p : p \geq 1\} \) be an increasing sequence of positive integers such that \( n_{p+1}/n_p \) is uniformly bounded. Let \( (a,b) \subset \mathbb{R} \) and \( f : (a,b) \to \mathbb{R} \) be smooth. If for each \( p \geq 1 \), the derivative \( f^{(n_p)} \) does not change sign in \( (a,b) \), then \( f \) is analytic in \( (a,b) \).

19.2. The stronger version of Schoenberg’s theorem. We now come to the main result of this part of the text: the promised strengthening of Schoenberg’s theorem.

\textbf{Theorem 19.10 (Schoenberg’s theorem, stronger version).} Given \( f : \mathbb{R} \to \mathbb{R} \), the following are equivalent:

1. The entrywise map \( f[-] \) preserves positivity on \( \mathbb{P}_n(\mathbb{R}) \) for all \( n \geq 1 \).
2. The entrywise map \( f[-] \) preserves positivity on the Hankel matrices in \( \mathbb{P}_n(\mathbb{R}) \) of rank at most 3, for all \( n \geq 1 \).
3. The function \( f \) equals a convergent power series \( \sum_{k=0}^{\infty} c_k x^k \) for all \( x \in \mathbb{R} \), with the Maclaurin coefficients \( c_k \geq 0 \) for all \( k \geq 0 \).

See also Theorem \[19.15\] for two a priori weaker, yet equivalent, assertions.
Remark 19.11. Recall the two definitions of positive definite functions, from Definition 16.13 and the discussion preceding Lemma 16.14. The metric-space version for Euclidean and Hilbert spheres was connected by Schoenberg, for functions of the form $f \circ \cos$, to kernels $f : S^{r-1} \times S^{r-1}$ satisfying: $(f((x_j, x_k)))_{j,k \geq 0}$ is positive semidefinite. A third notion of positive definite kernels on Hilbert spaces $\mathcal{H}$ arises from here, and is important in machine learning among other areas: one says $f : \mathbb{R} \to \mathbb{R}$ is positive definite on $\mathcal{H}$ if, for any choice of finitely many points $x_j, j \geq 0$, the matrix $(f((x_j, x_k)))_{j,k \geq 0}$ is positive semidefinite. Since Gram matrix ranks are bounded above by $\dim \mathcal{H}$, this shows that Rudin’s 1959 theorem 16.3 classifies the positive definite functions on $\mathcal{H}$ for any real Hilbert space of dimension $\geq 3$.

The stronger Schoenberg theorem 19.10 above, provides a second proof.

Returning to the stronger Schoenberg theorem 19.10, clearly (1) $\implies$ (2), and (3) $\implies$ (1) by the Pólya–Szegő observation 16.1. Thus, the goal over the next few sections is to prove (2) $\implies$ (3). The proof is simplified when some of the arguments below are formulated in the language of moment-sequences and their preservers. We begin by defining these and explaining the dictionary between moment-sequences and positive-semidefinite Hankel matrices, due to Hamburger (among others).

Definition 19.12. Recall that given an integer $k \geq 0$ and a real measure $\mu$ supported on a subset of $\mathbb{R}$, $\mu$ has $k$th moment equal to the following (if it converges):

$$s_k(\mu) := \int_{\mathbb{R}} x^k \, d\mu.$$ 

Henceforth we only work with admissible measures, i.e. such that $\mu$ is non-negative on $\mathbb{R}$ and $s_k(\mu)$ converges for all $\mu$. The moment-sequence of such a measure $\mu$ is the sequence

$$s(\mu) := (s_0(\mu), s_1(\mu), \ldots).$$

We next define transforms of moment-sequences: a function $f : \mathbb{R} \to \mathbb{R}$ acts entrywise, to take moment-sequences to real sequences:

$$f[s(\mu)] := (f(s_k(\mu)))_{k \geq 0}. \quad (19.13)$$

We are interested in examining when the transformed sequence (19.13) is also the moment-sequence of an admissible measure supported on $\mathbb{R}$. This connects to the question of positivity preservers via the following classical result.

Theorem 19.14 (Hamburger). A real sequence $(s_k)_{k \geq 0}$ is the moment-sequence of an admissible measure, if and only if the semi-infinite Hankel matrix $H := (s_{j+k})_{j,k \geq 0}$ is positive semidefinite.

Recall that the easy half of this result was proved long ago, in Lemma 2.21.

Thus, entrywise functions preserving positivity on Hankel matrices are intimately related to moment-sequence preservers. Also note that if a measure $\mu$ has finite support in the real line, then by examining e.g. (2.20), the Hankel moment matrix $H_\mu$ (i.e., every submatrix) has rank at most the size of the support. From this and Hamburger’s theorem, we deduce all but the last sentence of the following result:

Theorem 19.15. Theorem 19.10(2) implies the following a priori weaker statement:

(4) For each measure

$$\mu = a\delta_1 + b\delta_{u_0} + c\delta_{-1}, \quad \text{with } u_0 \in (0, 1), \ a, b, c \geq 0,$$

there exists an admissible (non-negative) measure $\sigma = \sigma_\mu$ on $\mathbb{R}$ such that $f(s_k(\mu)) = s_k(\sigma) \ \forall k \geq 0$. 


In turn, this implies the still-weaker statement:

(5) For each measure \( \mu \) as in (19.16), with semi-infinite Hankel moment matrix \( H_\mu \), the matrix \( f[H_\mu] \) is positive semidefinite.

In fact, these statements are equivalent to the assertions in Theorem 19.10.

Remark 19.17. In this text, we do not prove Hamburger’s theorem; but we have used it to state Theorem 19.15(4) – i.e., in working with the admissible measure \( \sigma = \sigma_\mu \). A closer look reveals that the use of Hamburger’s theorem and moment-sequences is not required to prove Schoenberg’s theorem, or even its stronger form in Theorem 19.15 which is explained in (proving) Theorem 19.15(5). Our workaround is explained in the next section, via a ‘positivity-certificate trick’ involving limiting sum-of-squares representations of polynomials. That said, moment-sequences help simplify the presentation of the proof, and hence we will continue to use them in the proof, in later sections.

The next few sections are devoted to proving \((4) \implies (3)\) (in Theorems 19.10 and 19.15). Here is an outline of the steps in the proof:

1. All matrices \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbb{P}_2((0, \infty)) \) with \( a \geq c \) occur as leading principal submatrices of the Hankel moment matrices \( H_\mu \), where \( \mu \) is as in (19.16).
2. Apply the stronger Horn–Loewner theorem 17.1 and Bernstein’s theorem to deduce that \( f|_{(0, \infty)} = \sum_{k=0}^{\infty} c_k x^k \) for some \( c_k \geq 0 \).
3. If \( f \) satisfies assertion (5) in Theorem 19.15, then \( f \) is continuous on \( \mathbb{R} \).
4. If \( f \) is smooth and satisfies assertion (5) in Theorem 19.15, then \( f \) is real analytic.
5. Real analytic functions satisfy the desired implication above: \((5) \implies (3)\).
6. Using mollifiers and complex analysis, one can go from smooth functions to continuous functions.

Notice that Steps 3, 4–5, and 6 resemble the three steps in the proof of the stronger Horn–Loewner theorem 17.1.

In this section, we complete the first two steps in the proof.

**Step 1:** For the first step, suppose \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbb{P}_2((0, \infty)) \) with \( a \geq c \). There are three cases. First, if \( b = \sqrt{ac} \) then use \( \mu = a\delta_{b/a} \), since \( 0 < b/a \leq 1 \).

Henceforth, assume \( 0 < b < \sqrt{ac} \leq a \). (In particular, \( 2b < 2\sqrt{ac} \leq a + c \).) The second case is if \( b > c \); we then find \( t > 0 \) such that \( A - t1_{2 \times 2} \) is singular. This condition amounts to a linear equation in \( t \), with solution (to be verified by the reader):

\[
t = \frac{ac - b^2}{a + c - 2b} > 0.
\]

Then \( c - t = \frac{(b - c)^2}{a + c - 2b} > 0 \), whence \( a - t, b - t > 0 \) and \( A = \begin{pmatrix} s_0(\mu) & s_1(\mu) \\ s_1(\mu) & s_2(\mu) \end{pmatrix} \), where

\[
\mu = \frac{ac - b^2}{a + c - 2b} \delta_1 + \frac{(a - b)^2}{a + c - 2b} \delta_{b-c}, \quad \text{with} \quad \frac{b - c}{a - b} \in (0, 1).
\]

The third case is when \( 0 < b \leq c \leq \sqrt{ac} \leq a \), with \( b < \sqrt{ac} \). Now find \( t > 0 \) such that the matrix \( \begin{pmatrix} a - t & b + t \\ b + t & c - t \end{pmatrix} \in \mathbb{P}_2((0, \infty)) \) and is singular. To do so requires solving a linear
equation, which yields:
\[
\begin{align*}
t &= \frac{ac - b^2}{a + c + 2b}, \\
c - t &= \frac{(b + c)^2}{a + c + 2b}, \\
a - t &= \frac{(a + b)^2}{a + c + 2b}, \\
b + t &= \frac{(a + b)(b + c)}{a + c + 2b},
\end{align*}
\]
and all of these are strictly positive. So \(a, b, c > 0\) are the first three moments of
\[
\mu = \frac{ac - b^2}{a + c + 2b} \delta_{-1} + \frac{(a + b)^2}{a + c + 2b} \delta_{\frac{b+c}{a+b}}, \quad \text{with } \frac{b + c}{a + b} \in (0, 1].
\]

**Step 2:** Observe that the hypotheses of the stronger Horn–Loewner theorem \[17.1\] can be rephrased as saying that \(f[-]\) sends \(P_2((0, \infty))\) to \(P_2(\mathbb{R})\), and that assertion (4) in Theorem \[19.15\] holds for all measures \(a\delta_1 + b\delta_{u_0}\), where \(u_0 \in (0, 1)\) is fixed and \(a, b > 0\). By Step 1 and the hypotheses, we can apply the stronger Horn–Loewner theorem in our setting for each \(n \geq 3\), whence \(f|_{(0, \infty)}\) is smooth and absolutely monotonic. As in the proof of the stronger Vasudeva theorem \[19.1\] extend \(f\) continuously to the origin, say to a function \(\tilde{f}\), and apply Bernstein’s theorem \[19.3\]. It follows that \(\tilde{f}|_{[0, \infty)}\) is a power series with non-negative Maclaurin coefficients, and Step 2 follows by restricting to \(\tilde{f}|_{(0, \infty)} = f|_{(0, \infty)}\). \(\square\)

**Remark 19.18.** From Step 2 above, it follows that assertions (1), (2) in the stronger Vasudeva theorem \[19.1\] can be further weakened, to deal only with the rank-one matrices in \(P_2(I)\), the Toeplitz matrices in \(P_2(I)\), and with the (Hankel \(TN\) moment matrices of) measures \(a\delta_1 + b\delta_{u_0}\), for a single fixed \(u_0 \in (0, 1)\) and all \(a, b \geq 0\) with \(a + b > 0\).

We continue with the proof of the stronger Schoenberg theorem \[19.10\]. Previously, we have shown the first two of the six steps in the proof (these are listed following Theorem \[19.15\]).

**Step 3:** The next step is to show that if assertion (4) (or (5)) in Theorem \[19.15\] holds, then \( f \) is continuous on \( \mathbb{R} \). Notice from Steps 1, 2 of the proof that \( f \) is absolutely monotonic, whence continuous, on \( (0, \infty) \).

### 20.1. Integration trick and proof of continuity

At this stage, we transition to moment-sequence preservers, via Hamburger’s theorem \[19.14\]. The following ‘integration trick’ will be used repeatedly in what follows: Suppose \( p(t) \) is a real polynomial that takes non-negative values for \( t \in [-1, 1] \). Write \( p(t) = \sum_{k=0}^{\infty} a_k t^k \) (with only finitely many \( a_k \) nonzero, but not necessarily all positive, note). If \( \mu \geq 0 \) is an admissible measure – in particular, non-negative by Definition \[19.12\] – then by assumption and Hamburger’s theorem we have \( f(s_k(\mu)) = s_k(\sigma_\mu) \forall k \geq 0 \), for some admissible measure \( \sigma_\mu \geq 0 \) on \( \mathbb{R} \). Now assuming \( \sigma_\mu \) is supported on \([-1, 1]\) (which is not \textit{a priori} clear from the hypotheses), we have:

\[
0 \leq \int_{-1}^{1} p(t) \, d\sigma_\mu = \sum_{k=0}^{\infty} \int_{-1}^{1} a_k t^k \, d\sigma_\mu = \sum_{k=0}^{\infty} a_k s_k(\sigma_\mu) = \sum_{k=0}^{\infty} a_k f(s_k(\mu)).
\]

**Example 20.2.** Suppose \( p(t) = 1 - t^d \) on \([-1, 1]\), for some integer \( d \geq 1 \). Then \( f(s_0(\mu)) - f(s_d(\mu)) \geq 0 \). As a further special case, if \( \mu = a\delta_1 + b\delta_{w_0} + c\delta_{-1} \) as in Theorem \[19.15(4)\], if \( \sigma_\mu \) is supported on \([-1, 1]\) then this would imply:

\[
f(a + b + c) \geq f(a + b u_0^d + c(-1)^d), \quad \forall u_0 \in (0, 1), \ a, b, c \geq 0.
\]

It is not immediately clear how the preceding inequalities can be obtained by considering only the preservation of matrix positivity by \( f[-] \) (or more involved such assertions). As we will explain shortly, this has connections to real algebraic geometry; in particular, to a well-known program of Hilbert.

Returning to the proof of continuity in Schoenberg’s theorem, we suppose without further mention that \( f \) satisfies only Theorem \[19.15(4)\] above – and hence is absolutely monotonic on \((0, \infty)\). We begin by showing two preliminary lemmas, which are used in the proof of continuity.

**Lemma 20.3.** \( f \) is bounded on compact subsets of \( \mathbb{R} \).

**Proof.** If \( K \subset \mathbb{R} \) is compact, say \( K \subset [-M, M] \) for some \( M > 0 \), then note that \( f|_{(0, \infty)} \) is non-decreasing, whence \( 0 \leq |f(x)| \leq f(M), \ \forall x \in (0, M] \). Now apply \( f[-] \) to the matrix \( \begin{pmatrix} x & -x \\ -x & x \end{pmatrix} \), arising from \( \mu = x\delta_{-1} \), with \( x > 0 \). This implies \( |f(-x)| \leq f(x) \leq f(M) \).

Similarly considering \( \mu = \frac{M}{2}\delta_1 + \frac{M}{2}\delta_{-1} \) shows that \( |f(0)| \leq f(M) \). \( \square \)

Now say \( \mu = a\delta_1 + b\delta_{w_0} + c\delta_{-1} \) as above, or more generally, \( \mu \) is any non-negative measure supported in \([-1, 1]\). It is easily seen that its moments \( s_k(\mu), \ k \geq 0 \) are all uniformly bounded in absolute value – in fact, by the mass \( s_0(\mu) \). Our next lemma shows that the converse is also true.

**Lemma 20.4.** Given an admissible measure \( \sigma \) on \( \mathbb{R} \), the following are equivalent:

1. The moments of \( \sigma \) are all uniformly bounded in absolute value.
2. The measure \( \sigma \) is supported on \([-1, 1]\).
Proof. As discussed above, (2) \implies (1). To show the converse, suppose (1) holds but (2) fails. Then \( \sigma \) has positive mass in \((1, \infty) \cup (-\infty, -1)\). We obtain a contradiction in the first case; the proof is similar in the other case. Thus, suppose \( \sigma \) has positive mass on 
\[(1, \infty) = [1 + \frac{1}{2^n}, 1 + \frac{1}{n+1}] \cup [1 + \frac{1}{2^n}, 1 + \frac{1}{n+1}] \cup \cdots \cup [1 + \frac{1}{2^n}, 1 + \frac{1}{n+1}] \cup \cdots , \]
where \( 1/0 := \infty \). Then \( \sigma(I_n) > 0 \) for some \( n \geq 0 \), where we denote \( I_n := [1 + \frac{1}{n+1}, 1 + \frac{1}{n}] \) for convenience. But now we obtain the desired contradiction:
\[ s_{2k}(\sigma) = \int_1 \int_{1 + \frac{1}{n+1}} x^{2k} \, \mathrm{d}\sigma \geq \int_{1 + \frac{1}{n+1}}^{1 + \frac{1}{n}} x^{2k} \, \mathrm{d}\sigma \geq \int_{1 + \frac{1}{n+1}}^{1 + \frac{1}{n}} (1 + \frac{1}{n+1})^k \, \mathrm{d}\sigma \geq \sigma(I_n)(1 + \frac{1}{n+1})^k , \]
and this is not uniformly bounded over all \( k \geq 0 \).

With these basic lemmas in hand, we have:

Proof of Step 3 for the stronger Schoenberg theorem: continuity. Given a function \( f \) satisfying Theorem [19.15](4), and any measure \( \mu = a \delta_1 + b \delta_{u_0} + c \delta_{-1} \) for \( u_0 > 0 \) and \( a, b, c \geq 0 \), note that \( |s_k(\mu)| \leq s_0(\mu) = a + b + c \). Hence by Lemma [20.3] the moments \( s_k(\sigma) \) are uniformly bounded over all \( k \). By Lemma [20.4] it follows that \( \sigma_{\mu} \) must be supported in \([-1, 1]\). In particular, we can apply the integration trick [20.1] above.

We use this trick to prove continuity at \(-\beta\) for \( \beta \geq 0 \). (By Step 2, this proves the continuity of \( f \) on \( \mathbb{R} \).) Thus, fix \( \beta \geq 0 \), \( u_0 \in (0, 1) \), and \( b > 0 \), and define
\[ \mu := (\beta + bu_0) \delta_{-1} + b \delta_{u_0} . \]
Let \( p_{\pm, 1}(t) := (1 \pm t)(1 - t^2) \); note that these polynomials are non-negative on \([-1, 1]\). By the integration trick [20.1],
\[ \int_{-1}^1 p_{\pm, 1}(t) \, \mathrm{d}\sigma_{\mu}(t) \geq 0 \]
\[ \implies s_0(\sigma_{\mu}) - s_2(\sigma_{\mu}) \geq s_1(\sigma_{\mu}) - s_3(\sigma_{\mu}) \]
\[ \implies f(s_0(\mu)) - f(s_2(\mu)) \geq \left| f(s_1(\mu)) - f(s_3(\mu)) \right| \]
\[ \implies f(\beta + b(1 + u_0)) - f(\beta + b(u_0 + u_0^2)) \geq \left| f(-\beta) - f(-\beta - bu_0(1 - u_0^2)) \right| . \]
Now let \( b \to 0^+ \). Then the left-hand side goes to zero by Step 2 (in the previous section), whence so does the right-hand side. This implies \( f \) is left-continuous at \(-\beta\) for all \( \beta \geq 0 \). To show \( f \) is right-continuous at \(-\beta\), use \( \mu' := (\beta + bu_0^3) \delta_{-1} + b \delta_{u_0} \) instead of \( \mu \).

Remark 20.5. Akin to its use in proving the continuity of \( f \), the integration trick [20.1] can also be used to prove the boundedness of \( f \) on compact sets \([-M, M] \), as in Lemma [20.3].

To do so, work with the polynomials \( p_{\pm, 0}(t) := 1 \pm t \), which are also non-negative on \([-1, 1]\). Given \( 0 \leq x < M \), applying [20.1] to \( \mu = M \delta_{x/M} \) and \( \mu' = x \delta_{-1} \) shows Lemma [20.3].

20.2. The integration trick explained: semi-algebraic geometry. Earlier in this section, we used the following 'integration trick': if \( \sigma \geq 0 \) is a real measure supported in \([-1, 1]\) with all moments finite, i.e. the Hankel moment matrix \( H_{\sigma} := (s_{j+k}(\sigma))_{j,k=0}^{\infty} \) is positive semidefinite; and if a polynomial \( p(t) = \sum_{k \geq 0} a_k t^k \) is non-negative on \([-1, 1]\), then
\[ 0 \leq \int_{-1}^1 p(t) \, \mathrm{d}\sigma = \sum_{k \geq 0} \int_{-1}^1 a_k t^k \, \mathrm{d}\sigma = \sum_{k \geq 0} a_k s_k(\sigma) . \]

This integration trick is at the heart of the link between moment problems and (Hankel) matrix positivity. This trick is now explained; namely, how this integral inequality can be
understood purely in terms of the positive semidefiniteness of $H_\sigma$. This also has connections to real algebraic geometry and Hilbert’s seventeenth problem.

The basic point is as follows: if a $d$-variate polynomial (in one or several variables) is a sum of squares of real polynomials – also called a s.o.s. polynomial – then it is automatically non-negative on $\mathbb{R}^d$. However, Hilbert showed in [130] – following the doctoral dissertation of Hermann Minkowski – that for $d \geq 2$, there exist polynomials that are not sums of squares, but are non-negative on $\mathbb{R}^d$. The first such example was constructed in 1967, and is the well-known Motzkin polynomial $M(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$. Such phenomena are also studied on polytopes (results of Farkas, Pólya, and Handelman), and on more general ‘semi-algebraic sets’ including compact ones (results of Stengle, Schmüdgen, Putinar, and Vasilescu, among others).

Now given say a one-variable polynomial that is non-negative on a semi-algebraic set such as $[-1, 1]$, one would like a positivity certificate for it, meaning a sum-of-squares ("s.o.s.") representation mentioned above, or more generally, a limiting s.o.s. representation. To make this precise, define the $L^1$-norm, or the Wiener norm, of a polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ as:

$$\|p(t)\|_{1,+} := \sum_{k \geq 0} |a_k|. \quad (20.6)$$

One would thus like to find a sequence $p_n$ of s.o.s. polynomials such that $\|p_n(t) - p(t)\|_{1,+} \to 0$ as $n \to \infty$. Two simple cases are if there exist polynomials $q_n(t)$ such that (i) $p_n(t) = q_n(t)^2 \forall n$, or (ii) $p_n(t) = \sum_{j=1}^{n} q_k(t)^2 \forall n$.

How does this connect to matrix positivity? It turns out that in our given situation, what is required is precisely a positivity certificate. For example, say $p(t) = (3-t)^2 = 9 - 6t + t^2 \geq 0$

Indeed, as explained e.g. in [244], by the AM–GM inequality we have $M(x, y) \geq 2t^3 - 3t^2 + 1 = (2t^2 - t - 1)(t - 1)$, where $t = x^2y^2$. Now either $t \in [0, 1]$, so both factors on the right are non-positive; or $t > 1$, whence both factors are positive. Next, suppose $M(x, y) = \sum f_j(x,y)^2$ is a sum of squares; since $M(x, 0) = M(0, y) = 1$, it follows that $f_j(x, 0), f_j(0, y)$ are constants, whence the $f_j$ are of the form $f_j(x, y) = a_j + b_jx + c_jx^2 + d_jxy$. Now equating the coefficient of $x^2y^2$ in $M = \sum f_j^2$ gives: $-3 = \sum b_j^2$, a contradiction.

Hilbert then showed in [131] that every non-negative polynomial on $\mathbb{R}^2$ is a sum of four squares of rational functions; e.g. the Motzkin polynomial equals $(x^2+y^2)^2/2$ if $x^2+y^2 = 1$. For more on this problem, see e.g. [244].

The reader may recall the name of Motzkin from Theorem 3.20 above, in an entirely different context. As a historical diergression, we mention several relatively ‘disconnected’ areas of mathematics, in all of which, remarkably, Motzkin made fundamental contributions. His thesis [202] was a landmark work in the area of linear inequalities / linear programming, introducing in particular the Motzkin transposition theorem and the Fourier–Motzkin Elimination (FME) algorithm. Additionally, he proved in the same thesis the fundamental fact in geometric combinatorics that, a convex polyhedral set is the Minkowski sum of a compact (convex) polytope and a convex polyhedral cone. Third, in his thesis Motzkin also characterized the matrices that satisfy the variation diminishing property; see Theorem 4.20.

Then in [203], Motzkin studied what is now called the Motzkin number in combinatorics: this is the number of different ways to draw non-intersecting chords between $n$ marked points on a circle. In [204], he provided the first examples of a principal ideal domain that is not a Euclidean domain: $\mathbb{Z}[1 + \sqrt{-19}]$ [2]. In [205], he provided an ideal-free short proof of Hilbert’s Nullstellensatz, together with degree bounds. Motzkin also provided in [206] the aforementioned polynomial $M(x, y)$ in connection to Hilbert’s 17th problem.
on \( \mathbb{R} \). Then
\[
\int_{-1}^{1} p \, d\sigma = 9s_0(\sigma) - 6s_1(\sigma) + s_2(\sigma) = (3, -1) \begin{pmatrix} s_0(\sigma) & s_1(\sigma) \\ s_1(\sigma) & s_2(\sigma) \end{pmatrix} (3, -1)^T
\]
\[
= (3e_0 - e_1)^T H_\sigma (3e_0 - e_1),
\]
where \( e_0 = (1, 0, 0, \ldots)^T, e_1 = (0, 1, 0, 0, \ldots)^T, \ldots \) comprise the standard basis for \( \mathbb{R}^{N_\mathbb{N}(0)} \), and \( H_\sigma \) is the semi-infinite, positive semidefinite Hankel moment matrix for \( \sigma \). From this calculation, it follows that \( \int_{-1}^{1} p \, d\sigma \) is non-negative – and this holds more generally, whenever there exists a (limiting) s.o.s. representation for \( p \).

We now prove the existence of such a limiting s.o.s. representation in two different ways for general \( p(t) \), and in a constructive third way for the special family of polynomials
\[
p_{\pm,n}(t) := (1 \pm t)(1 - t^2)^n, \quad n \geq 0.
\]
(Note, we used \( p_{\pm,0} \) and \( p_{\pm,1} \) to prove the local boundedness and continuity of \( f \) on \( \mathbb{R} \), respectively; and next time we will use \( p_{\pm,n} \) to prove that smoothness implies real analyticity.)

**Proof 1:** We claim more generally that for any dimension \( d \geq 1 \), every polynomial that is non-negative on \([-1,1]^d\) has a limiting s.o.s. representation. This is proved at the end of the 1976 paper of Berg, Christensen, and Ressel in *Math. Ann.*

**Proof 2:** Here is a constructive proof of a positivity certificate for the polynomials \( p_{\pm,n}(t) = (1 \pm t)(1 - t^2)^n, \quad n \geq 0 \). (It turns out, we only need to work with these in order to show the stronger Schoenberg theorem.) First notice that
\[
\text{Proof 2:}
\]
\[
(1 \pm t)(1 - t^2)^n = (1 \pm t)(1 - t^2)^n.
\]
and so on. Thus, if we show that \( p_{\pm,0}(t) = 1 \pm t \) are limits of s.o.s polynomials, then so are \( p_{\pm,n}(t) \) for all \( n \geq 0 \) (where limits are taken in the Wiener norm). But we have:
\[
\frac{1}{2} (1 \pm t)^2 = \frac{1}{2} t^2 + \frac{t^4}{4},
\]
\[
\frac{1}{4} (1 - t^2)^2 = \frac{1}{4} - \frac{t^2}{2} + \frac{t^4}{4},
\]
\[
\frac{1}{8} (1 - t^4)^2 = \frac{1}{8} - \frac{t^4}{4} + \frac{t^8}{8},
\]
and so on. Adding the first \( k \) of these equations shows that the partial sum
\[
p_k^\pm(t) := (1 - \frac{1}{2^k}) \pm t + \frac{t^{2k}}{2^k} = (1 \pm t) + \frac{t^{2k} - 1}{2^k}
\]
is a s.o.s. polynomial, for every \( k \geq 1 \). This provides a positivity certificate for \( 1 \pm t \), as desired. It also implies the sought-for interpretation of the integration trick in Step 3 above:
\[
\left| \int_{-1}^{1} p_k^\pm(t) - (1 \pm t) \, d\sigma \right| \leq \int_{-1}^{1} |p_k^\pm(t) - (1 \pm t)| \, d\sigma \leq \int_{-1}^{1} \frac{1}{2^k} \, d\sigma + \int_{-1}^{1} \frac{t^{2k}}{2^k} \, d\sigma \leq \frac{1}{2^k} \cdot 2s_0(\sigma),
\]
which goes to 0 as \( k \to \infty \). Hence, using the notation following (20.7),
\[
\int_{-1}^{1} (1 \pm t) \, d\sigma = \lim_{k \to \infty} \int_{-1}^{1} p_k^\pm(t) \, d\sigma = \frac{1}{2}(e_0 \pm e_1)^T H_\sigma (e_0 \pm e_1) + \sum_{j=2}^{\infty} \frac{1}{2^j}(e_0 - e_{2j-1})^T H_\sigma (e_0 - e_{2j-1}),
\]

The positivity-certificate trick.

and this is non-negative because $H_\sigma$ is positive semidefinite.

**Proof 3:** If we only want to interpret the integration trick (20.1) in terms of the positivity of the Hankel moment matrix $H_\sigma$, then the restriction of using the Wiener norm $\| \cdot \|_1$ can be relaxed, and one can work instead with the weaker notion of the uniform norm. With this metric, we claim more generally that every continuous function $f(t_1, \ldots, t_d)$ that is non-negative on a compact subset $K \subset \mathbb{R}^d$ has a limiting s.o.s. representation on $K$. (Specialized to $d = 1$ and $K = [-1, 1]$, this proves the integration trick.)

To see the claim, observe that $\sqrt{f}(t_1, \ldots, t_d) : K \to [0, \infty)$ is continuous, so by the Stone–Weierstrass theorem, there exists a polynomial sequence $q_n$ converging uniformly to $\sqrt{f}$ in $L^\infty(K)$. Thus $q_n^2 \to f$ in $L^\infty(K)$, as desired. Explicitly, if $d = 1$ and $q_n(t) = \sum_{k=0}^\infty c_{n,k}t^k$, then define the semi-infinite vectors

$$u_n := (c_{n,0}, c_{n,1}, \ldots)^T, \quad n \geq 1.$$  

Now compute for any admissible measure $\sigma$ supported in $K$:

$$\int_K f \ d\sigma = \lim_{n \to \infty} \int_K q_n^2(t) \ d\sigma = \lim_{n \to \infty} u_n^T H_\sigma u_n \geq 0, \quad (20.10)$$

which is a positivity certificate for all continuous, non-negative functions on compact $K \subset \mathbb{R}$.

This reasoning extends to all dimensions $d \geq 1$ and compact $K \subset \mathbb{R}^d$, by Lemma 2.23.

20.3. From the integration trick to the positivity-certificate trick. The above ‘Proof 2’ is the key to understanding why Hamburger’s theorem is not required to prove the stronger Schoenberg theorem 19.15 (namely, (5) $\implies$ (3)). Specifically, we only need to use the following fact:

For each fixed $n \geq 0$, if $\sum_k a_k t^k$ is the expansion of $p_{\pm,n}(t) = (1 \pm t)(1 - t^2)^n \geq 0$ on $[-1, 1]$, then $\sum_k a_k f(s_k(\mu)) \geq 0$.

This was derived above using the integration trick (20.1) via the auxiliary admissible measure $\sigma_\mu$, which exists by Theorem 19.15(4). We now explain a workaround via a related ‘positivity-certificate trick’ that requires using only that $f(H_\mu)$ is positive semidefinite, hence allowing us to work with the weaker hypothesis, Theorem 19.15(5) instead. In particular, one can avoid using Hamburger’s theorem and requiring the existence of $\sigma_\mu$.

The positivity-certificate trick is as follows:

**Theorem 20.11.** Fix a semi-infinite real Hankel matrix $H = (f_{j+k})_{j,k \geq 0}$ that is positive semidefinite (i.e., its principal minors are), with all entries $f_j$ uniformly bounded. If a polynomial $p(t) = \sum_{j \geq 0} a_j t^j$ has a positivity certificate – i.e. a Wiener-limiting s.o.s. representation – then $\sum_{j \geq 0} a_j f_j \geq 0$.

By the ‘Proof 2’ above (see the discussion around 20.9), Theorem 20.11 applies to $p = p_{\pm,n}$ for all $n \geq 0$ and $H = f(H_\mu)$, where $f, \mu$ are as in Theorem 19.15(5). This implies the continuity of the entrywise positivity preserver $f$ in the above discussion, and also suffices to complete the proof below, of the stronger Schoenberg theorem.

**Proof.** As an illustrative special case, if $p(t)$ is the square of a polynomial $q(t) = \sum_{j \geq 0} c_j t^j$, then as in 20.7,

$$\sum_{j \geq 0} a_j f_j = \sum_{j,k \geq 0} c_j c_k f_{j+k} = u^T H u \geq 0, \quad \text{where } u = (c_0, c_1, \ldots)^T.$$
By additivity, the result therefore also holds for a sum of squares of polynomials. The subtlety in working with a limiting s.o.s. is that the degrees of each s.o.s. polynomial in the limiting sequence need not be uniformly bounded. Nevertheless, suppose in the Wiener norm (20.6) that

\[ p(t) = \lim_{n \to \infty} q_n(t), \]  
where \( q_n(t) = \sum_{k=0}^{K_n} q_{n,k}(t)^2 \)

is a sum of squares of polynomials for each \( n \).

Define the linear functional \( \Psi_H \) (given the Hankel matrix \( H \)) that sends a polynomial \( p(t) = \sum_{j \geq 0} a_j t^j \) to the scalar \( \Psi_H(p) := \sum_{j \geq 0} a_j f_j \). Now define the vectors \( u_{n,k} \) via:

\[ q_{n,k}(t) = \sum_{j \geq 0} q_{n,k}^{[j]} t^j \in \mathbb{R}[t], \quad u_{n,k} := (q_{n,k}^{[0]}, q_{n,k}^{[1]}, \ldots)^T. \]

Similarly, define \( q_n(t) = \sum_{j \geq 0} q_n^{[j]} t^j \). Then for all \( n \geq 1 \),

\[ \sum_{j \geq 0} q_n^{[j]} f_j = \Psi_H(q_n) = \sum_{k=0}^{K_n} \Psi_H(q_{n,k}^2) = \sum_{k=0}^{K_n} u_{n,k}^T H u_{n,k} \geq 0. \]

Finally, taking the limit as \( n \to \infty \), and writing \( p(t) = \sum_{j \geq 0} a_j t^j \), we claim that

\[ \sum_{j \geq 0} a_j f_j = \lim_{n \to \infty} \sum_{j \geq 0} q_n^{[j]} f_j \geq 0. \]

Indeed, the (first) equality holds because if \( M \geq \sup_j |f_j| \) is a uniform (and finite) upper bound, then

\[ \left| \sum_{j \geq 0} q_n^{[j]} f_j - \sum_{j \geq 0} a_j f_j \right| \leq \sum_{j \geq 0} |q_n^{[j]} - a_j| |f_j| \leq M \|q_n - p\|_{1,+}, \]

and this goes to zero as \( n \to \infty \). \qed
21. PROOF OF STRONGER SHOENBERG THEOREM: II. SMOOTHNESS IMPLIES REAL ANALYTICITY.

Having explained the positivity-certificate trick, we return to the proof of the stronger Schoenberg theorem. The present goal is to prove that if a smooth function \( f : \mathbb{R} \to \mathbb{R} \) satisfies assertion (5) in Theorem \([19.15]\) then \( f \) is real analytic and hence satisfies assertion (3) in Theorem \([19.10]\). (See Steps (4), (5) in the discussion following Theorem \([19.15]\)). To show these results, we first discuss the basic properties of real analytic functions that are required in the proofs.

21.1. Preliminaries on real analytic functions.

**Definition 21.1.** Suppose \( I \subset \mathbb{R} \) is an open interval, and \( f : I \to \mathbb{R} \) is smooth, denoted \( f \in C^\infty(I) \). Recall that the *Taylor series* of \( f \) at a point \( x \in I \) is

\[
(Tf)_x(y) := \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} (y-x)^j, \quad y \in I,
\]

if this sum converges at \( y \). Notice that this sum is not equal to \( f(y) \) in general.

Next, we say that \( f \) is real analytic on \( I \), denoted \( f \in C^\omega(I) \), if \( f \in C^\infty(I) \) and for all \( x \in I \) there exists \( \delta_x > 0 \) such that the Taylor series of \( f \) at \( x \) converges to \( f \) on \( (x - \delta_x, x + \delta_x) \).

Clearly, real analytic functions on \( I \) form a real vector space. Less obvious is the following useful property, which is stated without proof:

**Proposition 21.2.** Real analytic functions are closed under composition. More precisely, if \( I \xrightarrow{f} J \xrightarrow{g} \mathbb{R} \), and \( f, g \) are real analytic on their domains, then so is \( g \circ f \) on \( I \).

We also develop a few preliminary results on real analytic functions, which are needed to prove the stronger Schoenberg theorem. We begin with an example of real analytic functions, which depicts what happens in our setting.

**Lemma 21.3.** Suppose \( I = (0, R) \) for \( 0 < R \leq \infty \), and \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) on \( I \), where \( c_k \geq 0 \forall k \). Then \( f \in C^\omega(I) \) and \((Tf)_a(x)\) converges at all \( x \in I \) such that \( |x - a| < R - a \).

In particular, if \( R = \infty \) and \( a > 0 \), then \((Tf)_a(x) = f(x)\) on the domain of \( f \).

**Proof.** Note that \( \sum_{k=0}^{\infty} c_k x^k \) converges on \((-R, R)\). Thus, we show more generally that \((Tf)_a(x)\) converges to \( f(x) \) for \( |x - a| < R - a \), \( |x| < R \), \( a \geq 0 \) (whenever \( f \) is defined at \( x \)). Indeed,

\[
f(x) = \sum_{k=0}^{\infty} c_k ((x-a) + a)^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} c_k (x-a)^j a^{k-j}.
\]

Notice that this double sum is absolutely convergent, since

\[
\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} c_k |x-a|^j a^{k-j} = f(a + |x-a|) < \infty.
\]

Hence we can rearrange the double sum (e.g. by Fubini’s theorem), to obtain

\[
f(x) = \sum_{j=0}^{\infty} \left( \sum_{m=0}^{\infty} \binom{m+j}{j} c_{m+j} a^m \right) (x-a)^j = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x-a)^j = (Tf)_a(x)
\]

using standard properties of power series. In particular, \( f \) is real analytic on \( I \). \( \square \)
We also require the following well-known result on zeros of real analytic functions.

**Theorem 21.4 (Identity theorem).** Suppose $I \subset \mathbb{R}$ is an open interval and $f, g : I \to \mathbb{R}$ are real analytic. If the subset of $I$ where $f = g$ has an accumulation point in $I$, then $f \equiv g$ on $I$.

In other words, the zeros of a nonzero (real) analytic function form a discrete set.

**Proof.** Without loss of generality, we may suppose $g \equiv 0$. Suppose $c \in I$ is an accumulation point of the zero set of $f$. Expand $f$ locally at $c$ into its Taylor series, and claim that $f^{(k)}(c) = 0$ for all $k \geq 0$. Indeed, suppose for contradiction that $f^{(0)}(c) = \cdots = f^{(k-1)}(c) = 0 \neq f^{(k)}(c)$ for some $k \geq 0$. Then, $f(x) = \frac{f^{(k)}(c)}{k!} + o(x-c)$, whence $f$ is nonzero close to $c$, and this contradicts the hypotheses. Thus, $f^{(k)}(c) = 0 \forall k \geq 0$, which in turn implies that $f \equiv 0$ on an open interval around $c$.

Now consider the set $I_0 := \{x \in I : f^{(k)}(x) = 0 \forall k \geq 0\}$. Clearly $I_0$ is a closed subset of $I$. Moreover, if $c_0 \in I_0$ then $f \equiv (Tf)_{c_0} \equiv 0$ near $c_0$, whence the same happens at any point near $c_0$ as well. Thus $I_0$ is also an open subset of $I$. Since $I$ is connected, $I_0 = I$, and $f \equiv 0$. □

21.2. **Proof of the stronger Schoenberg theorem for smooth functions.** We continue with the proof of the stronger Schoenberg theorem ((5) \(\Rightarrow\) (2) in Theorems 19.10 and 19.15).

Akin to the proof of the stronger Horn–Loewner theorem 17.1, we have shown that any function satisfying the hypotheses in Theorem 19.15(5) must be continuous. Hence by the first two steps in the proof – listed after Remark 19.17 – we have that $f(x) = \sum_{k=0}^{\infty} c_k x^k$ on $[0, \infty)$, with all $c_k \geq 0$.

Again similar to the proof of the stronger Horn–Loewner theorem 17.1, we next prove the stronger Schoenberg theorem for smooth functions. The key step here is:

**Theorem 21.5.** Let $f \in C^\infty(\mathbb{R})$ be as in the preceding discussion, and define the family of smooth functions

$$H_a(x) := f(a + e^x), \quad a, x \in \mathbb{R}.$$  

Then $H_a$ is real analytic on $\mathbb{R}$, for all $a \in \mathbb{R}$.

For ease of exposition, we break up the proof into several steps.

**Lemma 21.6.** For all $n \geq 1$, we have

$$H_a^{(n)}(x) = a_{n,1} f'(a + e^x) e^x + a_{n,2} f''(a + e^x) e^{2x} + \cdots + a_{n,n} f^{(n)}(a + e^x) e^{nx},$$

where $a_{n,j}$ is a positive integer for all $1 \leq j \leq n$.

**Proof and remarks.** One shows by induction on $n \geq 1$ (with the base case of $n = 1$ immediate) that the array $a_{n,j}$ forms a ‘weighted variant’ of Pascal’s triangle, in that:

$$a_{n,j} = \begin{cases} 1, & \text{if } j = 1, n, \\ a_{n-1,j-1} + ja_{n-1,j}, & \text{otherwise,} \end{cases}$$
This concludes the proof. Notice that some of the entries of the array $a_{n,j}$ are easy to compute:

$$a_{n,1} = 1, \quad a_{n,2} = 2^{n-1} - 1, \quad a_{n,n-1} = \binom{n}{2}, \quad a_{n,n} = 1.$$ 

An interesting combinatorial exercise may be to seek a closed-form expression and a combinatorial interpretation for the other entries. □

**Lemma 21.7.** We have the following bound:

$$|H_a^{(n)}(x)| \leq H|_{[a]}^{(n)}(x), \quad \forall a, x \in \mathbb{R}, \ n \in \mathbb{Z}_{\geq 0}. \tag{21.8}$$

**Proof.** By Lemma 21.6 we have that $H|_{[a]}^{(n)}(x) \geq 0$ for all $a, x, n$ as in (21.8), so it remains to show the inequality. For this, we assume $a < 0$, and use the ‘positivity-certificate trick’ from the previous section – i.e. Theorem 20.11 applied to the polynomials $p_{\pm,n}(t) := (1 \pm t)(1 - t^2)^n, \quad n \geq 0$ and the admissible measure

$$\mu := |a|\delta_{-1} + e^x\delta_{-h}, \quad a, h > 0, \ x \in \mathbb{R}.$$ 

Notice that $p_{\pm} \geq 0$ on $[-1, 1]$. Hence by Theorem 20.11 and akin to the calculation in the previous section to prove continuity – we get:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(|a| + e^{x-2kh}) \geq \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(a + e^{x-(2k+1)h}).$$ 

Dividing both sides by $(2h)^n$ and sending $h \to 0^+$, we obtain:

$$|H|_{[a]}^{(n)}(x) \geq |H_a^{(n)}(x)|,$$

which proves (21.8), as desired. □

**Remark 21.9.** In this computation, we do not need to use the measures $\mu = |a|\delta_{-1} + e^x\delta_{-h}$ for all $h > 0$. It suffices to fix a single $u_0 \in (0, 1)$ and consider the sequence $h_n := -\log(u_0)/n$, whence we work with $\mu = |a|\delta_{-1} + e^x\delta_{1/n}$ (supported at 1, $u_0^{1/n}$) for $a > 0, x \in \mathbb{R}, n \geq 1$.

**Lemma 21.10.** For all integers $n \geq 0$, the assignment $(a, x) \mapsto H_a^{(n)}(x)$ is non-decreasing in both $a \geq 0$ and $x \in \mathbb{R}$. In particular if $a \geq 0$, then $H_a$ is absolutely monotonic on $\mathbb{R}$, and its Taylor series at $b \in \mathbb{R}$ converges absolutely at all $x \in \mathbb{R}$.

**Proof.** The monotonicity in $a \geq 0$ follows from the absolute monotonicity of $f|_{[0, \infty)}$ mentioned above. The monotonicity in $x$ for a fixed $a \geq 0$ follows because $H_a^{(n+1)}(x) \geq 0$ by Lemma 21.6.

To prove the (absolute) convergence of $(TH_a)_b$ at $x \in \mathbb{R}$, notice that

$$|(TH_a)_b (x)| = \left| \sum_{n=0}^{\infty} H_a^{(n)}(b) \frac{(x-b)^n}{n!} \right| \leq \sum_{n=0}^{\infty} H_a^{(n)}(b) \frac{|x-b|^n}{n!} = (TH_a)_b (b + |x-b|).$$ 

We claim that this final sum is bounded above by $H_a(b + |x-b|)$, which would complete the proof. Indeed, by Taylor’s theorem, the nth Taylor remainder term for $H_a(b + |x-b|)$ can be written as (see e.g. (19.7))

$$\int_{b}^{b+|x-b|} \frac{(b + |x-b| - t)^n}{n!} H_a^{(n+1)}(t) \ dt,$$

which is non-negative from above. This shows the claim and completes the proof. □
Now we can prove the real-analyticity of $H_a$:

**Proof of Theorem 21.5.** Fix scalars $a, \delta > 0$. We show that for all $b \in [-a, a]$ and $x \in \mathbb{R}$, the $n$th remainder term for the Taylor series $TH_b$ around the point $x$ converges to zero as $n \to \infty$, uniformly near $x$. More precisely, define

$$
\Psi_n(x) := \sup_{y \in [x-\delta, x+\delta]} |R_n((TH_b)_x)(y)|.
$$

We then claim $\Psi_n(x) \to 0$ as $n \to \infty$, for all $x$. This will imply that at all $x \in \mathbb{R}$, $(TH_b)_x$ converges to $H_b$ on a neighborhood of radius $\delta$. Moreover, this holds for all $\delta > 0$ and at all $b \in [-a, a]$ for all $a > 0$.

Thus, it remains to prove for each $x \in \mathbb{R}$ that $\Psi_n(x) \to 0$ as $n \to \infty$. By the above results, we have:

$$
|H_a^{(n)}(y)| \leq H_b^{(n)}(y) \leq H_a^{(n)}(x+\delta), \quad \forall b \in [-a, a], \ y \in [x-\delta, x+\delta], \ n \in \mathbb{Z}^+.
$$

Using a standard estimate for the Taylor remainder, for all $b, y, n$ as above, it follows that

$$
|R_n((TH_b)_x)(y)| \leq H_a^{(n+1)}(x+\delta) \frac{|y-x|^{n+1}}{(n+1)!} \leq H_a^{(n+1)}(x+\delta) \frac{\delta^{n+1}}{(n+1)!}.
$$

But the right-hand term goes to zero by the calculation in Lemma 21.10 since

$$
0 \leq \sum_{n=-1}^{\infty} H_a^{(n+1)}(x+\delta) \frac{\delta^{n+1}}{(n+1)!} \leq H_a(x+\delta+\delta) = f(a + e^{x+2\delta}) < \infty.
$$

Hence we obtain:

$$
\lim_{n \to \infty} \sup_{y \in [x-\delta, x+\delta]} |R_n((TH_b)_x)(y)| \to 0, \quad \forall x \in \mathbb{R}, \ \delta > 0, \ b \in [-a, a], \ a > 0.
$$

From above, this shows that the Taylor series of $H_b$ converges locally to $H_b$ at all $x \in \mathbb{R}$, for all $b$ as desired. (In fact, the ‘local’ neighborhood of convergence around $x$ is all of $\mathbb{R}$.)

With the above analysis in hand, we can prove Steps 4, 5 of the proof of the stronger Schoenberg theorem (see the list after Remark 19.17):

**Suppose $f: \mathbb{R} \to \mathbb{R}$ satisfies assertion (4) of Theorem 19.15.**

(4) If $f$ is smooth on $\mathbb{R}$, then $f$ is real analytic on $\mathbb{R}$.

(5) If $f$ is real analytic on $\mathbb{R}$, then $f(x) = \sum_{k=0}^{\infty} c_k x^k$ on $\mathbb{R}$, with $c_k \geq 0$ $\forall k$.

**Proof of Step 4 for the stronger Schoenberg theorem.** Given $x \in \mathbb{R}$, we want to show that the Taylor series $(Tf)_x$ converges to $f$ locally around $x$. Choose $a > |x|$ and define

$$
L_a(y) := \log(a + y) = \log(a) + \log(1 + y/a), \quad y \in (-a, a).
$$

This is real analytic on $(-a, a)$ (e.g. akin to Lemma 21.3). Hence by Proposition 21.2 and Theorem 21.5, the composite

$$
y \xrightarrow{L_a} \log(a + y) \xrightarrow{H_a} H_a(L_a(y)) = f(-a + \exp(\log(a + y))) = f(y)
$$

is also real analytic on $(-a, a)$, whence around $x \in \mathbb{R}$. 

**Proof of Step 5 for the stronger Schoenberg theorem.** By Step 4, $f$ is real analytic on $\mathbb{R}$; and as observed above, by Steps 1 and 2 $f(x) = \sum_{k=0}^{\infty} c_k x^k$ on $(0, \infty)$, with $c_k \geq 0$ $\forall k$. Let $g(x) := \sum_{k=0}^{\infty} c_k x^k \in C^\omega(\mathbb{R})$. Since $f \equiv g$ on $(0, \infty)$, it follows by the Identity Theorem 21.4 that $f \equiv g$ on $\mathbb{R}$. 

\[\square\]

We can now complete the proof of the final Step 6 (listed after Remark 19.17) of the stronger Schoenberg theorem. Namely, suppose \( f : \mathbb{R} \to \mathbb{R} \) is such that for each measure

\[
\mu = a\delta_1 + b\delta_{u_0} + c\delta_{-1}, \quad \text{with } u_0 \in (0, 1), \, a, b, c \geq 0,
\]

with semi-infinite Hankel moment matrix \( H_\mu \), the matrix \( f[H_\mu] \) is positive semidefinite.

Under these assumptions, we have previously shown (in Steps 1, 2; 3, 4, 5, etc.,

- There exist real scalars \( c_0, c_1, \cdots \geq 0 \) such that \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) for all \( x \in (0, \infty) \).
- \( f \) is continuous on \( \mathbb{R} \).
- If \( f \) is smooth, then \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) on \( \mathbb{R} \).

We now complete the proof by showing that one can pass from smooth functions to continuous functions. The tools we will use are the “three M’s”: Montel, Morera, and Mollifiers. We first discuss some basic results in complex analysis that are required below.

### Definition 22.1

Suppose \( D \subset \mathbb{C} \) is open and \( f : D \to \mathbb{C} \) is a continuous function.

1. (Holomorphic.) A function \( f \) is holomorphic at a point \( x \in D \) if the limit \( \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \) exists. A function \( f \) is holomorphic on \( D \) if it is holomorphic at every point of \( D \).
2. (Complex analytic.) \( f \) is said to be complex analytic around \( c \in D \) if \( f \) can be expressed as a power series locally around \( c \), which converges to \( f(x) \) for every \( x \) sufficiently close to \( c \). Similarly, \( f \) is analytic on \( D \) if it is so at every point of \( D \).
3. Let \( \mathcal{F} \) be a family of holomorphic functions \( : D \to \mathbb{C} \). Then \( \mathcal{F} \) is normal if given any compact \( K \subset D \) and a sequence \( \{f_n : n \geq 1\} \subset \mathcal{F} \), there exists a subsequence \( f_{n_k} \) and a function \( f : K \to \mathbb{C} \) such that \( f_{n_k} \to f \) uniformly on \( K \).

### Remark 22.2

Note that it is not specified that the limit function \( f \) be holomorphic. However, this will turn out to be the case, as we shall see later.

We use without proof the following results (and Cauchy’s theorem, which we do not state).

#### Theorem 22.3

Let \( D \subset \mathbb{C} \) be an open subset.

1. A function \( f : D \to \mathbb{C} \) is holomorphic if and only if \( f \) is complex analytic.
2. (Montel.) Let \( \mathcal{F} \) be a family of holomorphic functions on \( D \). If \( \mathcal{F} \) is uniformly bounded on \( D \), then \( \mathcal{F} \) is normal on \( D \).
3. (Morera.) Suppose that for every closed oriented piecewise \( C^1 \) curve \( \gamma \) in \( D \), we have \( \oint_\gamma f \, dz = 0 \). Then \( f \) is holomorphic on \( D \).

22.2. Proof of the stronger Schoenberg theorem: conclusion. Let \( f : \mathbb{R} \to \mathbb{R} \) be as described above; in particular, \( f \) is continuous on \( \mathbb{R} \) and absolutely monotonic on \( (0, \infty) \). As discussed in the proof of the stronger Horn–Loewner theorem 17.1, we mollify \( f \) with the family \( \phi_{\delta}(u) = \phi(u/\delta) \) for \( \delta > 0 \) as in Proposition 18.5. As shown in 18.8, \( f_{\delta} \) satisfies assertion (5) in Theorem 19.13 so (e.g. by the last bulleted point above, and Steps 4, 5,)

\[
f_{\delta}(x) = \sum_{k=0}^{\infty} c_{k, \delta} x^k \quad \forall x \in \mathbb{R}, \quad \text{with } c_{k, \delta} \geq 0 \quad \forall k \geq 0, \quad \delta > 0.
\]

Since \( f_{\delta} \) is a power series with infinite radius of convergence, it extends analytically to an entire function on \( \mathbb{C} \) (see e.g. Lemma 21.3). Let us call this \( f_{\delta} \) as well; now define

\[
\mathcal{F} := \{f_{1/n} : n \geq 1\}.
\]
We claim that for any $0 < r < \infty$, the family $\mathcal{F}$ is uniformly bounded on the complex disc $D(0, r)$. Indeed, since $f_\delta \to f$ uniformly on $[0, r]$ by Proposition 18.5, we have that $|f_{1/n} - f|$ is uniformly bounded over all $n$ and on $[0, r]$, say by $M_r > 0$. Now if $z \in D(0, r)$, then
\[
|f_{1/n}(z)| \leq \sum_{k=0}^{\infty} c_k 1/n |z|^k = f_{1/n}(|z|) \leq M_r + f(|z|) \leq M_r + f(r) < \infty,
\]
and this bound (uniform over $z \in D(0, r)$) does not depend on $n$.

By Montel’s theorem, the previous claim implies that $\mathcal{F}$ is a normal family on $D(0, r)$ for each $r > 0$. Hence on the closed disc $\overline{D}(0, r)$, there is a subsequence $f_{1/n_l}$ with $n_l$ increasing, which converges uniformly to some (continuous) $g = g_r$. Since $f_{1/n_l}$ is holomorphic for all $l \geq 1$, by Cauchy’s theorem we obtain for every closed oriented piecewise $C^1$ curve $\gamma \subset D(0, r)$:
\[
\oint_\gamma g_r \, dz = \oint_\gamma \lim_{l \to \infty} f_{1/n_l} \, dz = \lim_{l \to \infty} \oint_\gamma f_{1/n_l} \, dz = 0.
\]
It follows by Morera’s theorem that $g_r$ is holomorphic, whence analytic, on $D(0, r)$. Moreover, $g_r \equiv f$ on $(-r, r)$ by the properties of mollifiers; thus, $f$ is real analytic on $(-r, r)$ for every $r > 0$. Now apply the previous step of the proof – i.e., the Identity Theorem 21.4. □

22.3. Concluding remarks and variations. We conclude with several generalizations of the above results. First, the results by Horn–Loewner, Vasudeva, and Schoenberg (more precisely, their stronger versions) that were shown in this part, together with the proofs given above, can be adapted to versions with bounded domains $(0, \rho)$ or $(-\rho, \rho)$ for $0 < \rho < \infty$. The small change is to use admissible measures with bounded mass:
\[
\mu = a \delta_1 + b \delta_{u_0} + c \delta_{-1},
\]
where $u_0 \in (0, 1)$, $a, b, c \geq 0$;
but moreover, one now imposes the condition that $s_0(\mu) = a + b + c < \rho$.

Second, all of these results, including for bounded domains (i.e., masses of the underlying measures), can be extended to studying functions of several variables. In this case, given a domain $I \subset \mathbb{R}$ and integers $m, n \geq 1$, a function $f : I^m \to \mathbb{R}$ acts entrywise on an $m$-tuple of $n \times n$ matrices $A_1 = (a_j^{(1)}), \ldots, A_m = (a_j^{(m)})$ in $I^{m \times m}$, via:
\[
f[A_1, \ldots, A_m] := (f(a_j^{(1)}), \ldots, a_j^{(m)})_{j=1}^n.
\]
One can now ask the multivariable version of the same question as above:
“Which functions, when applied entrywise to $m$-tuples of matrices, preserve positivity?”

Observe that the coordinate functions $f(x_1, \ldots, x_m) := x_l$ work for all $1 \leq l \leq m$. Hence by the Schur product theorem and the Pólya–Szegő observation (Lemma 16.1, since $\mathbb{P}_n$ is a closed convex cone for all $n \geq 1$), every convergent multi-power series of the form
\[
f(x) := \sum_{n \geq 0} c_n x^n,
\]
preserves positivity in all dimensions (where $x^n := x_1^n \cdots x_m^n$, etc.). Akin to the Schoenberg–Rudin theorem in the one-variable case, it was shown by FitzGerald, Micchelli, and Pinkus in *Linear Algebra Appl.* (1995) that the functions (22.5) are the only such preservers.

One can ask if the same result holds when one restricts the test set to $m$-tuples of Hankel matrices of rank at most 3, as in the treatment above. While this does turn out to yield the same classification, the proofs get more involved and now require multivariable machinery. For these stronger multivariate results, we refer the reader to the 2016 preprint “Moment-sequence transforms” by Belton, Guillot, Khare, and Putinar.
In this section we reproduce the complete proof of the theorem by Boas and Widder on functions with non-negative forward differences (Duke Math. J., 1940). This result was stated as Theorem 18.10(2) above, and we again write down its statement here for convenience. In it (and below), recall from just before Theorem 18.10 that given an interval \( I \subset \mathbb{R} \) and a function \( f : I \to \mathbb{R} \), the \( k \)th order forward differences of \( f \) with step size \( h > 0 \) are defined as follows:

\[
(\Delta^0_h f)(x) := f(x), \quad (\Delta^k_h f)(x) := (\Delta^k_h f)(x+h) - (\Delta^{k-1}_h f)(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x+jh),
\]

whenever \( k > 0 \) and \( x, x + kh \in I \). It is easily seen that these difference operators commute:

\[
\Delta^m \delta (\Delta^n \epsilon f(x)) = \Delta^n \epsilon (\Delta^m \delta f(x)), \quad \text{whenever } x, x + m\delta + n\epsilon \in I,
\]

and so we will omit parentheses and possibly permute these operators below, without further reference. Now we (re)state the theorem of interest:

**Theorem 23.1** (Boas–Widder). Suppose \( k \geq 2 \) is an integer, \( I \subset \mathbb{R} \) is an open interval, bounded or not, and \( f : I \to \mathbb{R} \) is a function that satisfies the following condition:

\[
(\Delta^k_h f)(x) \geq 0 \quad \text{whenever } h > 0 \text{ and } x, x + kh \in I, \quad \text{and } f \text{ is continuous on } I. \quad (H_k)
\]

(In other words, \( f \) is continuous and has all forward differences of order \( k \) non-negative on \( I \).) Then on all of \( I \), the function \( f^{(k-2)} \) exists, is continuous and convex, and has non-decreasing left and right hand derivatives.

This is a ‘finite-order’ result; for completeness, an order-free result can be found in Bernstein’s theorem 37.10 below.

**23.1. Further remarks and results.** Before writing down Boas and Widder’s proof of Theorem 23.1, we make several additional observations beyond the result and its proof. The first observation (which was previously mentioned following Theorem 18.10(2)) is that while \( f^{(k-1)}_\pm \) is non-decreasing by the above theorem, it is not always true that any other lower-order derivatives \( f, \ldots, f^{(k-2)} \) are non-decreasing on \( I \). For example, let \( 0 \leq l \leq k-2 \) and consider \( f(x) := -x^{l+1} \) on \( I \subset \mathbb{R} \); then \( f^{(l)} \) is strictly decreasing on \( I \).

Second, it is natural to seek examples of non-smooth functions satisfying the differentiability conditions of Theorem 23.1, but no more – in other words, to explore if Theorem 23.1 is indeed “sharp”. This is now verified to be true:

**Example 23.2.** Let \( I = (a, b) \subset \mathbb{R} \) be an open interval, where \(-\infty \leq a < b \leq \infty\). Consider any function \( g : I \to \mathbb{R} \) that is non-decreasing, whence Lebesgue integrable. For any interior point \( c \in I \), the function \( f_2(x) := \int_c^x g(t) \, dt \) satisfies \((H_2)\), but clearly not every monotone \( g \) gives rise to an anti-derivative that is differentiable on all of \( I \). Also, for completeness we
write down the proof of \((H_2)\):

\[
\Delta_h^2 f_2(x) = \int_c^x g(t) \, dt - 2 \int_c^{x+h} g(t) \, dt + \int_c^{x+2h} g(t) \, dt
\]

\[
= \int_{x+h}^{x+2h} g(t) \, dt - \int_x^{x+h} g(t) \, dt
\]

\[
= \int_x^{x+h} (\Delta_h g)(t) \, dt \geq 0.
\]

Finally, to see that the condition \([H_k]\) is sharp for all \(k > 2\) as well, define \(f\) to be the \((k - 1)\)-fold indefinite integral of \(g\). We claim that \(f\) satisfies \([H_k]\). Continuity is obvious; and to study the \(k\)th order divided differences of \(f\), first note by the fundamental theorem of calculus that \(f\) is \((k - 2)\)-times differentiable, with \(f^{(k-2)}(x) = f_2(x) = f_x^x g(t) \, dt\). In particular, \(\Delta_h^k f \in C^{k-2}(a, b - kh)\) whenever \(a < x < x + kh < b\) as in \([H_k]\).

Now given such \(x, h\), we compute using the Cauchy mean-value theorem \ref{18.10}(1) for divided differences (and its notation):

\[
\Delta_h^k f(x) = \Delta_h^{k-2}(\Delta_h^2 f)(x) = h^{k-2} D_h^{k-2}(\Delta_h^2 f)(x) = \frac{h^{k-2}}{(k-2)!}(\Delta_h^2 f)^{(k-2)}(y),
\]

for some \(y \in (a, b - 2h)\). But this is easily seen to equal

\[
\frac{h^{k-2}}{(k-2)!}(\Delta_h^2 f)^{(k-2)}(y) = \frac{h^{k-2}}{(k-2)!}\Delta_h^2 f_2(y),
\]

and we showed above that this is non-negative.

\[\square\]

The final observation in this subsection is that there are natural analogues for \(k = 0, 1\) of the Boas–Widder theorem (which is stated for \(k \geq 2\)). For this, we make the natural definition: for \(k < 0\), \(f^{(k)}\) will denote the \(|k|\)-fold anti-derivative of \(f\). Since \(f\) is assumed to be continuous, this is just the iterated indefinite Riemann integral starting at an \((y)\) interior point of \(I\). With this notation at hand:

**Proposition 23.3.** The Boas–Widder theorem \ref{23.1} also holds for \(k = 0, 1\).

**Proof.** In both cases, the continuity of \(f^{(k-2)}\) is immediate by the fundamental theorem of calculus. Next, suppose \(k = 1\) and choose \(c \in I\). Now claim that if \(f\) is continuous and non-decreasing (i.e., \((H_1)\)), then \(f^{(-1)}(x) := \int_c^x f(t) \, dt\) is convex on \(I\). Indeed, given \(x_0 < x_1 \in I\), define \(x_\lambda := (1 - \lambda)x_0 + \lambda x_1\) for \(\lambda \in [0, 1]\), and compute:

\[
(1 - \lambda)f^{(-1)}(x_0) + \lambda f^{(-1)}(x_1) - f^{(-1)}(x_\lambda)
\]

\[
= (1 - \lambda) \int_{x_0}^{x_\lambda} \mathbf{1}(t \leq x_0) f(t) \, dt + \lambda \int_{c}^{x_1} \mathbf{1}(t \leq x_1) f(t) \, dt - \int_{c}^{x_1} \mathbf{1}(t \leq x_\lambda) f(t) \, dt
\]

\[
= - (1 - \lambda) \int_{x_0}^{x_\lambda} f(t) \, dt + \lambda \int_{x_0}^{x_\lambda} f(t) \, dt.
\]

But since \(f\) is non-decreasing, each integral – together with the accompanying sign – is bounded below by the corresponding expression where \(f(t)\) is replaced by \(f(x_\lambda)\). An easy computation now yields:

\[
(1 - \lambda)f^{(-1)}(x_0) + \lambda f^{(-1)}(x_1) - f^{(-1)}(x_\lambda) \geq f(x_\lambda)(\lambda(x_1 - x_\lambda) - (1 - \lambda)(x_\lambda - x_0)) = 0;
\]

therefore \(f^{(-1)}\) is convex, as desired.
23. Appendix A: The Boas–Widder theorem on functions with positive differences. 135

This shows the result for \( k = 1 \). Next, if \( k = 0 \) then \( f \) is continuous and non-negative on \( I \), whence \( f^{(-1)} \) is non-decreasing on \( I \). Now the above computation shows that \( f^{(-2)} \) is convex; the remaining assertions are obvious. □

23.2. Proof of the main result. In this subsection, we reproduce Boas and Widder’s proof of Theorem 23.1. We first make a few clarifying remarks about this proof.

1. As Boas and Widder mention, Theorem 23.1 was shown earlier by T. Popoviciu (Mathematica, 1934) via an alternate argument using divided differences involving unequally spaced points. Here we will only explain Boas and Widder’s proof.

2. There is a minor error in the arguments of Boas and Widder, which is resolved by adding one word. See Remark 23.11 and the proof of Lemma 23.13 for more details. (There are other minor typos in the writing of Lemmas 23.6 and 23.10 and in some of the proofs; these are corrected in the exposition below.)

3. Boas and Widder do not explicitly write out a proof of the convexity of \( f \) (in the case \( k = 2 \)). This is addressed below as well.

Notice that Theorem 23.1 follows for the case of unbounded domain \( I \) from that for bounded domains, so we assume henceforth that

\[ I = (a, b), \quad \text{with } -\infty < a < b < \infty. \]

We now reproduce a sequence of fourteen lemmas shown by Boas and Widder, which culminate in the above theorem. These lemmas are numbered “Lemma 23.1”, . . . , “Lemma 23.14” and will be referred to only in this Appendix. The rest of the results, equations, and remarks – starting from Theorem 23.1 and ending with Proposition 23.12 – are numbered using a common (separate) counter. None of the results in this Appendix are cited elsewhere in the text.

The first of the fourteen lemmas by Boas and Widder says that if the \( k \)th order “equi-spaced” forward differences are non-negative, then so are the \( k \)th order “possibly non-equi-spaced” differences (the converse is immediate):

**Lemma 23.1.** If \( f(x) \) satisfies \((H_k)\) in \((a, b)\) for some \( k \geq 2 \), then for any \( k \) positive numbers \( \delta_1, \ldots, \delta_k > 0 \),

\[
\Delta_{\delta_1} \Delta_{\delta_2} \cdots \Delta_{\delta_k} f(x) \geq 0, \quad \text{whenever } a < x < x + \delta_1 + \delta_2 + \cdots + \delta_k < b.
\]

**Proof.** The key step is to prove using \((H_k)\) that

\[
\Delta_{\delta_1}^{k-1} f(x) \geq 0, \quad \text{whenever } a < x < x + (k - 1)h + \delta_1 < b. \tag{23.4}
\]

After this, the lemma is proved using induction on \( k \geq 2 \). Indeed, \((23.4)\) is precisely the assertion in the base case \( k = 2 \); and using \((23.4)\) we can show the induction step as follows: for a fixed \( \delta_1 \in (0, b - a) \), it follows that \( \Delta_{\delta_1} f \) satisfies \((H_{k-1})\) in the interval \((a, b - \delta_1)\). Therefore,

\[
\Delta_{\delta_2} \cdots \Delta_{\delta_k} (\Delta_{\delta_1} f(x)) \geq 0, \quad \text{whenever } a < x < x + \delta_1 + \cdots + \delta_k < b.
\]

Since the \( \Delta_{\delta} \) commute, and since \( \delta_1 \) was arbitrary, the induction step follows.

Thus, it remains to show \((23.4)\). Let \( h > 0 \) and \( n \in \mathbb{N} \) be such that \( a < x < x + h/n + (k - 1)h < b \). One checks using an easy telescoping computation that

\[
\Delta_{h} f(x) = \sum_{i=0}^{n-1} \Delta_{h/n} f(x + ih/n);\]
Lemma 23.3. Suppose required. Then by Lemma 23.1 – or simply (23.4) – it follows that

\[ \Delta_{h/n}^{k-1} f(x) = \sum_{i_1=0}^{n-1} \cdots \sum_{i_{k-1}=0}^{n-1} \Delta_{h/n}^{k-1} f(x + [i_1 + \cdots + i_{k-1}]h/n). \]  

(23.5)

(This works by induction on \( k \geq 2 \); the previous telescoping identity is the base case for \( k = 2 \), and for the induction step we evaluate the innermost sum using the base case.)

From the above computations, it further follows that

\[ \Delta_{h/n} \Delta_{h}^{k-1} f(x) = \sum_{i_1=0}^{n-1} \cdots \sum_{i_{k-1}=0}^{n-1} \Delta_{h/n}^{k} f(x + [i_1 + \cdots + i_{k-1}]h/n) \geq 0, \]

where the final inequality uses the assumption \((H_k)\). From this it follows that \(\Delta_{h}^{k-1} f(x) \leq \Delta_{h}^{k-1} f(x + h/n)\).

Now suppose \( x \) is such that \( a < x < x + mh/n + (k-1)h < b \). Applying the preceding inequality to \( x, x + h/n, \ldots, x + (m-1)h/n, \) we obtain:

\[ \Delta_{h}^{k-1} f(x) \leq \Delta_{h}^{k-1} f(x + h/n) \leq \cdots \leq \Delta_{h}^{k-1} f(x + mh/n). \]  

(23.6)

We can now prove (23.4). As in it, choose \( \delta_1 > 0 \) such that \( a < x < x + \delta_1 + (k-1)h < b \); and choose sequences \( m_j, n_j \) of positive integers such that \( m_j/n_j \to \delta_1/h \) and \( x + m_jh/n_j + (k-1)h < b \) for all \( j \geq 1 \).

Since \( f(x) \) is continuous, \( f(x + m_jh/n_j) \) converges to \( f(x + \delta_1) \), and \( \Delta_{h}^{k-1} f(x + m_jh/n_j) \) to \( \Delta_{h}^{k-1} f(x + \delta_1) \), as \( j \to \infty \). Hence using (23.6) with \( m_j, n_j \) in place of \( m, n \) respectively, we obtain by taking limits:

\[ \Delta_{h}^{k-1} f(x) \leq \Delta_{h}^{k-1} f(x + \delta_1). \]

But this is equivalent to (23.4), as desired. \( \square \)

Lemma 23.2. If \( f(x) \) satisfies \((H_k)\) in \( (a, b) \) for some \( k \geq 2 \), then \( \Delta_{\epsilon}^{k-1} f(x) \) and \( \Delta_{\epsilon}^{k-1} f(x-\epsilon) \) are non-decreasing functions of \( x \) in \( (a, b - (k-1)\epsilon) \) and \( (a + \epsilon, b - (k-2)\epsilon) \) respectively.

Proof. For the first part, suppose \( y < z \) are points in \( (a, b - (k-1)\epsilon) \), and set

\[ \delta_1 := z - y, \quad \delta_2 = \cdots = \delta_k := \epsilon. \]

Then by Lemma 23.1 – or simply (23.4) – it follows that

\[ \Delta_{\epsilon}^{k-1} f(z) - \Delta_{\epsilon}^{k-1} f(y) = \Delta_{\delta_1, \delta_2, \ldots, \delta_k}^{k-1} f(y) \geq 0, \]

which is what was asserted.

Similarly, for the second part we suppose \( y < z \) are points in \( (a + \epsilon, b - (k-2)\epsilon) \). Then \( y - \epsilon < z - \epsilon \) are points in \( (a, b - (k-1)\epsilon) \), so we are done by the first part. (Remark: Boas and Widder repeat the computations of the first part in this second part; but this is not required.) \( \square \)

We assume for the next four lemmas that \( f \) satisfies \((H_2)\) in the interval \( x \in (a, b) \).

Lemma 23.3. Suppose \( f \) satisfies \((H_2)\) in \( (a, b) \), and \( x \in (a, b) \). Then \( h^{-1} \Delta_{h} f(x) \) is a non-decreasing function of \( h \) in \( (a - x, b - x) \).

Remark 23.7. Notice that \( h = 0 \) lies in \( (a - x, b - x) \), and at this point the expression \( h^{-1} \Delta_{h} f(x) \) is not defined. Hence the statement of Lemma 23.3 actually says that \( h \mapsto h^{-1} \Delta_{h} f(x) \) is non-decreasing for \( h \) in \((0, x - a) \) and separately for \( h \) in \((a, x, 0) \). The latter can be reformulated as follows: since \( \Delta_{-\epsilon} f(x) = -\Delta_{\epsilon} f(x - \epsilon) \), Lemma 23.3 asserts that the map \( h \mapsto h^{-1} \Delta_{h} f(x - h) \) is a non-increasing function of \( h \) in \((0, x - a) \).
Proof of Lemma 23.3. We first prove the result for \( h \in (0, b - x) \). Thus, suppose \( 0 < \epsilon < \delta < b - x \). By condition \((H_2)\), for all integers \( n \geq 2 \) we have:

\[ \Delta_{\delta/n}^2 f(x) \geq 0, \quad \Delta_{\delta/2}^2 f(x + \delta/n) \geq 0, \quad \cdots, \quad \Delta_{\delta/n}^2 f(x + (n - 2)\delta/n) \geq 0 \]

implies that \( a < z < x < y < b \)

\[ \Rightarrow \Delta_{\delta/n} f(x) \leq \Delta_{\delta/n} f(x + \delta/n) \leq \cdots \leq \Delta_{\delta/n} f(x + (n - 1)\delta/n). \]

If \( 0 < m < n \), then the average of the first \( m \) terms here cannot exceed the average of all \( n \) terms. Therefore,

\[ \frac{f(x + m\delta/n) - f(x)}{m\delta/n} \leq \frac{f(x + \delta/n) - f(x)}{\delta}. \]

Now since \( \epsilon \in (0, \delta) \), choose integer sequences \( 0 < m_j < n_j \) such that \( m_j/n_j \to \epsilon/\delta \) as \( j \to \infty \). Applying the preceding inequality (with \( m, n \) replaced respectively by \( m_j, n_j \)) and taking limits, it follows that \( \epsilon^{-1}\Delta_{\epsilon} f(x) \leq \delta^{-1}\Delta_{\delta} f(x) \), since \( f \) is continuous. This proves the first part of the lemma, for positive \( h \).

The proof for negative \( h \in (a - x, 0) \) is similar, and is shown using the reformulation of the assertion in Remark 23.7. Given \( 0 < \epsilon < \delta < x - a \), by condition \((H_2)\) it follows for all integers \( 0 < m < n \) that

\[ \Delta_{\delta/n} f(x - \delta) \leq \Delta_{\delta/n} f(x - (n - 1)\delta/n) \leq \cdots \leq \Delta_{\delta/n} f(x - \delta/n) \]

implies

\[ \frac{f(x) - f(x - \delta/n)}{\delta} \leq \frac{f(x) - f(x - \delta)}{m\delta/n}. \]

this time using the last \( m \) terms instead of the first. Now work as above: using integer sequences \( 0 < m_j < n_j \) such that \( m_j/n_j \to \epsilon/\delta \), it follows from the continuity of \( f \) that \( \delta^{-1}\Delta_{\delta} f(x - \delta) \leq \epsilon^{-1}\Delta_{\epsilon} f(x - \epsilon) \), as desired.

We next define the one-sided derivatives of functions.

Definition 23.8. Let \( f \) be a real-valued function on \((a, b)\). Define:

\[ f'_+(x) := \lim_{\delta \to 0^+} \frac{\Delta_{\delta} f(x)}{\delta}, \quad f'_-(x) := \lim_{\delta \to 0^-} \frac{\Delta_{\delta} f(x)}{\delta} = \lim_{\delta \to 0^+} \frac{\Delta_{\delta} f(x - \delta)}{\delta}. \]

Lemma 23.4. Suppose \( f \) satisfies \((H_2)\) in \((a, b)\). Then \( f'_+, f'_- \) exist and are finite and non-decreasing on all of \((a, b)\).

Proof. That \( f'_+ \) exist on \((a, b)\) follows from Lemma 23.3 though the limits may possibly be infinite. Now fix scalars \( \delta, \epsilon, x, y, z \) satisfying:

\[ 0 < \delta < \epsilon \quad \text{and} \quad a < z - \epsilon < x - \epsilon < x < x + \epsilon < y + \epsilon < b, \]

which implies that \( a < z < x < y < b \). Then we have:

\[ \frac{\Delta_{\epsilon} f(z - \epsilon)}{\epsilon} \leq \frac{\Delta_{\epsilon} f(x - \epsilon)}{\epsilon} \leq \frac{\Delta_{\delta} f(x - \delta)}{\delta} \leq \frac{\Delta_{\delta} f(x)}{\delta} \leq \frac{\Delta_{\epsilon} f(y)}{\epsilon}, \]

where the five inequalities follow respectively using Lemma 23.2, Remark 23.7, Lemma 23.3, and Lemma 23.2.

Now let \( \delta \to 0^+ \) keeping \( \epsilon, x, y, z \) fixed; this yields:

\[ \frac{\Delta_{\epsilon} f(z - \epsilon)}{\epsilon} \leq f'_+(x) \leq f'_+(x) \leq \frac{\Delta_{\epsilon} f(y)}{\epsilon}, \]

which implies that \( f'_+(x) \) are finite on \((a, b)\). In turn, letting \( \epsilon \to 0^+ \) yields:

\[ f'_-(z) \leq f'_-(x) \leq f'_+(x) \leq f'_+(y), \]

which shows that \( f'_\pm \) are non-decreasing on \((a, b)\).
Lemma 23.5. If \( f \) satisfies (H2) in \((a, b)\) then \( f \) approaches a limit in \((-\infty, +\infty)\) as \( x \) goes to \( a^+ \) and \( x \) goes to \( b^- \).

Proof. Note by Lemma 23.2 that \( \Delta_\delta f(x) \) is non-decreasing in \( x \in (a, b-\delta) \). Hence \( \lim_{x \to a^+} \Delta_\delta f(x) \) exists or equals \(-\infty\). (The key point is that it is not \(+\infty\).) Therefore, since \( f \) is continuous,

\[
+\infty > \lim_{x \to a^+} \Delta_\delta f(x) = \lim_{x \to a^+} (f(x+\delta) - f(x)) = f(a+\delta) - f(a^+).
\]

It follows that \( f(a^+) \) exists and cannot equal \(-\infty\).

By the same reasoning, the limit \( \lim_{x \to (b^-)-} \Delta_\delta f(x) \) exists or equals \(+\infty\), whence

\[
-\infty < \lim_{x \to (b^-)-} \Delta_\delta f(x) = f(b^-) - f(b - \delta).
\]

It follows that \( f(b^-) \) exists and cannot equal \(-\infty\). \( \square \)

Lemma 23.6. Suppose \( f(x) \) satisfies (H2) in \((a, b)\).

1. If \( f(a^+) < +\infty \), define \( f(a) := f(a^+) \). Then \( f_+'(a) \) exists and is finite or \(-\infty\).
2. If \( f(b^-) < +\infty \), define \( f(b) := f(b^-) \). Then \( f_-'(b) \) exists and is finite or \(+\infty\).

Proof. First, if \( f(a^+) \) or \( f(b^-) \) are not \(+\infty\) then they are finite by Lemma 23.5. To show (1), by Lemma 23.3 for \( h \in (0, b-a) \) the map \( h \mapsto h^{-1}\Delta_h f(x) \) is non-decreasing. Therefore \( h \mapsto h^{-1}\Delta_h f(a) \) is the limit of a set of non-decreasing functions in \( h \), whence it too is non-decreasing in \( h \). This proves (1).

The second part is proved similarly, using that \( h \mapsto h^{-1}\Delta_h f(b-h) \) is a non-increasing function in \( h \). \( \square \)

Common hypothesis for Lemmas 7–14: \( f \) satisfies (Hk) in \((a, b)\), for some \( k > 2 \).

(We use this hypothesis below without mention or any other reference.)

Lemma 23.7. For any \( a < x < b \), the map \( h \mapsto h^{-k+1}\Delta_h^{k-1} f(x) \) is a non-decreasing function of \( h \) in \((0, (b-x)/(k-1))\).

Proof. First note that the given map is indeed well-defined. Now we prove the result by induction on \( k \geq 2 \); the following argument is similar in spirit to (for instance) computing by induction the derivative of \( x^{k-1} \).

For \( k = 2 \) the result follows from Lemma 23.3. To show the induction step, given fixed \( 0 < h < (b-a)/(k-2) \) and \( \delta \in (0, b-a) \), it is clear by Lemma 23.1 that if \( f \) satisfies (Hk) in \((a, b)\), then we have, respectively:

\[
\Delta_h^{k-2} f \text{ satisfies (H2) in } (a, b-(k-2)h),
\]

\[
\Delta_\delta f \text{ satisfies (H}_{k-1}) \text{ in } (a, b-\delta).
\]

In particular, if \( 0 < \delta < \epsilon < (b-x)/(k-1) \), then we have:

\[
\frac{\Delta_\delta \Delta_h^{k-2} f(x)}{\epsilon^{k-2} \delta} \geq \frac{\Delta_\delta \Delta_h^{k-2} f(x)}{\epsilon^{k-2} \delta} = \frac{\Delta_h^{k-2} \Delta_\delta f(x)}{\epsilon^{k-2} \delta} \geq \frac{\Delta_h^{k-2} \Delta_\delta f(x)}{\delta^{k-2} \delta}.
\]

Indeed, the first inequality is by the assertion for \( k = 2 \), which follows via Lemma 23.3 from the first condition in (23.9); and the second inequality is by the induction hypothesis (i.e., the assertion for \( k-1 \)) applied using the second condition in (23.9).

We saw in the preceding calculation that \( \epsilon^{-k+1} \Delta_h^{k-1} f(x) \geq \delta^{-k+1} \Delta_\delta^{k-1} f(x) \). But this is precisely the induction step. \( \square \)

Lemma 23.8. There is a point \( c \in [a, b] \), such that \( f(x) \) satisfies (H_{k-1}) in \((c, b)\) and \(-f(x)\) satisfies (H_{k-1}) in \((a, c)\).
Proof. Define subsets $A, B \subset (a, b)$ via:

$$
A := \{ x \in (a, b) : \Delta^k_h f(x) \geq 0 \text{ for all } \delta \in (0, (b-x)/(k-1)) \},
$$

$$
B := (a, b) \setminus A.
$$

If both $A, B$ are non-empty, and $z \in A, y \in B$, then we claim that $y < z$. Indeed, since $y \notin A$, there exists $0 < \epsilon < (b-y)/(k-1)$ such that $\Delta^k_{\epsilon} f(y) < 0$. By Lemma 23.2 if $z' \in (a, y]$ then $\Delta^k_{\epsilon} f(z') < 0$, whence $z' \notin A$. Now conclude that $z > y$.

The above analysis implies the existence of $c \in [a, b]$ such that $(a, c) \subset B \subset (a, c]$ and $(c, b) \subset A \subset [c, b)$. It is also clear that $f$ satisfies $(H_{k-1})$ in $(c, b)$.

It remains to show that if $a < c$ then $-f$ satisfies $(H_{k-1})$ in $(a, c)$. Begin by defining a map $\varepsilon : (a, c) \to (0, \infty)$ as follows: for $x \in (a, c)$, there exists $\epsilon \in (0, (c-x)/(k-1))$ such that $\Delta^k_{\epsilon} f(x) < 0$. By Lemmas 23.2 and 23.7 this implies that

$$
\Delta^k_{\epsilon} f(y) < 0, \quad \forall a < y \leq x, \quad 0 < \delta \leq \epsilon.
$$

Now define $\varepsilon : (a, c) \to (0, \infty)$ by setting

$$
\varepsilon(x) := \sup\{ \epsilon \in (0, \frac{c-x}{k-1}) : \Delta^k_{\epsilon} f(x) < 0 \}.
$$

By the reasoning just described, $\varepsilon$ is a non-increasing function on $(a, c)$.

With the function $\varepsilon$ in hand, we now complete the proof by showing that $-f(x)$ satisfies $(H_{k-1})$ in $(a, c)$. Let $x \in (a, c)$ and let $h > 0$ be such that $x + (k-1)h < c$. Choose any $y \in (x + (k-1)h, c)$ as well as an integer $n > h/\varepsilon(y)$. It follows that $\Delta^k_{h/n} f(y) < 0$.

Now recall from Equation (23.5) that

$$
\Delta^k_{h} f(x) = \sum_{i_1=0}^{n-1} \cdots \sum_{i_{k-1}=0}^{n-1} \Delta^k_{h/n} f(x + [i_1 + \cdots + i_{k-1}]h/n).
$$

But in each summand, the argument $x + [i_1 + \cdots + i_{k-1}]h/n < y$, whence by Lemmas 23.2 and 23.7 the previous paragraph implies that each summand is negative. It follows that $\Delta^k_{h} f(x) < 0$. This shows that $-f(x)$ satisfies $(H_{k-1})$ in $(a, c)$, as desired, and concludes the proof.

**Lemma 23.9.** There are points

$$
a = x_0 < x_1 < \cdots < x_p = b, \quad \text{with } 1 \leq p \leq 2^{k-1},
$$

such that in each interval $x_j < x < x_{j+1}$, either $f(x)$ or $-f(x)$ satisfies $(H_2)$.

This follows immediately from Lemma 23.8 by induction on $k \geq 2$.

**Lemma 23.10.** The derivatives $f'_\pm$ both exist and are finite on all of $(a, b)$.

We remark here that $f'_\pm$ are both needed in what follows; yet Boas and Widder completely avoid discussing $f'$ in this lemma or its proof (or in the sequel). For completeness, the proof for $f'_-$ is also now described.

Proof. By Lemmas 23.9, 23.4, and 23.6, the functions $f'_{\pm}$ exist on all of $(a, b)$, and are finite, possibly except at the points $x_1, \ldots, x_p$ in Lemma 23.9. We now show that $f'_{\pm}$ are finite at each of these points $x_j$.

First suppose $f'_{+}(x_j)$ or $f'_{-}(x_j)$ equals $+\infty$. Choose $\delta > 0$ small enough such that

$$
x_{j-1} < x_j - (k-2)\delta < x_j < x_j + \delta < x_{j+1}.
$$
Now if \( f'_+(x_j) = +\infty \), then

\[
\Delta_{k-1}^\delta f'_+(x_j - (k - 2)\delta) = -\infty
\]

\[
\implies \lim_{h \to 0^+} \frac{1}{h} \Delta_{k-1}^\delta f(x_j - (k - 2)\delta - h) = -\infty
\]

\[
\implies \Delta_{k-1}^\delta f(x_j - (k - 2)\delta) < 0 \quad \text{for all small positive } h.
\]

But this contradicts Lemma 23.1. Similarly, if \( f'_-(x_j) = +\infty \), then

\[
\Delta_{k-1}^\delta f'_-(x_j - (k - 2)\delta) = -\infty
\]

\[
\implies \lim_{h \to 0^+} \frac{1}{h} \Delta_{k-1}^\delta f(x_j - (k - 2)\delta - h) = -\infty
\]

\[
\implies \Delta_{k-1}^\delta f(x_j - (k - 2)\delta) < 0 \quad \text{for all small positive } h,
\]

which again contradicts Lemma 23.1.

The other case is if \( f'_+(x_j) \) or \( f'_-(x_j) \) equals \(-\infty\). The first of these sub-cases is now treated; the sub-case \( f'_-(x_j) = -\infty \) is similar. Begin as above by choosing \( \delta > 0 \) such that

\[
x_{j-1} < x_j - (k - 1)\delta < x_j < x_{j+1}.
\]

Now if \( f'_+(x_j) = +\infty \), then a similar computation to above yields:

\[
\Delta_{k-1}^\delta f'_+(x_j - (k - 1)\delta) = -\infty
\]

\[
\implies \lim_{h \to 0^+} \frac{1}{h} \Delta_{k-1}^\delta f(x_j - (k - 1)\delta - h) = -\infty
\]

\[
\implies \Delta_{k-1}^\delta f(x_j - (k - 1)\delta) < 0 \quad \text{for all small positive } h,
\]

which contradicts Lemma 23.1. \( \Box \)

The above trick of studying \( \Delta_n^\delta g(y - p\delta) \) where \( p = k - 1 \) or \( k - 2 \) (and \( n = k - 1 \), \( g = f'_\pm \) so that we deal with the \( k \)th order divided differences / derivatives of \( f \)) is a powerful one. Boas and Widder now use the same trick to further study the derivative of \( f \), and show its existence, finiteness, and continuity in Lemmas 23.11 and 23.13.

Lemma 23.11. \( f' \) exists and is finite on \((a,b)\).

Proof. We fix \( x \in (a,b) \), and work with \( \delta > 0 \) small such that \( a < a + k\delta < x < b - 2\delta < b \). Let \( p \in \{0,1,\ldots,k\} \); then

\[
0 \leq \frac{1}{\delta} \Delta_k^\delta f(x - p\delta) = \frac{1}{\delta} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + (i-p)\delta).
\]

Subtract from this the identity \( 0 = \delta^{-1} f(x) (1 - 1)^k = \delta^{-1} f(x) \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \), so that the \( i = p \) term cancels; and multiply and divide the remaining terms by \((i-p)\) to obtain:

\[
0 \leq \frac{1}{\delta} \Delta_k^\delta f(x - p\delta) = \sum_{i=0, i \neq p}^{i=k} (-1)^{k-i} \binom{k}{i} \frac{f(x + (i-p)\delta) - f(x)}{(i-p)\delta} (i-p).
\]
Letting $\delta \to 0^+$, it follows that
\[ A_p f_-(x) + B_p f_+(x) \geq 0, \quad \text{where } A_p := \sum_{i=0}^{p-1} (-1)^{k-i} \binom{k}{i} (i-p), \]
\[ B_p := \sum_{i=p+1}^{k} (-1)^{k-i} \binom{k}{i} (i-p); \] (23.10)

note here that

\[ A_p + B_p = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (i-p) = k \sum_{i=1}^{k-1} (-1)^{k-i} \binom{k-1}{i-1} - p \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} = 0. \]

Now specialize $p$ to be $k-1$ and $k-2$. In the former case $B_p = 1$, whence $A_p = -1$, and by (23.10) we obtain: $f_+(x) \geq f_-(x)$. In the latter case $p = k-2$ (with $k > 2$), we have $B_p = 2 - k < 0$. Thus $A_p = k - 2 > 0$, and by (23.10) we obtain: $f'_-(x) \geq f'_+(x)$. Therefore $f'(x)$ exists, and by Lemma 23.10 it is finite.

**Lemma 23.12.** If $a < x < x + (k-1)h < b$, then $\Delta_h^{k-1} f'(x) \geq 0$.

**Proof.** $\Delta_h^{k-1} f'(x) = \lim_{\delta \to 0^+} \frac{\Delta_{\delta} \Delta_h^{k-1} f(x)}{\delta}$, and this is non-negative by Lemma 23.1.

**Lemma 23.13.** $f'$ is continuous on $(a,b)$.

**Remark 23.11.** We record here a minor typo in the Boas–Widder paper. Namely, the authors begin the proof of Lemma 23.13 by claiming that $f$ is continuous on $I$. However, this is not true as stated: for any $k > 2$, the function $f(x) = x^3$ satisfies (H) on $I = (-1,1)$ but $f'$ is not monotone on $I$. The first paragraph of the following proof addresses this issue, using that $f'$ is piecewise monotone on $(a,b)$.

**Proof of Lemma 23.13.** By Lemmas 23.9 and 23.4 there are finitely many points $x_j$, $0 \leq j \leq p \leq 2^{k-1}$, such that on each $(x_j, x_{j+1})$, $f'_\pm = f'$ is monotone (where this last equality follows from Lemma 23.11). Thus $f'$ is piecewise monotone on $(a,b)$.

Now define the limits
\[ f'(x^\pm) := \lim_{h \to 0^+} f'(x \pm h), \quad x \in (a,b). \]

It is clear that $f'(x^\pm)$ exists on $(a,b)$, including at each $x_j \neq a,b$. Note that $f'(x^\pm) \in [-\infty, +\infty]$, while $f'(x^\pm) \in \mathbb{R}$ for all other points $x \neq x_j$. First claim that $f'(x^+) = f'(x^{-})$, where this common limit is possibly infinite – and then that $f'(x^+) = f'(x)$, which will rule out the infinitude using Lemma 23.11 and complete the proof.

For each of the two steps, we proceed as in the proof of Lemma 23.11. Begin by fixing $x \in (a,b)$, and let $\delta > 0$ be such that $a < x - k\delta < x < x + 2\delta < b$. Let $p \in \{0,1,\ldots,k\}$; then by Lemma 23.12
\[ 0 \leq \Delta_h^{k-1} f'(x - (p - \frac{1}{2})\delta) = \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} f'(x + (i-p + \frac{1}{2})\delta). \]

Let $\delta \to 0^+$; then,
\[ A_p f'(x^-) - A_p f'(x^+) \geq 0, \quad \text{where } A_p := \sum_{i=0}^{p-1} (-1)^{k-1-i} \binom{k-1}{i} = - \sum_{i=p}^{k-1} (-1)^{k-1-i} \binom{k-1}{i}. \]
Now specialize \( p \) to be \( k - 1 \) and \( k - 2 \). In the former case \( A_p = -1 \), whence \( f'(x^-) \leq f'(x^+) \); whereas if \( p = k - 2 \) then \( A_p = k - 2 > 0 \), whence \( f'(x^-) \geq f'(x^+) \). These inequalities and the trichotomy of the extended real line \([-\infty, +\infty]\) imply that \( f'(x^-) = f'(x^+) \).

Using the same \( \delta \in ((x - a)/k, (b - x)/2) \), and \( p \in \{0, 1, \ldots, k\} \), Lemma 23.12 also implies:

\[
0 \leq \Delta_x^{k-1} f'(x - p\delta).
\]

Taking \( \delta \to 0^+ \) and using that \( f'(x^-) = f'(x^+) \) yields:

\[
B_p f'(x) - B_p f'(x^+) \geq 0, \quad \text{where } B_p := (-1)^{k-1-p} \left( \begin{array}{c} k-1 \\ p \end{array} \right) = -\sum_{i=0, i \neq p}^{k-1} (-1)^{k-1-i} \left( \begin{array}{c} k-1 \\ i \end{array} \right).
\]

Now specialize \( p \) to be \( k - 1 \) and \( k - 2 \). In the former case \( B_p = 1 \), whence \( f'(x) \geq f'(x^+) \); whereas if \( p = k - 2 \) then \( B_p = 1 - k < 0 \), whence \( f'(x) \leq f'(x^+) \). These inequalities imply that \( f'(x^+) = f'(x^-) \) equals \( f(x) \), and in particular is finite, for all \( x \in (a, b) \). □

The final lemma simply combines the preceding two:

**Lemma 23.14.** \( f' \) satisfies the condition \((H_{k-1})\) in \((a, b)\).

**Proof.** This follows immediately from Lemmas 23.12 and 23.13. □

Having shown the fourteen lemmas above, we conclude with:

**Proof of the Boas--Widder Theorem 23.1.** The proof is by induction on \( k \geq 2 \). The induction step is clear: use Lemma 23.14. We now show the base case of \( k = 2 \). By Lemma 23.4, the functions \( f'_+ \) exist and are non-decreasing on \((a, b)\). Moreover, \( f \) is continuous by assumption. To prove its convexity, we make use of the following basic result from one-variable calculus:

**Proposition 23.12.** Let \( f : [p, q] \to \mathbb{R} \) be a continuous function whose right-hand derivative \( f'_+ \) exists on \([p, q]\) and is Lebesgue integrable. Then,

\[
f(y) = f(p) + \int_p^y f'_+(t) \, dt, \quad \forall y \in [p, q].
\]

Proposition 23.12 applies to our function \( f \) satisfying \((H_2)\), since \( f'_+ \) is non-decreasing by Lemma 23.4, whence Lebesgue integrable. Therefore \( f(y) - f(x) = \int_x^y f'_+(t) \, dt \) for \( a < x < y < b \). Now repeat the proof of Proposition 23.3 to show that \( f \) is convex on \((a, b)\). This completes the base case of \( k = 2 \), and concludes the proof. □

Dimension-free non-absolutely-monotonic preservers.

In this section, we explore a variant of the question of classifying the ‘dimension-free’ preservers. Recall that Schoenberg’s original motivation in proving his result was to classify the entrywise positivity preservers \( f[-] \) on correlation/Gram matrices – with or without rank-constraints – since these are the matrices that arise as distance matrices on Euclidean space.

In a similar vein, and motivated by modern applications via high-dimensional covariance estimation, Guillot and Rajaratnam in Trans. Amer. Math. Soc. (2015) classified entrywise maps that operate only on off-diagonal entries, and preserve positivity in all dimensions:

**Theorem 24.1** (Guillot–Rajaratnam). Let \( 0 < \rho \leq \infty \) and \( f : (-\rho, \rho) \to \mathbb{R} \). Given a square matrix \( A \in \mathbb{P}_n((-\rho, \rho)) \), define \( f^*[A] \in \mathbb{R}^{n \times n} \) to be the matrix with \((j,k)\)-entry \( f(a_{jk}) \) if \( j \neq k \), and \( a_{jj} \) otherwise. Then the following are equivalent:

1. \( f^*[-] \) preserves positivity on \( \mathbb{P}_n((-\rho, \rho)) \) for all \( n \geq 1 \).
2. There exist scalars \( c_k \geq 0 \) such that \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) and \( |f(x)| \leq |x| \) on all of \((-\rho, \rho)\). (Thus if \( \rho = \infty \) then \( f(x) = cx \) on \( \mathbb{R} \), for some \( c \in [0,1] \).)

Once again, the robust characterization of absolute monotonicity emerges out of this variant of entrywise operations.

The main result of this section provides – in a closely related setting – an example of a ‘dimension-free’ preserver that is not absolutely monotonic. To elaborate: Theorem 24.1 was recently strengthened by Vishwakarma in Trans. Amer. Math. Soc., where he introduced the more general model where a different function \( g(x) \) acts on the diagonal entries. Even more generally, Vishwakarma allowed \( g[-] \) to act on ‘prescribed’ principal submatrices / diagonal blocks, and \( f[-] \) to act on the remaining entries. To explain his results, we adopt the following notation throughout this section:

**Definition 24.2.** Fix \( 0 < \rho \leq \infty \), \( I = (-\rho, \rho) \), and \( f, g : I \to \mathbb{R} \). Also fix families of subsets \( T_n \subset 2^{[n]} \) for each \( n \geq 1 \), such that all elements in a fixed family \( T_n \) are pairwise incomparable. Now given \( n \geq 1 \) and a matrix \( A \in I^{n \times n} \), define \( (g,f)_{T_n}[A] \in \mathbb{R}^{n \times n} \) to be the matrix with \((j,k)\)-entry \( g(a_{jk}) \) if there is some \( E \in T_n \) containing \( j,k \); and \( f(a_{jk}) \) otherwise.

Adopting this notation, Vishwakarma classifies the pairs \((g,f)\) which preserve positivity according to a given sequence \( \{T_n : n \geq 1 \} \). Notice that if \( T_n = \{[n] \} \) for \( n > n_0 \) and \( T_n \) is empty for \( n \leq n_0 \), this implies from the previous sections that \( g(x) \) is absolutely monotonic as in Schoenberg–Rudin’s results; and that \( f[-] \) preserves positivity on \( \mathbb{P}_{n_0}((-\rho, \rho)) \). Such functions \( f \) do not admit a known characterization for \( n_0 \geq 3 \); and the following result will also not consider them. Thus, below we require \( T_n \neq \{[n] \} \) for infinitely many \( n \geq 1 \).

**Theorem 24.3** (Vishwakarma). Notation as in Definition 24.2. Suppose \( \{T_n \} \) is such that \( T_n \neq \{[n] \} \) for infinitely many \( n \geq 1 \). Then \((g,f)_{T_n}[-]\) preserves positivity on \( \mathbb{P}_n(I) \) for all \( n \geq 1 \), if and only if exactly one of the following occurs:

1. If \( T_n \) is the empty collection, i.e. \( (g,f)_{T_n}[-] = f[-] \) for all \( n \geq 1 \), then \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) on \( I \), where \( c_k \geq 0 \) for all \( k \geq 0 \).
2. If some \( T_n \) contains two non-disjoint subsets of \([n]\), then \( g(x) = f(x) \), and \( f(x) \) is a power series as in the preceding sub-case.
3. If \( T_n \subset \{1, \ldots, n \} \) for all \( n \geq 1 \), and some \( T_n \) is non-empty, then \( f \) is as in (1), and \( 0 \leq f \leq g \) on \([0, \rho]\).
(4) If \( T_2 = \{1,2\} \) and \( T_n \subset \{1,\ldots,n\} \) for all \( n \geq 3 \), then \( f \) is as in (1), \( g(x) \) is non-negative, non-decreasing, and multiplicatively mid-convex on \([0,\rho]\), and \(|g(x)| \leq g(|x|)\) for all \( x \). If moreover some \( T_n \) is non-empty for \( n \geq 3 \), then \( 0 \leq f \leq g \) on \([0,\rho]\).

(5) Otherwise \( T_n \not\subset \{1,\ldots,n\} \) for some \( n \geq 3 \); and \( T_n \) is a partition of some subset of \([n]\) for each \( n \geq 1 \). In this case, with the additional assumption that \( g(x) = \alpha x^k \) for some \( \alpha \geq 0 \) and \( k \in \mathbb{Z}_{\geq 0} \):

(a) If for all \( n \geq 3 \) we have \( T_n = \{n\} \) or \( \{1,\ldots,n\} \), then \( f \) is as in (1) and \( 0 \leq f \leq g \) on \([0,\rho]\).

(b) If \( T_n \) is not a partition of \([n]\) for some \( n \geq 3 \), then \( f(x) = cg(x) \) for some \( c \in [0,1] \).

(c) If neither (a) nor (b) holds, then \( f(x) = cg(x) \) for some \( c \in [-1/(K - 1),1] \), where

\[
K := \max_{n \geq 1} |T_n| \in \{2, 3, 4, \ldots\} \sqcup \{+\infty\}.
\]

In fact, the assertions in the above cases are equivalent to the weaker assertion (than above) that \( (g,f)_{T_n}[-] \) preserves positivity on the rank \( \leq 3 \) positive matrices of all dimensions.

We refer the reader to Vishwakarma’s work for similar results with the domain \( I \) replaced by \((0,\rho), \{0,\rho\}\), or even the complex disc \( D(0,\rho) \). As mentioned above, one interesting feature here is that in the final assertion (5)(c), we find perhaps the first example of a non-absolutely-monotonic function that is a ‘dimension-free’ preserver, in this setting.

To prove Theorem 24.3 we require two well-known preliminaries, and a couple of additional results, shown below.

**Proposition 24.4.** Given a Hermitian matrix \( A_{n \times n} \), denote its largest and smallest eigenvalues by \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \) respectively.

1. (Rayleigh–Ritz theorem.) If \( A \in \mathbb{C}^{n \times n} \) is Hermitian, then the ratio \( v^*Av/v^*v \), as \( v \) runs over \( \mathbb{C}^n \setminus \{0\} \), attains its maximum and minimum values, which equal \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \) respectively.

2. (Weyl’s inequality, special case.) If \( A,B \in \mathbb{C}^{n \times n} \) are Hermitian, then

\[
\lambda_{\min}(A) + \lambda_{\min}(B) \leq \lambda_{\min}(A + B) \leq \lambda_{\min}(A) + \lambda_{\max}(B).
\]  

(24.5)

The second assertion holds more generally; we do not state/prove/require it below.

**Proof.** For the first part, it suffices to show the minimum bound. (That the bound is attained follows from the compactness of the unit complex sphere.) The matrix \( A - \lambda_{\min}(A) I_n \) is Hermitian with smallest eigenvalue zero, hence is positive semidefinite. Thus we compute for nonzero \( v \in \mathbb{C}^n \):

\[
0 \leq \frac{v^*(A - \lambda_{\min}(A) I_n)v}{v^*v} = \frac{v^*Av}{v^*v} - \lambda_{\min}(A).
\]

This shows the first assertion. For the second, let \( v \in \ker(A - \lambda_{\min}(A) I_n) \) be nonzero. Applying the previous part twice,

\[
\lambda_{\min}(A + B) \leq \frac{v^*(A + B)v}{v^*v} = \frac{v^*Av}{v^*v} + \frac{v^*Bv}{v^*v} \leq \lambda_{\min}(A) + \lambda_{\max}(B).
\]

Similarly, if \( v \) is a nonzero eigenvector for \( A + B \) with eigenvalue \( \lambda_{\min}(A + B) \), then by the previous part applied twice,

\[
\lambda_{\min}(A + B) = \frac{v^*(A + B)v}{v^*v} = \frac{v^*Av}{v^*v} + \frac{v^*Bv}{v^*v} \geq \lambda_{\min}(A) + \lambda_{\min}(B). \]  
\[\square\]
We also require the following special case of the main result.

**Lemma 24.6.** Let $0 < \rho \leq \infty$, $I = (-\rho, \rho)$, and $f : I \to \mathbb{R}$. Let $g(x) = \alpha x^k$ for $\alpha \geq 0$ and $k \in \mathbb{Z}^{\geq 0}$. Finally, let $T_3 = \{\{1, 2\}\}$ and $c_0 = 0$. The following are equivalent:

1. $(g, f)_{T_3}[A] \in \mathbb{P}_3$ for all matrices $A \in \mathbb{P}_3(I)$.
2. $(g, f)_{T_3}[A] \in \mathbb{P}_3$ for all rank-one matrices $A \in \mathbb{P}_3(I)$.
3. $f(x) \equiv cg(x)$ on $I$, for some $c \in [c_0, 1]$.

The same equivalence holds if $T_3 = \{\{1, 2\}, \{3\}\}$ and $c_0 = -1$.

**Proof.** First suppose $T_3 = \{\{1, 2\}\}$ and $c_0 = 0$. Clearly (1) $\implies$ (2). Now suppose (2) holds. If $f \equiv 0$ or $g \equiv 0$ then the result is immediate, so suppose $f, g \not\equiv 0$ (hence $\alpha > 0$). Note also that $f(0) = 0$ by considering $(g, f)_{T_3}[\mathbf{0}_{3 \times 3}]$. Now given $0 < |z| \leq w < \rho$, define

$$A(w, z) := \begin{pmatrix} z^2/w & z & z \\ z & w & w \\ z & w & w \end{pmatrix} = \frac{1}{w} uu^T, \quad \text{where} \quad u = (z, w, w)^T. \quad (24.7)$$

By choice of $z, w$, we have $A \in \mathbb{P}_3(I)$. Applying $(g, f)_{T_3}[\cdot]$ and expanding along the third row, a straightforward computation yields:

$$0 \leq \det(g, f)_{T_3}[A(w, z)] = -\frac{\alpha}{w^k} \left( w^k f(z) - z^k f(w) \right)^2. \quad (24.8)$$

Thus we have $f(z)/z^k = f(w)/w^k$ whenever $0 < |z| \leq w < \rho$. By using an increasing sequence $w_n \uparrow \rho^-$, this shows $f(z)/z^k$ is constant on $I \setminus \{0\}$, say $c \in \mathbb{R}$. By considering $A(w, w) = w I_{3 \times 3}$ for $w > 0$, it is not hard to see that $c \in [0, 1]$, which proves (3). Finally, if (3) holds, then $(g, f)_{T_3}[A]$ is the sum of $cg[A]$ and $(1 - c)g[B]$ (padded by a zero row and column at the end), where $B$ is the leading principal $2 \times 2$ submatrix of $A$. This shows (1) by the Schur product theorem.

The proof is similar if $T_3 = \{\{1, 2\}, \{3\}\}$ and $c_0 = -1$. Clearly (1) $\implies$ (2); similarly, the proof of (2) $\implies$ (3) is unchanged (including the computation $[24.8]$) until the very last step, at which point we can only conclude $c \in [-1, 1]$. Finally, we assume (3) holds and show (1). The point is that for any scalar $c \in [-1, 1]$ and any matrix $A \in \mathbb{P}_3(I)$, the principal minors of $(g, cg)_{T_3}[A]$ equal those of $(g, |c|g)_{T_3}[A]$, so that we may work with $|c| \in [0, 1]$ instead of $c \in [-1, 1]$. Now one shows (1) similarly as the previous case. \(\square\)

A final preliminary result; the second part easily follows from the first, and in turn strengthens Lemma 24.6.

**Proposition 24.9.** Suppose for an integer $n \geq 3$ that $T_n \subset 2^{[n]}$ is a partition of $[n]$ into $k \geq 2$ non-empty subsets.

1. Let $g(0) = 1$ and $f(0) = c$. Then $(g, f)_{T_n}[\mathbf{0}_{n \times n}]$ is positive semidefinite if and only if $c \in [-1/(k-1), 1)$.
2. Suppose $0 < \rho \leq \infty$, $I = (-\rho, \rho)$, and $f : I \to \mathbb{R}$. Also suppose $g : I \to \mathbb{R}$ is multiplicative and preserves positivity on $\mathbb{P}_n(I)$. If $T_n \neq \{\{1\}, \ldots, \{n\}\}$, then the following are equivalent:
   a. $(g, f)_{T_n}[\cdot]$ preserves positivity on $\mathbb{P}_n(I)$.
   b. $(g, f)_{T_n}[\cdot]$ preserves positivity on the rank-one matrices in $\mathbb{P}_n(I)$.
   c. $f(x) \equiv cg(x)$ on $I$, for some $c \in [-1/(k-1), 1]$. 


Dimension-free non-absolutely-monotonic preservers.

The nonzero functions in part (2) include the powers \( x^k, k \in \mathbb{Z}^{\ge 0} \) by the Schur product theorem; but also – by results of Hiai in Linear Algebra Appl. (2009) – the ‘powers’

\[
\phi_\alpha(x) := |x|^\alpha, \quad \psi_\alpha(x) := \text{sgn}(x)|x|^\alpha, \quad \alpha \ge n - 2.
\]

**Proof.** Let \( T_n = \{J_1, \ldots, J_k\} \) with \( \sqcup_j J_j = [n] \).

1. Choose elements \( j_1, \ldots, j_k \) with \( j_i \in J_i \). By possibly relabelling the rows and columns, we may assume without loss of generality that \( 1 \le j_1 < \cdots < j_k \le n \). Now if \( (g,f)_T_n[0_{n \times n}] \in \mathbb{P}_n \), then by considering the principal \( k \times k \) submatrix corresponding to the indices \( \{j_1, \ldots, j_k\} \), we obtain:

\[
C := c1_{k \times k} + (1-c) \text{Id}_k \in \mathbb{P}_k. \tag{24.10}
\]

Since this matrix has eigenvalues \((1-c)\) and \(1+(k-1)c\), we get \( c \in [-1/(k-1), 1]\), as desired.

For the converse, define the ‘decompression’ of \( C \), given by

\[
\tilde{C} := c1_{n \times n} + (1-c) \sum_{j=1}^k 1_{J_j \times J_j} = (g,f)_{T_n}[0_{n \times n}] \in \mathbb{C}^{n \times n}. \tag{24.11}
\]

We now show that if \( c \in [-1/(k-1), 1] \), then \( \tilde{C} \in \mathbb{P}_n \). Indeed, given a vector \( u \in \mathbb{C}^n \), define \( u_{T_n} \in \mathbb{C}^k \) to have \( j \)th coordinate \( \sum_{i \in J_j} u_i \). Then,

\[
u^* \tilde{C} u = u_{T_n}^* C u_{T_n} \ge 0, \quad \forall u \in \mathbb{C}^n,
\]

because the matrix \( C \) as in \(24.10\) is positive semidefinite as above.

2. If \( g \equiv 0 \) then the result is easy to prove, so we suppose henceforth that \( g \not\equiv 0 \). Clearly (a) implies (b). Next if (b) holds, then one can restrict to a suitable \( 3 \times 3 \) submatrix – without loss of generality indexed by \( 1,2,3 \), such that \( T_n \cap \{1,2,3\} = \{\{1,2\}, \{3\}\} \) by a slight abuse of notation. Hence \( f(x) \equiv cg(x) \) on \( I \) for some \( c \in [-1,1] \), by Lemma 24.6. Now if \( g(x_0) \neq 0 \) then \( (g,f)_{T_n}[x_01_{n \times n}] \) has as a principal submatrix, \( g(x_0)C \), where \( C \) is as in \(24.10\). Hence \( c \in [-1/(k-1), 1] \) by the previous part, proving (c). Finally, given any matrix \( A \in \mathbb{P}_n(I) \), we have

\[
(g,cg)_{T_n}[A] = g[A] \circ \tilde{C},
\]

where \( \tilde{C} \) is as in \(24.11\). Now if (c) holds, then \( \tilde{C} \in \mathbb{P}_n \) by the previous part, and this shows (a) by the assumptions on \( g,f \) as well as the Schur product theorem. \( \square \)

With these results in hand, we are ready to proceed.

**Proof of Theorem 24.3.** Clearly if \( (g,f)_{T_n}[-] \) preserves positivity on \( \mathbb{P}_n(I) \), then it does so on the rank \( \le 3 \) matrices in \( \mathbb{P}_n(I) \). Thus, we will prove that this latter assertion implies the conclusions on \( (g,f) \) in the various cases; and that these conclusions imply in turn that \( (g,f)_{T_n}[-] \) preserves positivity on \( \mathbb{P}_n(I) \). This is done in each of the sub-cases (which place constraints on the family \( T_n \)). First if (1) all \( T_n \) are empty sets, then the result follows from the stronger Schoenberg–Rudin Theorem 16.3 (which holds over \((-\rho,\rho)\) instead of \(\mathbb{R}\), as remarked in Section 22.3).

Next, suppose (2) some \( T_n \) contains \( I_1, I_2 \subset [n] \) that are not disjoint. Clearly if \( g = f \) and \( f \) is as in (1) then \( (g,f)_{T_n}[-] = f[-] \) preserves positivity by the Schur product theorem. Conversely, if \( (g,f)_{T_n}[-] \) preserves positivity even on the rank-one matrices in \( \mathbb{P}_n((-\rho,\rho)) \) for all \( n \ge 3 \), then there exist integers \( n \ge 3 \) and \( a,b,c \in [n] \) such that

\[
a,b \in I_1, \quad c \not\in I_1, \quad b,c \in I_2, \quad a \not\in I_2.
\]
By relabelling indices if needed, we will assume without loss of generality that \( a = 1, b = 2, c = 3. \) Now let \( x \in (-\rho, \rho) \) and define \( A := \begin{pmatrix} x & x & x \\ x & |x| & |x| \\ x & |x| & |x| \end{pmatrix} \oplus 0_{(n-3) \times (n-3)} \in \mathbb{P}_n(I) \). If \( B \) denotes the leading principal \( 3 \times 3 \) submatrix of \((g, f)_{T_n}[A]\), then

\[
0 \leq \det B = \det \begin{pmatrix} g(|x|) & g(x) & f(x) \\ g(x) & g(|x|) & g(|x|) \\ f(x) & g(|x|) & g(|x|) \end{pmatrix} = -g(|x|)(f(x) - g(x))^2.
\]

If \( g(|x|) = 0 \) then by considering the \( 2 \times 2 \) submatrices of \( B \), we see that \( f(x) = g(x) = 0. \) If \( g(|x|) \neq 0 \), then it is positive, whence \( f(x) = g(x) \). Either way, we have \( f \equiv g \) on \((-\rho, \rho)\). Hence \((g, f)_{T_n}[\cdot] = f[\cdot]\), and we reduced to case (1). This proves the equivalence for case (2).

Next suppose (3) holds. First assume \( f \) is as in (1) and \( 0 \leq f \leq g \) on \([0, \rho)\). If \( A \in \mathbb{P}_n((-\rho, \rho)) \), then \((g, f)_{T_n}[A] \) is the sum of \( f[A] \) and a diagonal matrix with non-negative entries. Hence \((g, f)_{T_n}[A] \) is positive semidefinite by the Schur product theorem. The converse has two sub-cases. Let \( s_n := \# \cup_{E \in T_n} E \), so \( 0 \leq s_n \leq n \), and hence either \( n - s_n \) or \( s_n \) is an unbounded sequence. If the former, then by restricting to the corresponding principal submatrices (padded by zeros), we are done by case (1) – considering the \( 2 \times 2 \) matrix

\[
\begin{pmatrix} g(x) & f(x) \\ f(x) & g(x) \end{pmatrix}
\]

for \( x \in [0, \rho) \), we obtain \( f(x) \leq g(x) \) as desired.

Thus we henceforth assume the latter holds, i.e. \( s_n \) is unbounded; restricting to these principal submatrices, we may assume without loss of generality that \( T_n = \{1, \ldots, n\} \) for all \( n \geq 1 \). We claim that \( f[\cdot] \) preserves positivity on rank \( \leq 3 \) matrices in \( \mathbb{P}_n(I) \) for all \( n \). This would finish the proof in case (3), since now \( f \) is as in (1), and as above, this implies \( 0 \leq f(x) \leq g(x) \) for \( x \in [0, \rho) \).

To prove the claim, let \( A \in \mathbb{P}_n((-\rho, \rho)) \), and let \( D_A \) be the diagonal matrix with \((j, j)\)-entry \( g(a_{jj}) - f(a_{jj}) \). If \( 1_{m \times m} \) denotes the all-ones \( m \times m \) matrix, then \( 1_{m \times m} \otimes A = \begin{pmatrix} A & \cdots & A \\ \vdots & \ddots & \vdots \\ A & \cdots & A \end{pmatrix} \), a matrix in \( \mathbb{P}_{mn}(I) \). Also note that if \( A \) has rank \( \leq 3 \), then by (3.11), so does \( 1_{m \times m} \otimes A \). Now applying \((g, f)_{T_n}[\cdot] \) yields:

\[
(g, f)_{T_n}[1_{m \times m} \otimes A] = 1_{m \times m} \otimes f[A] + \text{Id}_m \otimes D_A \geq 0.
\]

Hence by (24.5),

\[
0 \leq \lambda_{\min}((g, f)_{T_n}[1_{m \times m} \otimes A]) \leq \lambda_{\max}(1_{m \times m} \otimes f[A]) + \lambda_{\max}(\text{Id}_m \otimes D_A) = m\lambda_{\min}(f[A]) + \max_{1 \leq j \leq n} \{g(a_{jj}) - f(a_{jj})\},
\]

where the equality holds because of (3.11) and since the eigenvalues of \( 1_{m \times m} \) are 0, \( m \). From this it follows that \( \lambda_{\min}(f[A]) \geq -\max_{1 \leq j \leq n} \{(g(a_{jj}) - f(a_{jj})) / m \} \) for all \( m \geq 1 \). This shows \( f[A] \) is positive semidefinite, and concludes the proof in case (3).

If (4) holds, the proof in the preceding case shows \( f \) is as in (1); and using \((g, f)_{T_2}[\cdot] = g[\cdot] \) via an argument similar to Theorem 12.7 shows the desired constraints on \( g \). This is left to the reader to work out.) The converse is shown using (variations of) the same proofs.

It remains to prove the equivalence in case (5); here we are also given that \( g(x) = \alpha x^k \) for \( \alpha, k \geq 0 \) (and \( k \) an integer). If \( \alpha = 0 \) then the result is easy, so we suppose henceforth without loss of generality that \( \alpha = 1 \). In sub-case (a), since \( T_n = \{1, \ldots, n\} \) for infinitely
many \( n \) by assumption, we can repeat the proof for case (3) to show that any preserver-pair \( (g, f) \) must satisfy \( 0 \leq f \leq g \) on \([0, \rho]\) and \( f \) is as in (1). Conversely, given such \( (g, f) \), if \( T_n = \{[n]\} \) then \( (g, f)_{T_n}[-] = g[-] \), which preserves positivity by the Schur product theorem.

Otherwise for \( A \in \mathbb{P}_n(I) \), we compute:

\[
(g, f)_{T_n}[A] = f[A] + \text{diag}(g(a_{jj}) - f(a_{jj}))_{j=1}^n,
\]

and both matrices are positive, whence so is \( (g, f)_{T_n}[A] \), as desired.

Next for (b), we fix \( n_1 \geq 3 \) such that \( T_{n_1} \not\subseteq \{\{1\}, \ldots, \{n_1\}\} \); also fix \( n_0 \geq 3 \) such that \( T_{n_0} \) is not a partition of \([n_0]\). If \( f(x) = cg(x) \) for \( c \in [0, 1] \), then \( (g, f)_{T_{n_0}}[A] \) is the sum of \( cA^{\otimes k} \) and matrices of the form \((1-c)B^{\otimes k}\), where \( B \) is a principal submatrix of \( A \in \mathbb{P}_{n_0} \), whence positive semidefinite. It follows by the Schur product theorem that \( (g, f)_{T_{n_0}}[-] \) preserves positivity.

Conversely, suppose \( (g, f)_{T_n}[-] \) preserves positivity for all \( n \geq 1 \), on rank \( \leq 3 \) matrices in \( \mathbb{P}_n(I) \). At \( n = n_1 \), we can find three indices – labelled 1, 2, 3 without loss of generality – such that for all \( A \in \mathbb{P}_{n_1}(I) \), the leading \( 3 \times 3 \) submatrix of \( (g, f)_{T_{n_1}}[A] \) equals \((g, f)_{\{\{1,2\}\}}[A_{3\times 3}]\) or \((g, f)_{\{\{1,2,3\}\}}[A_{3\times 3}]\). Now using rank-one matrices via Lemma 24.6 shows \( f(x) = cg(x) \) for \( c \in [-1, 1] \). Finally, considering matrices in \( \mathbb{P}_{n_0}(I) \) yields \( c \geq 0 \), as desired.

The remaining sub-case is (5)(c), whence every \( T_n \) is a partition of \([n]\). Also note by the hypotheses that \( K > 1 \); and there exists \( n_1 \geq 3 \) and three indices – labelled 1, 2, 3 without loss of generality – such that for all \( A \in \mathbb{P}_{n_1}(I) \), the leading \( 3 \times 3 \) submatrix of \( (g, f)_{T_{n_1}}[A] \) equals \((g, f)_{\{\{1,2,3\}\}}[A_{3\times 3}]\). Now using rank-one matrices via Lemma 24.6 or Proposition 24.9 implies \( f \equiv cg \), with \( c \in [-1/(K-1), 1] \). Conversely, if \( f, g \) are as specified and \( T_n = \{[n]\} \) then \((g, f)_{T_n}[-] = g[-]\), which preserves positivity by the Schur product theorem. Else we are done by Proposition 24.9 since \( k \leq K \).
25. Appendix C. Preservers of positivity on kernels.

We conclude this part with an Appendix on the transforms that preserve positive (semi)definiteness, Loewner monotonicity and convexity, and total non-negativity and positivity, primarily on kernels on infinite domains. We begin with preservers of positive semidefinite and positive definite kernels.

**Definition 25.1.** Let $X, Y$ be non-empty sets, and $K : X \times Y \to \mathbb{R}$ a kernel.

1. Given $x \in X^m$ and $y \in Y^n$ for integers $m, n \geq 1$, define $K[x; y]$ to be the $m \times n$ real matrix, with $(j, k)$ entry $K(x_j, y_k)$.
2. Given an integer $n \geq 1$, define $X^{n, \not\in}$ to be the set of all $n$-tuples in $X$ with pairwise distinct coordinates.
3. A kernel $K : X \times X \to \mathbb{R}$ is said to be positive semidefinite (respectively, positive definite) if $K$ is symmetric – i.e., $K(x, y) = K(y, x) \forall x, y \in X$ – and for all $n \geq 1$ and tuples $x \in X^{n, \not\in}$, the matrix $K[x; x]$ is positive semidefinite (respectively, positive definite).
4. Given an integer $n \geq 1$, and a totally ordered set $X$, define $X^{n, \uparrow}$ to be the set of all $n$-tuples $x = (x_1, \ldots, x_n) \in X$ with strictly increasing coordinates: $x_1 < \cdots < x_n$.

By ‘padding principal submatrices by the identity kernel’, it is easily seen that given subsets $X \subset Y$ and a positive (semi)definite kernel $K$ on $X \times X$, we can embed $K$ into a kernel $\tilde{K} : Y \times Y \to \mathbb{R}$ that is also positive (semi)definite: define $\tilde{K}(x, y)$ to be $1_{x=y}$ if $(x, y) \not\in X \times X$, and $K(x, y)$ otherwise.

Now given a set $X$ and a domain $I \subset \mathbb{R}$, we will study the inner transforms

$$
\mathcal{S}^\text{psd}_X(I) := \{F : I \to \mathbb{R} | \text{if } K : X \times X \to I \text{ is positive semidefinite, so is } F \circ K\},
$$

$$
\mathcal{S}^\text{pd}_X(I) := \{F : I \to \mathbb{R} | \text{if } K : X \times X \to I \text{ is positive definite, so is } F \circ K\}.
$$

Notice that if $X$ is finite then $\mathcal{S}^\text{psd}_X(I)$ is precisely the set of entrywise maps preserving positivity on $\mathbb{P}|X|(I)$; as mentioned above, this question remains open for all $|X| \geq 3$. If instead $X$ is infinite, then the answer follows from Schoenberg and Rudin’s results:

**Theorem 25.2.** Fix $0 < \rho \leq \infty$, and suppose $I$ is any of $(0, \rho)$, $[0, \rho)$, or $(-\rho, \rho)$. If $X$ is an infinite set, then $\mathcal{S}^\text{psd}_X(I)$ consists of all power series with non-negative Maclaurin coefficients, which are convergent on $I$.

**Proof.** For $I = (0, \rho)$ or $(-\rho, \rho)$ with $\rho = \infty$, the result follows by embedding every positive semidefinite matrix into a kernel on $X \times X$, and applying Theorems 16.4 and 16.3 respectively. If $I = [0, \rho)$ then from above we have the desired power series expansion on $(0, \infty)$, and it remains to show that any preserver $F$ is right-continuous at 0. To see why, first note that $F(0) \geq 0$, and $F$ is non-decreasing and non-negative on $(0, \infty)$, so $F(0^+) := \lim_{x \to 0^+} F(x)$ exists. Now consider a three-point subset $\{x_1, x_2, x_3\}$ of $X$, with complement $X$, and define the kernel

$$
K_0(x, y) := \begin{cases} 
3, & \text{if } x = y, \\
1, & \text{if } (x, y) = (x_1, x_2), (x_2, x_3), (x_3, x_2), (x_2, x_1), \\
0, & \text{otherwise}.
\end{cases} \quad (25.3)
$$
Thus $K_0$ is the ‘padding by the identity’ of a positive definite $3 \times 3$ matrix. It follows that $F \circ (cK_0)$ is positive semidefinite for $c > 0$, whence its principal submatrix \( \begin{pmatrix} F(3c) & F(0) \\ F(0) & F(3c) \end{pmatrix} \).

It follows by taking determinants and then $c \to 0^+$ that $F(0^+) \geq F(0) \geq 0$.

Finally, let $x := (x_1, x_2, x_3)$; then
\[
0 \leq \lim_{c \to 0^+} \det F[cK_0(x; x)] = -F(0^+)(F(0^+) - F(0))^2.
\]

Thus either $F(0^+) > 0$, in which case $F(0^+) = F(0)$, or else $F(0^+) = 0$, whence $F(0) = 0 = F(0^+)$ as well. This concludes the proof for $\rho = \infty$; for $\rho < \infty$, the result follows by the remarks in Section 22.3. \(\square\)

Having worked out the preservers of positive semidefinite kernels, we move on to the positive definite case. Once again, if $X$ is finite then the fixed-dimension case remains open; but for infinite $X$ we have:

**Theorem 25.4.** Fix $0 < \rho \leq \infty$, and suppose $I$ is any of $(0, \rho)$, $[0, \rho)$, or $(-\rho, \rho)$. If $X$ is an infinite set, then $\mathcal{P}_{X}^{\text{psd}}(I)$ consists of all non-constant power series with non-negative Maclaurin coefficients, which are convergent on $I$.

**Proof.** By the Schur product theorem, every monomial $x^k$ for $k \geq 1$ preserves positive definiteness. This observation shows one implication.

Conversely, suppose first that $F \in \mathcal{P}_{X}^{\text{psd}}(I)$ is continuous. Now every positive definite kernel $K : X \times X \to \mathbb{R}$ is the pointwise limit as $\epsilon \to 0^+$ of the family $K_{\epsilon} : X \times X \to \mathbb{R}$, given by $K_{\epsilon}(x, y) := K(x, y) + \epsilon \mathbf{1}_{x=y}$. Now since $F$ is continuous, it follows that $F$ preserves positive semidefinite kernels as well, reducing the problem to the preceding result. Moreover, $F$ is clearly not constant, e.g. by considering its action on the ‘identity kernel.

The rest of the proof is devoted to show that $F$ is continuous on $I$. First suppose $I = (0, \rho)$ and $A \in \mathcal{P}_2(I)$ is positive definite. Then there exists $\epsilon \in (0, 1)$ such that $A' := A - \epsilon \text{Id}_{2 \times 2}$ is still positive definite. Choose $x_1, x_2 \in X$ and define the kernel
\[
K : X \times X \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} a_{jk}, & \text{if } x = x_j, y = y_j, \quad 1 \leq j, j \leq 2; \\
1, & \text{if } x = y \notin \{x_1, x_2\}; \\
\epsilon, & \text{otherwise}. \end{cases}
\]

Clearly,
\[
K = \epsilon \mathbf{1}_{X \times X} + (A' \oplus (1 - \epsilon) \text{Id}_{X \setminus \{x_1, x_2\}}),
\]
and so $K$ is positive definite on $X$ with all values in $I = (0, \rho)$. Hence $F \circ K$ is also positive definite. It follows that the entrywise map $F[-]$ preserves the set of symmetric $2 \times 2$ TP matrices. Now invoke Lemma 21.1 to conclude that $F$ is continuous on $(0, \rho)$.

This concludes the proof for $I = (0, \rho)$. Next suppose $I = [0, \rho)$; by the preceding case, $F$ is given by a non-constant power series as asserted, and we just need to show $F$ is right-continuous at 0. Since $F$ is increasing on $(0, \rho)$, the limit $F(0^+) \equiv \lim_{x \to 0^+} F(x)$ exists and $F(0^+) \geq F(0) \geq 0$. Now use the kernel $K_0$ from (25.3) and repeat the subsequent arguments.

The final case is if $I = (-\rho, \rho)$. In this case we fix $u_0 \in (0, 1)$ and a countable subset $Y := \{x_0, x_1, \ldots\} \subset X$. Denote $Y^c := X \setminus Y$. Given $a, b > 0$ such that $a + b < \rho$, let the measure
\[
\mu = \mu_{a,b} := a\delta_{-1} + b\delta_{u_0}.
\]

The corresponding Hankel moment matrix is $H_\mu$, with $(j, k)$ entry $a(-1)^{j+k} + bu_0^{j+k}$, and this is positive semidefinite of rank two. Now for each $\epsilon > 0$, define the kernel $K_\epsilon : X \times X \to \mathbb{R}$,
via:
\[
K_{\epsilon}(x,y) := \begin{cases} 
H_{\mu}(j,j) + \epsilon, & \text{if } x = y = x_j, \ j \geq 0; \\
H_{\mu}(j,k), & \text{if } (x,y) = (x_j,x_k), \ j \neq k; \\
\epsilon, & \text{if } x = y \in Y^c; \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly, \( K_{\epsilon} \) is positive definite, with entries in \( I = (-\rho, \rho) \) for sufficiently small \( \epsilon > 0 \). It follows that \( F \circ K_{\epsilon} \) is positive definite. Since \( F \) is continuous on \([0,\rho)\) by the previous cases, \( \lim_{\epsilon \to 0^+} F \circ K_{\epsilon} = F[H_{\mu} \oplus 0_{Y^c \times Y^c}] \) is positive semidefinite, whence \( F[-] \) preserves positivity on the Hankel moment matrices \( H_{\mu} \) for all \( \mu = \mu_{a,b} \) as above. It follows by the proof of Step 3 for the stronger Schoenberg theorem above (see the computations following Lemma \[20.4\]) that \( F \) is continuous on \((-\rho, \rho)\), as desired. This concludes the proof in all cases. \( \square \)

For completeness, we remark that the above use of absolutely monotone functions to characterize positivity preservers of kernels on infinite domains (or all finite domains) parallels other results in the literature that feature completely monotone functions. Here are two such results for the interested reader, both of which are easily reformulated in the language of kernels:

1. In his 1974 paper in Manuscripta Math., Ressel characterized the functions that are positive definite in a different sense: given an abelian semigroup \((S,+)\), a function \( f : S \to \mathbb{R} \) is said to be positive semidefinite if \( f \) is bounded and for any finite set of elements \( s_1, \ldots, s_n \in S \), the matrix \( (f(s_j + s_k))_{j,k=1}^n \) is positive semidefinite. Ressel then showed for all \( p \geq 1 \) that the continuous and positive semidefinite functions on the semigroup \([0, \infty)^p\) are precisely Laplace transforms of finite non-negative Borel measures on \([0, \infty)^p\). In particular, for \( p = 1 \), this is further equivalent — by a result attributed to Bernstein, Hausdorff, and Widder — to \( f \) being completely monotone, i.e., \( f \) is continuous on \([0, \infty)\), smooth on \((0, \infty)\), and \((-1)^k f^{(k)}\) is non-negative on \((0, \infty)\) for all \( k \geq 0 \).

2. A related result to this was shown by Schoenberg in Ann. of Math. (1938). It says that a continuous function \( f : [0, \infty) \to \mathbb{R} \) satisfies the property that for all \( n \geq 1 \) and vectors \( x_1, \ldots, x_m \in \mathbb{R}^n \), the matrix \( (f(\|x_j - x_k\|))_{j,k=1}^n \) is positive semidefinite, if and only if there exists a finite non-negative measure \( \mu \geq 0 \) on \([0, \infty)\) such that
\[
f(x) = \int_0^\infty \exp(-x^2 t) \, d\mu(t), \quad \forall x \geq 0.
\]

A final related result: Christensen and Ressel also proved Schoenberg’s Theorem [16.2] using Choquet’s representation theorem, in 1978 in Trans. Amer. Math. Soc. They also proved the complex analogue: namely, for preservers of positivity on Gram matrices from unit complex spheres, in Math. Z. in 1982:

**Theorem 25.5** (Christensen–Ressel, 1982). Suppose \( f : \overline{D}(0,1) \to \mathbb{C} \) is continuous on the closed unit disc, and \( \mathcal{H} \) is an infinite-dimensional complex Hilbert space with unit ball \( S \). Then the following are equivalent:

1. \( f \) is ‘positive definite’ on \( S \), in that for all \( n \geq 1 \) and points \( z_1, \ldots, z_n \in S \), the matrix with \((j,k)\)-entry \( f(\langle z_j, z_k \rangle) \) is positive semidefinite.

2. \( f(z) \) has the unique series representation \( f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k \), with all \( c_{m,k} \geq 0 \) and \( \sum_{m,k} c_{m,k} < \infty \).
This resembles Herz’s Theorem 16.5 (which proved Rudin’s conjecture) similar to the relation between Schoenberg’s Theorem 16.22 and Rudin’s Theorem 16.3.
26. Appendix D. Preservers of Loewner monotonicity and convexity on kernels.

Thus far, we have studied the preservers of (total) positivity and related variants, with a brief look in Section 15 at entrywise powers preserving other Loewner properties. In this section, we return to these properties. Specifically, we classify all composition operators preserving Loewner monotonicity and convexity, on kernels on infinite domains. (The case of finite domains remains open, as for positivity preservers – see the preceding sections.)

The results for infinite kernels will crucially use the finite versions; thus we begin by reminding the reader of the definitions. Roughly speaking, a function is Loewner monotone (see Definition 14.7) if \( f[A] \geq f[B] \) whenever \( A \geq B \in \mathbb{R}^{n \times n} \). Similarly, a function is Loewner convex (see Definition 15.7) if \( f[\lambda A + (1 - \lambda)B] \leq \lambda f[A] + (1 - \lambda)f[B] \) whenever \( A \geq B \geq 0 \) and \( \lambda \in [0, 1] \).

As explained in Remark 14.8 for \( n = 1 \) the usual notion of a monotonically non-decreasing function coincides with Loewner monotonicity. The same holds for convex functions vis-a-vis Loewner convex functions, for \( n = 1 \). Now for \( n = 1 \), a differentiable function \( f : (0, \infty) \to \mathbb{R} \) is monotone (resp., convex) if and only if \( f' \) is positive, i.e., has image in \([0, \infty)\) (resp., monotone). The following result by Hiai in Linear Algebra Appl. (2009) extends this to the corresponding Loewner-properties, in every dimension:

**Theorem 26.1** (Hiai, fixed dimension). Suppose \( 0 < \rho \leq \infty \), \( I = (-\rho, \rho) \), and \( f : I \to \mathbb{R} \).

1. Given \( n \geq 2 \), the function \( f \) is Loewner convex on \( \mathbb{P}_n(I) \) if and only if \( f \) is differentiable on \( I \) and \( f' \) is Loewner monotone on \( \mathbb{P}_n(I) \). This result also holds if we restrict both test sets to rank \( \leq k \) matrices in \( \mathbb{P}_n(I) \), for every \( 2 \leq k \leq n \).

2. Given \( n \geq 3 \), the function \( f \) is Loewner monotone on \( \mathbb{P}_n(I) \) if and only if \( f \) is differentiable on \( I \) and \( f' \) is Loewner positive on \( \mathbb{P}_n(I) \).

Recall the related but somewhat weaker variant in Proposition 15.9. Here we show the first part and a weaker version of the second part of Theorem 26.1; see Hiai’s 2009 paper for the complete proof. (Note: Hiai showed the first part for \( k = n \); also, we do not use the second part in the present text.) First, as a consequence of the first part and the previous results, we obtain the following Schoenberg-type classification of the corresponding ‘dimension-free’ entrywise preservers:

**Theorem 26.2** (Dimension-free preservers of monotonicity and convexity). Suppose \( 0 < \rho \leq \infty \), \( I = (-\rho, \rho) \), and \( f : I \to \mathbb{R} \). The following are equivalent:

1. \( f \) is Loewner monotone on \( \mathbb{P}_n(I) \), for all \( n \).
2. \( f \) is Loewner monotone on the rank \( \leq 3 \) Hankel matrices in \( \mathbb{P}_n(I) \), for all \( n \).
3. \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) on \( I \), with \( c_1, c_2, \ldots \geq 0 \).

Similarly, the following are equivalent conditions characterizing Loewner convexity:

1. \( f \) is Loewner convex on \( \mathbb{P}_n(I) \), for all \( n \).
2. \( f \) is Loewner convex on the rank \( \leq 3 \) matrices in \( \mathbb{P}_n(I) \), for all \( n \).
3. \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) on \( I \), with \( c_2, c_3, \ldots \geq 0 \).

**Proof.** We begin with the dimension-free Loewner monotone maps. Clearly (1) \( \implies \) (2). To show (3) \( \implies \) (1), note that \( f(x) - c_0 \) is also Loewner monotone for any \( c_0 \in \mathbb{R} \) if \( f(x) \) is, so it suffices to consider \( f(x) = x^k \) for \( k \geq 1 \). But such a function is clearly monotone, by the Schur product theorem. This is an easy exercise, or see e.g. the proof of Theorem 14.9.

Finally, note from the definition of Loewner monotonicity that \( f - f(0) \) entrywise preserves positivity if \( f \) is Loewner monotone – on \( \mathbb{P}_n(I) \) or on subsets of these that contain the zero
matrix. In particular, if (2) holds then \( f - f(0) \) is a dimension-free positivity preserver, whence of the form \( \sum_{k \geq 0} c_k x^k \) with all \( c_k \geq 0 \) by Theorem 19.10 – or more precisely, its variant for restricted domains \((-\rho, \rho)\) as in Section 22.3. Since \( f - f(0) \) also vanishes at the origin, we have \( c_0 = 0 \), proving (3).

We next come to convexity preservers. Clearly (1) \( \Rightarrow \) (2). To show (3) \( \Rightarrow \) (1), note that \( f(x) - c_0 - c_1 x \) is also Loewner convex for any \( c_0, c_1 \in \mathbb{R} \) if \( f(x) \) is, so it again suffices to consider \( f(x) = x^k \) for integers \( k \geq 2 \). In fact we claim by induction that \( x^k \) is Loewner convex for all \( k \geq 0 \). The convexity of 1, \( x \) is immediate, and for the induction step, if \( x^k \) is convex, then for any integer \( n \geq 1 \), scalar \( \lambda \in [0, 1] \), and matrices \( A \geq B \geq 0_{n \times n}, \)

\[
(\lambda A + (1 - \lambda)B)^{\circ(k+1)} \leq (\lambda A + (1 - \lambda)B) \circ (\lambda A^\circ k + (1 - \lambda)B^\circ k) \\
= \lambda A^{\circ(k+1)} + (1 - \lambda)B^{\circ(k+1)} - \lambda(1 - \lambda)(A - B) \circ (A^\circ k - B^\circ k) \\
\leq \lambda A^{\circ(k+1)} + (1 - \lambda)B^{\circ(k+1)},
\]

where the final inequality follows from the Loewner monotonicity of \( x^k \) and the Schur product theorem. Finally, if (2) holds, then by Theorem 26.1(1) for \( k = 3 \), \( f' \) exists and is Loewner monotone on rank \( \leq 3 \) matrices in \( \mathbb{P}_n(I) \) for all \( n \), hence a power series as in the preceding set of equivalent statements. This immediately implies (3).

The remainder of this section is devoted to proving Theorem 26.1(1), beginning with some elementary properties of convex functions:

**Lemma 26.3 (Convex functions).** Suppose \( I \subset \mathbb{R} \) is an interval and \( f : I \to \mathbb{R} \) is convex.

1. The function \( (s, t) \mapsto \frac{f(t) - f(s)}{t - s} \), where \( t > s \), is non-decreasing in both \( t, s \in I \).
2. If \( I \) is open, then \( f' \) exist on \( I \). In particular, \( f \) is continuous on \( I \).
3. If \( I \) is open and \( z_1 < x < z_2 \) in \( I \), then \( f'_+(z_1) \leq f'_-(x) \leq f'_+(x) \leq f'_-(z_2) \). In particular, \( f'_+ \) are non-decreasing in \( I \), whence each continuous except at countably many points of jump discontinuity.
4. If \( I \) is open, then for all \( x \in I \),

\[
f'_+(x) = \lim_{z \to x^+} f'_+(z), \quad f'_-(x) = \lim_{z \to x^-} f'_-(z).
\]

5. If \( I \) is open, there exists a co-countable (whence dense) subset \( D \subset I \) on which \( f' \) exists. Moreover, \( f' \) is continuous and non-decreasing on \( D \).

Note that the assertions involving open intervals \( I \) may be carried over to the interiors of arbitrary intervals \( I \) on which \( f \) is convex.

**Proof.**

1. Suppose \( s < t < u \) lie in \( I \). One needs to show:

\[
\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.
\]

But both inequalities can be reformulated to say

\[
f(t) \leq \frac{u - t}{u - s} f(s) + \frac{t - s}{u - s} f(u),
\]

which holds as \( f \) is convex.
26. Appendix D. Preservers of Loewner monotonicity and convexity on kernels. 155

(2) Given \( t \in I \), choose \( s < t < u \) in \( I \), and note by the previous part that the ratio
\[
\frac{f(x) - f(t)}{x - t}, \quad x \in (t, u)
\]
is non-increasing in \( x \) as \( x \to t^+ \), and bounded below by \( \frac{f(t) - f(s)}{t - s} \). Thus \( f'_+(t) \) exists; a similar argument works to show \( f'_-(t) \) exists. In particular, the two limits \( \lim_{x \to t^\pm} f(x) - f(t) \) are both zero, proving \( f \) is continuous at \( t \in I \).

(3) The second sentence follows from the first, which in turn follows from the first part by taking limits, and is left to the reader.

(4) The preceding part implies \( f'_+(z) \) are non-decreasing as \( z \to x^- \) and non-increasing as \( z \to x^+ \), and shows ‘half’ of the desired inequalities. We now show \( f'_+(x) = \lim_{x \to x^+} f'_+(z) \); the remaining similar inequalities are shown similarly, and again left to the reader. Let \( y \in I \), \( y > x \); then the first part implies:
\[
\frac{f(y) - f(z)}{y - z} \leq \frac{f(y') - f(z)}{y' - z}, \quad \forall x < z < y < y'.
\]

Taking \( y' \to z_+ \), we have \( f'_+(z) \leq \frac{f(y) - f(z)}{y - z} \). From above, \( f \) is continuous on \( I \), so
\[
\frac{f(y) - f(x)}{y - x} = \lim_{z \to x^+} \frac{f(y) - f(z)}{y - z} \geq \lim_{z \to x^+} f'_+(z).
\]

Finally, taking \( y \to x^+ \) concludes the proof.

(5) Let \( D \subset I \) be the subset where \( f' \) exists, which is if and only if \( f'_+ \) is continuous (by the preceding part). In particular, \( D \) is co-countable from a previous part, and \( f' = f'_+ \) is continuous and non-decreasing on \( D \) by the same part. \hfill \Box

The next preliminary result shows the continuity (resp., differentiability) of monotone (resp. convex) functions on \( 2 \times 2 \) matrices:

**Proposition 26.4.** Suppose \( 0 < \rho \leq \infty \), \( I = (-\rho, \rho) \), and \( g : I \to \mathbb{R} \).

1. If \( g[\cdot] \) is monotone on \( P_2(I) \), then \( g \) is continuous on \( I \).
2. If \( g[\cdot] \) is convex on \( P_2(I) \), then \( g \) is differentiable on \( I \).  

**Proof.** We begin with the first assertion. It is easily verified that if \( g \) is monotone on \( P_2(I) \), then \( g - g(0) \), when applied entrywise to \( P_2(I) \), preserves positivity. Hence by (the bounded domain-variant of) Theorem [2.7], \( g \) is continuous on \((0, \rho)\). Moreover, we may assume without loss of generality that \( g(0) = 0 \).

Now let \( 0 < a < \rho \) and \( 0 < \epsilon < \rho - a \); then the monotonicity of \( g \) implies:
\[
\begin{pmatrix} a + \epsilon & a \\ a & a \end{pmatrix} \geq \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix} \geq 0_{2 \times 2} \implies \begin{pmatrix} g(a + \epsilon) - g(\epsilon) & g(a) \\ g(a) & g(a) \end{pmatrix} \geq 0_{2 \times 2}.
\]

Pre- and post-multiplying this last matrix by \( (1, -1) \) and \( (1, -1)^T \) respectively, we have \( g(a + \epsilon) - g(\epsilon) \geq g(\epsilon) \), and by the monotonicity of \( g \) (applied to \( \epsilon 1_{2 \times 2} \geq 0_{2 \times 2} \)), it follows that \( g \) is non-decreasing on \([0, \rho)\). Now taking the limit as \( \epsilon \to 0^+ \), we have:
\[
0 = g(a^+) - g(a) \geq g(0^+) \geq 0,
\]
where the first equality follows from the continuity of \( g \). Hence \( g \) is right-continuous at \( 0 \).

Next, for the continuity of \( g \) on \((-\rho, 0)\), let
\[
a \in (0, \rho), \quad 0 < \epsilon < \min(a, \rho - a), \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},
\]
26. Appendix D. Preservers of Loewner monotonicity and convexity on kernels.

and deduce from the monotonicity of \( g \):

\[(a + \epsilon)A \geq aA \geq (a - \epsilon)A \geq 0 \implies g[(a + \epsilon)A] \geq g[aA] \geq g[(a - \epsilon)A].\]

The positivity of the difference matrices implies, upon taking determinants:

\[|g(a \pm \epsilon) - g(a)| \geq |g(-a \mp \epsilon) - g(-a)|.\]

Let \( \epsilon \to 0^+ \); then the continuity of \( g \) at \( a \) implies that at \(-a\), as desired. A similar (one-sided) argument shows the left-continuity of \( g \) at 0, via the step \( g(\epsilon) - g(0) \geq |g(-\epsilon) - g(0)| \).

Next, we come to the second assertion. If \( g \) is convex on \( \mathbb{P}_2(I) \), then restricting to the matrices \( a_{12 \times 2} \) for \( a \in [0, \rho) \), it follows that \( g \) is convex on \([0, \rho)\). Hence \( g_+ \) exists on \((0, \rho)\) by Lemma \textbf{26.3}. Now suppose \( 0 < s < t < \rho \) and \( 0 < \epsilon < \rho - t \). Then by the convexity of \( g \),

\[
\begin{pmatrix} t + \epsilon & t \\ t & t - \epsilon \end{pmatrix} \geq \begin{pmatrix} s + \epsilon & s \\ s & s - \epsilon \end{pmatrix} \geq 0 \implies \begin{pmatrix} g(\lambda t + s + \epsilon) & g(\lambda t + (1 - \lambda)s) \\ g(\lambda t + (1 - \lambda)s) & g(\lambda t + (1 - \lambda)s) \end{pmatrix} \\
\leq \lambda \begin{pmatrix} g(t + \epsilon) & g(t) \\ g(t) & g(t) \end{pmatrix} + (1 - \lambda) \begin{pmatrix} g(s + \epsilon) & g(s) \\ g(s) & g(s) \end{pmatrix},
\]

for all \( \lambda \in [0, 1] \). Write this inequality in the following form:

\[
\begin{pmatrix} \alpha & \beta \\ \beta & \beta \end{pmatrix} \geq 0_{2 \times 2}. \]

As above, pre- and post-multiplying this last matrix by \((1, -1)\) and \((1, -1)^T\) respectively yields:

\[g(\lambda t + (1 - \lambda)s + \epsilon) - g(\lambda t + (1 - \lambda)s) \leq \lambda(g(t + \epsilon) - g(t)) + (1 - \lambda)(g(s + \epsilon) - g(s)).\]

Divide by \( \epsilon \) and let \( \epsilon \to 0^+ \); this shows \( g_+ \) is convex, whence continuous by Lemma \textbf{26.3} on \((0, \rho)\).

Next, denote by \( g_0, g_1 \) the even and odd parts of \( g \), respectively:

\[
g_0(t) := \frac{1}{2}(g(t) + g(-t)), \quad g_1(t) := \frac{1}{2}(g(t) - g(-t)).
\]

We claim that \( g_0, g_1 \) are convex on \([0, \rho)\). Indeed, by the convexity of \( g \) we deduce for \( 0 \leq s \leq t < \rho \) and \( \lambda \in [0, 1] \):

\[
\begin{pmatrix} t & -t \\ -t & t \end{pmatrix} \geq \begin{pmatrix} s & -s \\ -s & s \end{pmatrix} \geq 0 \implies \begin{pmatrix} g(c_\lambda) & g(-c_\lambda) \\ g(-c_\lambda) & g(c_\lambda) \end{pmatrix} \leq \lambda \begin{pmatrix} g(t) & g(-t) \\ g(-t) & g(t) \end{pmatrix} + (1 - \lambda) \begin{pmatrix} g(s) & g(-s) \\ g(-s) & g(s) \end{pmatrix},
\]

where \( c_\lambda = \lambda t + (1 - \lambda)s \). Pre- and post-multiplying this last inequality by \((1, \pm 1)\) and \((1, \pm 1)^T\) respectively, yields:

\[g(\lambda t + (1 - \lambda)s) \pm g(-\lambda t + (1 - \lambda)s) \leq \lambda(g(t) \pm g(-t)) + (1 - \lambda)(g(s) \pm g(-s)).\]

This yields: \( g_0, g_1 \) are convex on \([0, \rho)\). Next, note that if \( 0 \leq s < t < \rho \), and \( 0 < \epsilon \leq \min(t - s, \rho - t) \), then

\[
g(s + \epsilon) - g(s) - g(t - \epsilon) \leq g(t) - g(t - \epsilon),
\]

by Lemma \textbf{26.3}(1). Taking \( \epsilon \to 0^+ \) shows that \( g_+(s) \leq g_+(t) \) if \( 0 \leq s < t < \rho \) and \( g \) is convex. Similarly, \( g_-(t) \leq g_-(t) \); therefore,

\[
g_+(s) \leq g_+(t) = (g_0)'_+(t) + (g_1)'_+(t) \leq (g_0)'_+(t) + (g_1)'_+(t) = g_+(t).
\]

Since \( g_+ \) is continuous, letting \( s \to t^- \) shows \( (g_j)'_+(t) = (g_j)'_+(t) \) for \( j = 0, 1 \). Thus \( g_j \) is differentiable on \((0, \rho)\). Since \( g_0 \) is even and \( g_1 \) is odd, they are also differentiable on \((-\rho, 0)\). Hence \( g_0, g_1, g \) are differentiable on \( I \setminus \{0\} \). Finally, let \( I' := (-2\rho/3, 2\rho/3) \) and
define $h(x) := g(x + \rho/3)$. It is easy to check that $h$ is convex on $P_2(I')$, so it is differentiable at $-\rho/3$ by the above analysis, whence $g$ is differentiable at 0 as desired. \qed

With these preliminary results in hand, we now complete the remaining proof:

**Proof of Theorem 26.1** We begin by showing the first assertion. First suppose $f$ is differentiable on $I$ and $f'$ is monotone on the rank $\le k$ matrices in $P_n(I)$. Now suppose $A \ge B \ge 0_{n \times n}$ are matrices of rank $\le k$. Now follow the proof of Proposition 15.9(3) to show that $f[-]$ is Loewner convex on rank $\le k$ matrices in $P_n(I)$. Here we use that since $A \ge B \ge 0$, we have the chain of Loewner inequalities

$$A \ge \lambda A + (1 - \lambda)B \ge \lambda \frac{A + B}{2} + (1 - \lambda)B \ge B,$$

and hence the ranks of all matrices here are at most $\text{rk}(A) \le k$.

The converse is shown in two steps; in fact we will also prove that $f$ is continuously differentiable on $I$. The first step is to show the result for $n = k = 2$. Note by Proposition 26.4 that $f$ is differentiable on $I$. Now say $A \ge B \ge 0_{2 \times 2}$ with $A \neq B$ in $P_2(I)$. Writing $A = \begin{pmatrix} a_1 & a \\ a & a_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b \\ b & b_2 \end{pmatrix}$, we have $a_j \ge b_j \ge 0$ for $j = 1, 2$ and $(a - b)^2 \le (a_1 - b_1)(a_2 - b_2)$. Define $\delta \in [0, a_1 - b_1]$ and the matrix $C_{2 \times 2}$ via:

$$(a - b)^2 = (a_1 - b_1 - \delta)(a_2 - b_2), \quad C := \begin{pmatrix} b_1 + \delta & b \\ b & b_2 \end{pmatrix} \in P_2(I).$$

Clearly, $A \ge C \ge B$, all matrices are in $P_2(I)$, and $A - C, C - B$ have rank at most one. Thus we may assume without loss of generality that $A - B$ has rank one; write $A - B = \begin{pmatrix} a & \sqrt{ab} \\ \sqrt{ab} & b \end{pmatrix} \in P_2$. First if $ab = 0$, then $f'[A] - f'[B]$ is essentially a scalar on the main diagonal. Now since $f$ is convex on $[0, \rho)$ by considering $a1_{2 \times 2}$ for $a \in [0, \rho)$, we have $f'$ is non-decreasing on $(0, \rho)$, whence $f'[A] \ge f'[B]$.

The other case is if $c \neq 0$, whence $a, b > 0$. In this case $A > B \ge 0$ and $A - B$ is rank-one with no zero entries. Now follow the proof of Proposition 15.9(3) to infer $f'[A] \ge f'[B]$. Together, both cases show that $f'$ is monotone on $P_2(I)$, whence $f'$ is continuous on $I$ by Proposition 26.4(1).

This shows the result for $n = k = 2$. Now suppose $n > 2$. First, $f$ is convex on $P_2(I)$, whence $f'$ is monotone on $P_2(I)$ and hence continuous on $I$ by the previous case. Second, to show that $f'$ is monotone as asserted, suppose $A \ge B \ge 0_{n \times n}$ are matrices in $P_n(I)$ of rank $\le k$. Now claim that there is a chain of Loewner matrix inequalities

$$A = A_n \ge A_{n-1} \ge \cdots \ge A_0 = B,$$

satisfying: (1) $A_j \in P_n(I)$ for all $0 \le j \le n$, and (2) $A_j - A_{j-1}$ has rank at most one for each $1 \le j \le n$. Note that such a chain of inequalities would already imply the reverse inclusions for the corresponding null spaces, whence each $A_j$ has rank at most $k$.

To show the claim, spectrally decompose $A - B = UDU^T$, where $U$ is orthogonal and $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_j \ge 0$, and write

$$A_j := B + U \text{diag}(\lambda_1, \ldots, \lambda_j, 0, \ldots, 0)U^T, \quad 0 \le j \le n.$$

Note that $A_j \le A$, whence the same applies to each of their corresponding (non-negative) diagonal entries. Thus $0 \le (A_j)_{ll} \le a_{ll}$ for $1 \le l \le n$. Thus the diagonal entries of each $A_j$ lie in $I = (-\rho, \rho)$, whence so do the off-diagonal entries. This shows the claim.
Thus to show $f'(A) = f'(A_n) \geq f'(A_0) = f'(B)$, it suffices to assume, as in the previous case of $n = k = 2$, that $A - B$ has rank one. First if $A - B$ has no zero entries, then $f'[A] \geq f'[B]$ by Proposition 15.9(3). Otherwise, suppose $A - B = uu^T$, with $u \in \mathbb{R}^n$ a nonzero vector having zero entries. Without loss of generality, write $u = \begin{pmatrix} v \\ 0 \end{pmatrix}$, with $v \in \mathbb{R}^l$ having no zero entries for some $1 \leq l \leq n - 1$. Accordingly, write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, and similarly for $B$; it follows that $A_{ij} = B_{ij}$ for all $(i, j) \neq (1, 1)$, and $A_{11} = B_{11} + vv^T$. Now since $f$ is Loewner convex on $[B, A]$, it is so on $[B_{11}, A_{11}]$, where all matrices are positive semidefinite and also have rank $\leq k$. Moreover, $f'$ exists and is continuous on $I$ from above. Again by Proposition 15.9(3), it follows that $f'$ is Loewner monotone on $[B_{11}, A_{11}]$ (if $k = 1$ then this assertion is true by one-variable calculus). But then,

$$f'[A] - f'[B] = \begin{pmatrix} f'[A_{11}] - f'[B_{11}] \\ 0 \\ 0 \end{pmatrix} \geq 0.$$  

This proves the first assertion; we turn to the second. First suppose $A \succeq B \succeq 0_{n \times n}$, and $f'$ is Loewner positive on $\mathbb{P}_n(I)$. Then follow the proof of Proposition 15.9(2) to infer $f[A] \geq f[B]$. Conversely, we prove the result under a stronger hypothesis: namely, $f$ is differentiable. Now the proof of Proposition 15.9(2) again applies: given $A \in \mathbb{P}_n(I)$, we have $A + \epsilon 1_{n \times n} \in \mathbb{P}_n(I)$ for some $\epsilon > 0$. By monotonicity, it follows that

$$\frac{1}{\epsilon}(f[A + \epsilon 1_{n \times n}] - f[A]) \in \mathbb{P}_n.$$  

Taking $\epsilon \to 0^+$ proves $f'[A] \in \mathbb{P}_n$, as desired. \qed

We now move to kernels.

**Definition 26.5.** Suppose $X$ is a non-empty set, $I \subset \mathbb{R}$ a domain, and $\mathcal{V}$ is a set of (real symmetric) positive semidefinite kernels on $X \times X$, with values in $I$.

1. The **Loewner order** on kernels on $X \times X$ is: $K \succeq L$ for $K, L$ kernels on $X \times X$, if $K - L$ is a positive semidefinite kernel. (Note, if $X$ is finite, this specializes to the usual Loewner ordering on real $|X| \times |X|$ matrices.)

2. A function $F : I \to \mathbb{R}$ is **Loewner monotone** on $\mathcal{V}$ if $F \circ K \succeq F \circ L$ whenever $K \succeq L \succeq 0$ are kernels in $\mathcal{V}$.

3. A function $F : I \to \mathbb{R}$ is **Loewner convex** on $\mathcal{V}$ if whenever $K \succeq L \succeq 0$ are kernels in $\mathcal{V}$, we have

$$\lambda F \circ K + (1 - \lambda) F \circ L \succeq F \circ (\lambda K + (1 - \lambda) L), \quad \forall \lambda \in [0, 1].$$

The above results for matrices immediately yield the results for kernels:

**Theorem 26.6.** Suppose $0 < \rho < \infty$, $I = (-\rho, \rho)$, and $F : I \to \mathbb{R}$, and $X$ is an infinite set. The composition map $F \circ -$ is Loewner monotone (respectively, Loewner convex) on positive kernels on $X \times X$, if and only if $F$ satisfies the respective equivalent conditions on matrices of all sizes, in Theorem 26.3.

**Proof.** First, if $F$ is Loewner monotone or convex on kernels on $X \times X$, then by restricting the defining inequalities to kernels on $Y \times Y$ (padded by zeros) for finite sets $Y \subset X$, it follows that $F$ is respectively Loewner monotone or convex on $\mathbb{P}_n(I)$ for all $n \geq 1$.

To show the converse, suppose first that $F(y) = \sum_{k=0}^{\infty} c_k y^k$ on $I$, with $c_1, c_2, \cdots \geq 0$. To show that $F \circ -$ is Loewner monotone on kernels on $X \times X$, it suffices to do so on every ‘principal submatrix’ of such kernels – i.e., for every finite indexing subset of $X$. But this is indeed true for $F$, by Theorem 26.2. A similar proof holds for Loewner convex maps. \qed
Lemma 16.1 is taken from the well-known monograph [225] by Pólya and Szegő. Theorems 16.2, 16.3, 16.4, and 16.5 classifying the dimension-free entrywise positivity preservers on various domains, are due to Schoenberg [251], Rudin [241], Vasudeva [278], and Herz [128], respectively. Rudin’s and Vasudeva’s results were shown by Belton–Guillot–Khare–Putinar in [20] using significantly smaller test sets than all positive semidefinite matrices of all sizes; proving these results is the main focus of this part. These results turn out to be further useful in Part 4, in fully classifying the total positivity preservers on bi-infinite domains.

Loewner’s theorem on operator/matrix monotone functions is from [186]; see also Donoghue’s book [75], as well as the recent monograph by Simon [266] that contains a dozen different proofs.

The results in distance geometry are but a sampling from the numerous works of Schoenberg. Theorem 16.8 relating Euclidean embedding of a metric and the conditional negativity of the corresponding squared-distance matrix, is from [248], following then-recent works by Menger [200] and Fréchet [95]. Schoenberg’s Theorem 16.15 (resp. Proposition 16.17), characterizing Hilbert space (resp. the Hilbert sphere) in terms of positive definiteness of the Gaussian family (resp. the cosine), is from [250] (resp. from [248]). See also Schoenberg’s paper [248] and another with von Neumann [280] (and its related work [170] by Kolmogorov). For the works of Bochner in this context, we restrict ourselves to mentioning [41, 42]. Theorems 16.18 and 16.22 by Schoenberg on positive definite functions on spheres are from [251]. In this vein, see [59] and [60] by Christensen–Ressel, for characterizations of the positive definite functions on unit spheres in infinite-dimensional real and complex Hilbert spaces, respectively. The real case is precisely Theorem 16.22. Also see the survey [272] of positive definite functions by James Drewry Stewart (who is perhaps somewhat better known for his series of calculus textbooks).

While Schoenberg’s motivations in arriving at his theorem lay in metric geometry, as described above, Rudin’s motivations were from Fourier analysis. More precisely, Rudin was studying functions operating on spaces of Fourier transforms of $L^1$ functions on groups $G$, or of measures on $G$. Here, $G$ is a locally compact abelian group equipped with its Haar measure; Rudin worked with the torus $G = \mathbb{T}$, while Kahane and Katznelson worked with its dual group $\mathbb{Z}$. These authors together with Nelson proved a remarkable result in a converse direction to Wiener–Levy theory [124], in Acta Math. 1959. That same year, Rudin showed Schoenberg’s theorem without the continuity hypothesis, i.e. Theorem 16.3. For more details on this part, on the metric geometric motivations of Schoenberg, and other topics, the reader is referred to the detailed recent twin surveys of Belton–Guillot–Khare–Putinar [21, 22].

The Horn–Loewner Theorem 17.1 (in a special case) originally appeared in Horn’s paper [138], where he attributes it to Loewner. The theorem has since been extended by the author (jointly) in various ways; see e.g. [20, 116]; a common, overarching generalization of these and other variants has been achieved in [160]. Horn–Loewner’s determinant calculation in Proposition 17.4 was also extended to Proposition 17.8 by Khare [160]. Mollifiers were introduced by Friedrichs [96], following the famous paper of Sobolev [270], and their basic properties can be found in standard textbooks in analysis, as can Cauchy’s mean value theorem for divided differences. The remainder of the proof of the stronger Horn–Loewner theorem is from [138], and the Boas–Widder theorem 18.10(2) is from [40].

Bernstein’s theorem 19.3 is from his well-known memoir on absolutely monotone functions [32]; see also Widder’s textbook [295]. Boas’s Theorem 19.9 on the analyticity of
smooth functions with ‘strictly sign regular’ derivatives is from [39]. Hamburger’s Theorem [19.14] is a folklore result, found in standard reference books – see e.g. [6, 245, 263]. The remainder of Section 19 is from Belton–Guillot–Khare–Putinar [20]. Section 20 is taken from the same paper, with the exception of Theorem 20.11 which is new, as are ‘Proofs 1 and 2’ of the existence of a positivity certificate / limiting s.o.s. representation for \((1 \pm t)(1 - t^2)^n\). The former proof cites a result by Berg–Christensen–Ressel [28], and the latter, direct proof is new.

Sections 21 and 22 are again from [20], save for the standard Identity Theorem 21.4 and Proposition 21.2 on the closure of real analytic functions under composition; these can be found in e.g. [174]. The complex analysis basics, including Montel and Morera’s theorems, can be found in standard textbooks; we cite [65]. The multivariate Schoenberg–Rudin theorem was proved by FitzGerald, Michelli, and Pinkus in [90], and subsequently, under significantly weaker hypotheses in [20].

Appendix A on the Boas–Widder theorem is from [40] except for the initial observations. Boas–Widder mention Popoviciu [226] had proved the same result previously, using unequally-spaced difference operators. The very last ‘calculus’ Proposition 23.12 can be found in standard textbooks. Appendix B, classifying the dimension-free preservers of positivity when not acting on diagonal blocks, is from the recent work of Vishwakarma [279], with the exception of the textbook Proposition 24.4 and Theorem 24.1 by Guillot–Rajaratnam [118].

Theorems 26.1 and 26.2 in Appendix D, understanding and relating the Loewner positive, monotone, and convex maps, were originally proved without rank constraints on the test sets, by Hiai in [129]. Lemma 26.3 is partly taken from [129] and partly from Rockafellar’s book – see [238, Theorems 24.1 and 25.3]. The remainder of Section 26 (i.e. Appendix D), as well as Section 25 (i.e. Appendix C), are taken from Belton–Guillot–Khare–Putinar [23].
Part 4:
Pólya frequency functions and sequences
Part 4: Pólya frequency functions and sequences

27. Totally non-negative functions of finite order.

In this part and the next, we approach the preserver problem in the above settings through a more classical viewpoint: that of spaces of kernels and their endomorphisms. The material in this part of the text is drawn primarily from the work of Schoenberg and his coauthors (as well as its account in Karlin’s monograph), and a few recent preprints, all from 2020.

Before proceeding further, we briefly describe this part of the text and the next. We are interested in characterizing the preservers of totally non-negative or totally positive kernels, on $X \times Y$ for arbitrary totally ordered sets $X, Y$. The case of $X, Y$ finite was studied in Part 2, and the only such functions – up to rescaling – turn out to be powers if $|X|, |Y| \geq 2$.

Next, if exactly one of the sets $X, Y$ is finite, then a workaround can be achieved by using a generalization of Whitney’s density theorem, we show this in Section 38. The difficulty lies in the remaining case: classifying the total positivity preservers for kernels on $X \times Y$ when both $X, Y$ are infinite. In this case, the families of kernels we have encountered so far, do not suffice to yield a complete classification.

Thus, we begin by relaxing our goal, to classifying such preservers for structured kernels, on specific domains. Our first goal is to study the inner transforms of $TN/TP$ Toeplitz kernels on $\mathbb{R} \times \mathbb{R}$. This is not an arbitrary choice: indeed, such kernels have long been studied in the analysis literature, under the name of Pólya frequency functions, and so it is natural to study this test set – as well as the related Pólya frequency sequences – and to understand the endomorphisms of these classes. This understanding is achieved in the next part of the text.

The class of Pólya frequency functions is fundamental to time-frequency analysis and to interpolation theory, the latter via splines (a subject which begins with dozens of papers by Schoenberg and his coauthors). Pólya frequency functions possess beautiful properties that were established by Schoenberg and others in the twentieth century, and that allow us to exploit tools from harmonic analysis to try and classify their preservers. Looking ahead (and using these tools), we will find that Toeplitz $TN$ kernels turn out to be quite rigid, and the results in the next few sections will help resolve – again in the next part – the original problem, of classifying the $TN/TP$ kernel preservers on arbitrary domains. (See Section 38.)

Thus, a roadmap of this part and the next: We begin by discussing some preliminaries on Pólya frequency (PF) functions and sequences, including the variation diminishing property and its history. This is followed by a selection of results from the landmark paper of Schoenberg in *J. d’Analyse Math.* (1951), which establishes a host of properties of PF functions and surveys the development of the subject until that point. (Karlin’s book also develops the theory comprehensively.) In doing so, we discuss several examples of PF functions and sequences. We also briefly discuss the Laguerre–Pólya class of entire functions, and its connection to both Pólya frequency functions as well as the Pólya–Schur theory of multiplier sequences (and some well-known modern achievements). This part can be read from scratch, requiring only Sections 6 and 12 and Lemma 26.3 as pre-requisites.

In the next part: in Section 34, we will see results of Schoenberg and Karlin (and a converse to the latter) which reveal a ‘critical exponent’ phenomenon in total positivity, akin to Section 9. In Sections 35 and 36, we prove a host of classification results on preservers: of Pólya frequency functions and sequences, one-sided variants, and other structured Toeplitz kernels on various sub-domains of $\mathbb{R}$. In Section 37, we classify the preservers of $TN$ and $TP$ Hankel kernels. Finally, these results all come together in Section 38, along with discretization techniques and set-theoretic embedding arguments, to solve the overarching problem of classifying the preservers of totally positive kernels on all totally ordered domains.
27.1. **TN₂ functions – basic properties.** We begin with notation.

**Definition 27.1.** Given an integer \( p \geq 1 \), we term a function \( f : \mathbb{R} \to \mathbb{R} \) **totally non-negative of order** \( p \), or \( TN_p \), if \( f \) is Lebesgue measurable and the associated Toeplitz kernel

\[
T_f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto f(x - y)
\]

is \( TN_p \) (see Definition 5.10). We will say \( f \) is **totally non-negative**, or \( TN \), if \( f \) is Lebesgue measurable and \( T_f \) is \( TN \). This definition extends to \( T_f : X \times Y \to \mathbb{R} \) (where \( X, Y \subset \mathbb{R} \)) if \( f \) is only defined on the Minkowski difference \( X - Y \subset \mathbb{R} \).

We discuss examples in the next section; first, we explore basic properties of these functions.

**Lemma 27.3.** Suppose \( p \geq 1 \) is an integer, and \( f : \mathbb{R} \to \mathbb{R} \) is a \( TN_p \) function. Then \( cf(ax + b) \) and \( ce^{ax}f(x) \) are also \( TN_p \) functions, for any \( a, b \in \mathbb{R} \) and \( c \geq 0 \).

**Proof.** The first part is left as an exercise to the reader, noting if \( a < 0 \) that (the sign of) a determinant remains unchanged upon reversing all rows as well as columns. For the second part, if \( x, y \in \mathbb{R}^{r-1} \) for \( 1 \leq r \leq p \) (see Definition 25.1), and \( g(x) := ce^{ax}f(x) \), then

\[
T_g[x; y] = \text{diag}(ce^{ax})\sum_{j=1}^{r}T_f[x; y] \text{diag}(e^{-ay})_{k=1}^{r},
\]

and hence \( \det T_g[x; y] \geq 0 \) by the hypotheses. \( \square \)

The following result reveals the nature of \( TN_2 \) functions over symmetric intervals:

**Theorem 27.4.** Suppose an interval \( J \subset \mathbb{R} \) contains the origin and has positive length, and \( f : J - J \to \mathbb{R} \) is Lebesgue measurable. The following are equivalent:

1. There exists an interval \( I \subset J - J \) such that \( f \) is positive on \( I \) and vanishes outside \( I \), and \( \log(f) \) is concave on \( I \).

2. \( f \) is \( TN_2 \), i.e., given \( a < b \) and \( c < d \) in \( J \), the matrix \( \begin{pmatrix} f(a-c) & f(a-d) \\ f(b-c) & f(b-d) \end{pmatrix} \) is \( TN \).

If so, \( f \) is continuous in the interior of \( I \), so discontinuous in \( J - J \) at most at two points.

**Proof.** First if \( f \) is nonzero at most at one point then the result is clear, so assume throughout this proof that \( f \) does not vanish at two or more points.

Suppose (1) holds, and \( a < b \) and \( c < d \) index the rows and columns of a ‘2 × 2 submatrix’ drawn from the kernel \( T_f \) on \( J \times J \). By the hypotheses, all entries of the matrix \( M := \begin{pmatrix} f(a-c) & f(a-d) \\ f(b-c) & f(b-d) \end{pmatrix} \) are non-negative. Also,

\[
a - c, \ b - d \in (a - d, \ b - c).
\]

Now there are several cases. If either \( a - c \) or \( b - d \) lie outside \( I \), then \( M \) has a zero row or zero column, whence it is \( TN_2 \). Otherwise \( a - c, b - d \in I \). If now \( a - d \) or \( b - c \) lie outside \( I \) then \( M \) is a triangular matrix, hence \( TN_2 \). Else we may suppose all entries of \( M \) are positive. Now the above ordering and the concavity of \( \log(f) \) easily imply that \( \det(M) \geq 0 \), proving (2).

Conversely, suppose (2) holds, so that \( f \geq 0 \) on \( J - J \), and suppose \( f(x_0) > 0 \). First fix \( \delta > 0 \) such that either \( [0, \delta) \) or \( (-\delta, 0] \) is contained in \( J \). We now claim that if \( f \) vanishes at \( x_1 > x_0 \) (resp. \( x_2 < x_0 \)), then it vanishes on the intersection of \( J - J \) with \( [x_1, \infty) \) (resp., with \( (-\infty, x_2) \)). We only show this for \( x_1 > x_0 \); note moreover that it suffices to show this for \( y \in (x_1, x_1 + \delta) \cap (J - J) \). If \( (-\delta, 0] \subset J \), consider the \( TN_2 \) submatrix

\[
T_f([x_0, x_1); (x_1 - y, 0)] = \begin{pmatrix} f(x_0 - x_1 + y) & f(x_0) \\ f(y) & f(x_1) \end{pmatrix},
\]
where we note that \( x_0 - x_1 + y \in (x_0, y) \subset J - J \). If instead \([0, \delta) \subset J\), then

\[
T_f[(x_0 - x_1 + y); (0, y - x_1)] = \begin{pmatrix}
  f(x_0 - x_1 + y) & f(x_0) \\
  f(y) & f(x_1)
\end{pmatrix}
\]

yields the same \( TN_2 \) submatrix. In both cases, taking determinants gives: \(-f(x_0)f(y) \geq 0\), and since \( f(x_0) > 0 \), the claim follows. This shows the existence of an interval \( I \subset J - J \) of positive measure, such that \( f > 0 \) on \( I \) and \( f \equiv 0 \) outside \( I \).

We next claim that \(-\log(f)\) is mid-convex on \( I \). Indeed, given entries \( y - \epsilon < y < y + \epsilon \) in \( I \), it suffices to show \( f(y) \geq \sqrt{f(y + \epsilon)f(y - \epsilon)} \). The following argument owes its intuition to the theory of discrete-time Markov chains on a finite state space, but can also be made direct. Begin by defining \( n_0 := 2[\epsilon/\delta] \); thus \( \epsilon/n_0 \in (0, \delta) \). Now set \( z_k := f(y + k\epsilon/n_0) \) for \(-n_0 \leq k \leq n_0\); since all arguments lie in \( I \), \( z_k > 0 \) for all \( k \). There are now two cases: if \((-\delta, 0) \subset J\), then

\[
0 \leq \det T_f[(y - (k + 1)\epsilon/n_0, y - k\epsilon/n_0); (-\epsilon/n_0, 0)] = z_k^2 - z_{k-1}z_{k+1}, \quad \forall -n_0 < k < n_0.
\]

If instead \([0, \delta) \subset J\), use \( 0 \leq \det T_f[(y - k\epsilon/n_0, y - (k - 1)\epsilon/n_0); (0, \epsilon/n_0)]\) for the same values of \( k \). Thus, it follows that \( z_k \geq \sqrt{z_{k-1}z_{k+1}} \) for \(-n_0 < k < n_0\). Now one shows inductively:

\[
z_0 \geq (z_1z_{-1})^{1/2} \geq (z_2^{2/3}z_{-2})^{1/3} \geq \cdots \geq \prod_{j=0}^{n_0} (z_j^{2/j-n_0}) \geq \cdots
\]

Note at each step that the powers of \( z_{\pm k} \) are not touched, while all remaining terms \( z_j^{2/j-n_0} \) are lower-bounded by \( \sqrt{z_{k-1}z_{k+1}}^{1/2} \). Now think of each step as a (positive) integer time \( t > 0 \), and consider the exponents at each step. These give a probability distribution \( \pi_t \) on the set \( S := \{-n_0, \ldots, 0, 1, \ldots, n_0\} \). This is a well-studied model in probability theory: the simple random walk with absorbing barriers \( \pm n_0 \), on the state space \( S \). In particular, this is the Markov chain called the symmetric gambler’s ruin. Specifically, the transition probabilities are \( \pm 1/2 \) to go from a non-absorbing state \(-n_0 < k < n_0\) are \( \pm 1/2 \) to \( k \pm 1 \). Moreover, the probability distribution \( \pi_t \) on \( S \), at each time point \( t \in \mathbb{Z}^\geq 0 \), has equal mass at \( \pm n_0 \), so the same holds as \( t \to \infty \). Now by Markov chain theory, the limiting probability distribution as \( t \to \infty \) exists and has mass only at the absorbing states \( \pm n_0 \). As these masses must be equal, we obtain by translating back:

\[
f(y) = z_0 \geq \sqrt{z_{-n_0}z_{n_0}} = \sqrt{f(y - \epsilon)f(y + \epsilon)}.
\]

In this special case, the argument can be made direct as well; here is a sketch. Let \( c_t := \sum_{j=-n_0-1}^{n_0-1} \pi_t(j) \) for \( t = 1, 2, \ldots\). If \(-n_0 < j < n_0\), then after \( 2n_0 - 1 \) steps, the power \( z_{\pi_t(j)} \) will get replaced by either \( z_{-n_0}^{\pi_t(j)/2^{n_0+}} \) or by \( z_{n_0}^{\pi_t(j)/2^{n_0-}} \). From this it follows via the AM–GM inequality that

\[
c_{t+2n_0-1} \leq c_t(1 - 2^{1-n_0}), \quad \forall t \in \mathbb{Z}^\geq 0.
\]

Choosing \( t = m(2n_0 - 1) \) for \( m = 0, 1, \ldots\), and recalling that \( \pi_t(-n_0) = \pi_t(n_0) \) for all \( t \), we have the desired conclusion as \( m \to \infty \).

Thus \(-\log(f)\) is mid-convex on \( I \). Moreover, \(-\log(f)\) is Lebesgue measurable on \( I \) by assumption, so Theorem 12.4 implies \(-\log(f)\), whence \( f \), is continuous on the interior of \( I \).

(0, 1), \( a, b \in I \), and show that

\[
\log f(\lambda a + (1 - \lambda)b) \geq \lambda \log f(a) + (1 - \lambda) \log f(b).
\]

(27.5)
To show this, approximate \( \lambda \) by a sequence of dyadic rationals \( \lambda_n \in (0,1) \), and note by mid-concavity/mid-convexity that
\[
\log f(\lambda_n a + (1 - \lambda_n)b) \geq \lambda_n \log f(a) + (1 - \lambda_n) \log f(b), \quad \forall n \geq 1.
\]
Since \( \lambda_n, \lambda \in (0,1) \), the arguments on the left are always in the interior of \( I \), where \( f \) is positive and continuous. Thus, the preceding inequality shows (27.5) as \( n \to \infty \). \( \square \)

An immediate consequence of Theorem 27.4 is:

**Corollary 27.6.** If \( 0 \in J \subset \mathbb{R} \) is as in Theorem 27.4, and functions \( f, g : J - J \to \mathbb{R} \) are TN2, then so are \( f \cdot g \) and \( f^\alpha \) for \( \alpha \geq 0 \).

This follows from the fact that \( \log(f) + \log(g), \alpha \log(f) \) are concave if \( \log(f), \log(g) \) are.

We now revert to the ‘classical’ setting of TN2 functions, i.e., \( f : \mathbb{R} \to \mathbb{R} \). These functions decay exponentially at infinity, except for the exponentials themselves:

**Proposition 27.7.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is TN2 (whence measurable), and has unbounded support. Then either \( f(x) = e^{ax+b} \) for suitable scalars \( a, b \in \mathbb{R} \), or there exists \( \gamma \in \mathbb{R} \) such that the TN2 function \( e^{-\gamma x} f(x) \) tends to zero exponentially fast as \( |x| \to \infty \), whence is integrable.

**Proof.** Via Theorem 27.4 let \( I \) denote the largest interval on which \( f \) is positive, and set \( g(x) := \log f(x) \). We show the result for \( I = \mathbb{R} \); the proof is similar (but easier) for other \( I \). First note that \( -g(x) \) is convex on \( I \). Now recall Lemma 26.3: \( g' \) exists on a co-countable, dense subset of \( I \), and \( g'_\pm \) exist and are non-increasing on the interior of \( I \) (whence have only jump discontinuities).

Suppose \( g(x) \) is not linear (i.e. \( f(x) \) is not of the form \( e^{ax+b} \)). Then \( g' \) is not constant on \( I \), so there exist points \( x_1 < x_2 \) in \( I \) such that \( g'(x_1) > g'(x_2) \). It follows that there exist constants \( c_1, c_2 \in \mathbb{R} \) satisfying:
\[
\log f(x) \leq g'(x_j)x + c_j, \quad j = 1, 2.
\]
Choose \( \gamma \in (g'(x_2), g'(x_1)) \). Then,
\[
\log f(x) - \gamma x \leq \begin{cases} (g'(x_1) - \gamma)x + c_1, & x \in I \cap (-\infty, 0), \\ (g'(x_2) - \gamma)x + c_2, & x \in I \cap (0, \infty). \end{cases}
\]
Now exponentiating both sides gives the result. \( \square \)

### 27.2. Classification of TN\(_p\) functions for higher \( p \).

The preceding subsection saw a characterization of TN2 functions, by Schoenberg (1951). It has taken longer to obtain a characterization for TN\(_p\) functions for \( p \geq 3 \). (A 1983 result in Appl. Anal. of Weinberger [290] toward characterizing TN3 functions has a small gap; see Section 33.) We conclude this section by showing such a characterization: to check if a non-negative function is TN\(_p\) (with some decay properties), it suffices to check the signs of all \( p \times p \) minors, but no smaller ones. Remarkably, the first such result was discovered only in 2017, leading to the characterization in 2020. We begin with the 2017 result, for integrable functions:

**Lemma 27.8** (Förster–Kieburg–Kösters, [93]). Suppose \( f : \mathbb{R} \to \mathbb{R} \) is integrable and every \( p \times p \) matrix \( (f(x_j - y_k))_{j,k=1}^p \) has non-negative determinant, for \( x, y \in \mathbb{R}^p \). Then \( f \) is TN\(_p\).

Notice that not every TN\(_p\) (or even TN) function is integrable – for instance, \( e^{ax+b} \), which is a TN function by Lemma 27.3 since \( f(x) \equiv 1 \) is TN. Thus, Lemma 27.8 cannot characterize the TN\(_p\) functions. However, we shall presently see a characterization result along these lines, among other variants of Lemma 27.8. As a first variant, the result can be extended to hold for more general domains \( X, Y \subset \mathbb{R} \), and for functions that merely decay, and at one of \( \pm \infty \):
Proposition 27.9. Fix scalars \( t_*, \rho \in \mathbb{R} \) and a subset \( Y \subset \mathbb{R} \) unbounded above. Suppose \( X \subset \mathbb{R} \) contains \( t_* + y \) for all \( \rho < y \in Y \). Let \( f : X - Y \to [0, \infty) \) be such that \( f(t_*) > 0 \) and
\[
\lim_{y \in Y, \rho < y \to \infty} f(x - y)f(t_* + y - y_0) \to 0, \quad \forall x_0 \in X, \ y_0 \in Y.
\]
If \( \det T_f[x; y] \geq 0 \) for all \( x \in X^{p, \uparrow} \) and \( y \in Y^{p, \uparrow} \), then the Toeplitz kernel \( T_f \) is \( TN_p \).

Lemma 27.8 is the special case \( X = Y = \mathbb{R} \), where \( \rho \in \mathbb{R} \) is arbitrary and \( t_* \in \mathbb{R} \) any value at which \( f : \mathbb{R} \to \mathbb{R} \) is nonzero. (If no such \( t_* \) exists, i.e. \( f \equiv 0 \), the result is immediate.) Thus it applies to detect “Pólya frequency functions of order \( p \)” (i.e., integrable \( TN_p \) functions on \( \mathbb{R} \)). Proposition 27.9 is more general in two ways: first, it also specializes to other domains – for instance, \( X = Y = \mathbb{Z} \), i.e. to detect “Pólya frequency sequences of order \( p \)” that vanish at \( \pm \infty \) (with \( t_* \) an integer). More generally, one can specialize to \( X = Y = G \), for any additive subgroup \( G \subset (\mathbb{R}, +) \). Second, Proposition 27.9 can accommodate non-integrable functions such as \( e^{-|x|} \) (which is seen to be \( TN \) in (30.12) below).

Proof. It suffices to show that if \( \det T_f[x; y] \geq 0 \) for all \( x \in X^{p, \uparrow} \) and \( y \in Y^{p, \uparrow} \), then the same condition holds for all \( x' \in X^{p-1, \uparrow} \) and \( y' \in Y^{p-1, \uparrow} \). Thus, fix such \( x', y' \). We are to show
\[
\psi(x_p, y_p) := \det T_f[(x', x_p); (y', y_p)] \geq 0 \quad \forall x_p > x_{p-1}, y_p > y_{p-1} \implies \det T_f[x'; y'] \geq 0.
\]

To see why, first define the \((p-1) \times (p-1)\) matrix \( A := T_f[x'; y'] \) and \( A(j,k) \) to be the submatrix of \( A \) with the \( j \)th row and \( k \)th column removed. Note that the following maximum does not depend on \( x_p, y_p \):
\[
L := \max_{1 \leq j,k \leq p-1} |\det A(j,k)| \geq 0.
\]

Next, given \( m \geq 1 \), define \( t_m \in Y \) such that \( t_m > t_0 := \max\{x_{p-1} - t_*, y_{p-1} - \rho\} \) and
\[
f(x_j - t_m)f(t_* + t_m - y_k) < 1/m, \quad \forall 0 < j, k < p.
\]

We now turn to the proof. Expand the determinant \( \psi(x_p, y_p) \) along the last column, and apart from the cofactor for \((p, p)\), expand every other cofactor along the last row. This yields:
\[
\psi(x_p, y_p) = f(x_p - y_p) \det(A) + \sum_{j,k=1}^{p-1}(-1)^{j+k}f(x_j - y_p)f(x_p - y_k) \det A(j,k).
\]

Given \( t_* \) as above, and \( m \geq 1 \), let
\[
y_p^{(m)} := t_m \in Y, \quad x_p^{(m)} := t_* + t_m \in X.
\]

Then \( x_p^{(m)} > x_{p-1} \) and \( y_p^{(m)} > y_{p-1} \), and \( \psi(x_p^{(m)}, y_p^{(m)}) \geq 0 \) by the hypotheses, so from above,
\[
f(t_*) \det(A) \geq \psi(x_p^{(m)}, y_p^{(m)}) - L \sum_{j,k=1}^{p-1} f(x_j - y_p^{(m)})f(x_p^{(m)} - y_k) \geq -L\frac{(p-1)^2}{m}.
\]

As this holds for all \( m \geq 1 \), we have \( \det(A) = \det T_f[x'; y'] \geq 0 \), as desired. \( \square \)

This result specializes to provide a characterization of \( TN_p \) kernels for arbitrary \( p \geq 2 \):

Corollary 27.10. Given \( f : \mathbb{R} \to [0, \infty) \) and an integer \( p \geq 2 \), the following are equivalent:
\begin{enumerate}
\item Either \( f(x) = e^{ax+b} \) for \( a, b \in \mathbb{R} \), or: (a) \( f \) is Lebesgue measurable; (b) for all \( x_0, y_0 \in \mathbb{R} \), \( f(x_0 - y)f(y - y_0) \to 0 \) as \( y \to \infty \); and (c) \( \det T_f[x; y] \geq 0 \) \( \forall x, y \in \mathbb{R}^{p, \uparrow} \).
\item The function \( f : \mathbb{R} \to \mathbb{R} \) is \( TN_p \).
\end{enumerate}
This result improves on Lemma 27.8 in that not every $TN_p$ function is integrable or an exponential $e^{ax+b}$. For example, we will see in (30.12) below such a function, when $\alpha \beta < 0$.

**Proof.** The result is obvious for $f \equiv 0$ on $\mathbb{R}$, so we assume henceforth that this is not the case. That (1)(a)–(c) $\implies$ (2) is now immediate from Proposition 27.9 specialized to $X = Y = \mathbb{R}$ and arbitrary $\rho \in \mathbb{R}$. If instead $f(x) = e^{ax+b}$, then by Lemma 27.3 it is $TN$ because the constant function $f \equiv 1$ is obviously $TN$.

Conversely, suppose (2) holds and $f(x)$ is not of the form $e^{ax+b}$. Now (1)(a) and (1)(c) are immediate; if the support of $f$ is either bounded above or below, then (1)(b) is also immediate. Otherwise $f$ is $TN_2$ with support $\mathbb{R}$, whence by the proof of Proposition 27.7 there exist $\beta, \gamma \in \mathbb{R}$ and $\delta > 0$ such that $e^{-\gamma x} f(x) < e^{\beta - \delta|x|}$. Now a straightforward computation shows (1)(b).

We conclude this section with another variant for arbitrary positive-valued kernels on $X \times Y$, for arbitrary $X, Y \subset \mathbb{R}$:

**Proposition 27.11.** Fix nonempty subsets $X, Y \subset \mathbb{R}$, and suppose a kernel $K : X \times Y \to (0, \infty)$ satisfies one of the following four decay conditions:

1. Every $p \times p$ minor of $K$ is non-negative.
2. $K$ is $TN_p$ on $X \times Y$.

**Proof.** Clearly (2) $\implies$ (1); conversely, it suffices to show by induction that $\det K[\mathbf{x}'; \mathbf{y}'] \geq 0$ for $\mathbf{x}' \in X^{p-1,\uparrow}$ and $\mathbf{y}' \in Y^{p-1,\uparrow}$. We show this under the last of the four assumptions; the other cases are similar to the following proof and to the proof of Proposition 27.9. Thus, let

$$
\mathbf{x}' = (x_2, \ldots, x_p), \quad \mathbf{y}' = (y_2, \ldots, y_p)
$$

and $y_1 < y_2, y_1 \in Y$ be fixed. Let $A := K[\mathbf{x}'; \mathbf{y}']$; thus the maximum

$$
L := \max_{1 \leq j, k \leq p} |\det A^{(j,k)}| \geq 0.
$$

Next, construct a sequence $x_1^{(m)} \in X$, $m \geq 1$ such that $x_1^{(m)} < x_2$ and

$$
K(x_1^{(m)}, y_k) < 1/m, \quad \forall k = 2, \ldots, p.
$$

Now compute as in the proof of Proposition 27.9

$$
K(x_1^{(m)}, y_1) \det(A) \geq \det K[(x_1^{(m)}, \mathbf{x}'); (y_1, \mathbf{y}')] - L \sum_{j, k=2}^p K(x_j, y_1) K(x_1^{(m)}, y_k)
$$

$$
\geq \det K[(x_1^{(m)}, \mathbf{x}'); (y_1, \mathbf{y}')] - \frac{L(p-1)}{m} \sum_{j=2}^p K(x_j, y_1),
$$

where we expand the determinant along the first row and column. Since the first term on the right is non-negative, and $K > 0$ on $X \times Y$, taking $m \to \infty$ shows $\det(A) \geq 0$, as desired. \qed
28. Pólya frequency functions. The variation diminishing and sign non-reversal properties.

In this section, we introduce and study the distinguished class of Pólya frequency functions, as well as their variation diminishing property (including a look at its early history, from Descartes to Motzkin). Both of these studies were carried out by Schoenberg in his landmark 1951 paper [255]. The latter property will require first studying the variation diminishing property of TN matrices – carried out even earlier, by Schoenberg in 1930 in Math. Z.

We begin by introducing the titular class of functions in this part of the text.

**Definition 28.1.** A function \( \Lambda : \mathbb{R} \to \mathbb{R} \) is said to be a **Pólya frequency (PF) function** if \( \Lambda \) is an integrable TN function that does not vanish at least at two points.

Some historical remarks on terminology follow. The term *frequency function* traditionally meant being integrable, or (up to normalization) a density function. Pólya frequency functions were introduced by Schoenberg in his landmark 1951 paper in *J. d’Analyse Math.*

His definition means that the class of ‘Dirac’ TN functions \( \Lambda(x) = 1_{x \in c} \) (see Example 28.6 below) are not Pólya frequency functions. In fact, Schoenberg also studied a wider class of TN functions in *loc. cit.*, again excluding the Dirac functions. Specifically, he worked with (what he called) **totally positive functions** – which are (measurable) TN functions that do not vanish at least at two points.

**Remark 28.2.** We also refrain here from discussing either Schoenberg’s motivations or the prior results by Laguerre, Pólya, Schur, and Hamburger that led Schoenberg to developing the theory of PF functions. This discussion will take place in Sections 31 and 32.

We begin by specializing the results in Section 27 to Pólya frequency (PF) functions:

**Proposition 28.3.**

1. The results in Section 27.1 hold for all PF functions.
2. The class of PF functions is closed under the change of variables \( x \mapsto ax + b \) for \( a \neq 0 \), and under convolution.
3. If \( f \) is a TN function that is non-vanishing at least at two points, and \( f \) is not of the form \( e^{ax+b} \) for \( a, b \in \mathbb{R} \), then there exists \( \gamma \in \mathbb{R} \) such that \( e^{-\gamma x} f(x) \) is a PF function. In other words, there is a strict trichotomy for (measurable) TN functions \( f : \mathbb{R} \to \mathbb{R} \):
   
   - (a) \( f(x) \) is monotone, or equivalently, an exponential \( e^{ax+b} \) for some \( a, b \in \mathbb{R} \).
   - (b) \( f \) is supported (and positive) at a single point in the line.
   - (c) Up to an exponential factor \( e^{\gamma x} \) with \( \gamma \in \mathbb{R} \), the function \( f \) is integrable, whence
     a Pólya frequency function – in fact, this latter decays exponentially as \( |x| \to \infty \).

The final trichotomy holds more generally for all TN functions, by Proposition 27.7.

We next provide examples. The results in the preceding section studied TN functions for \( p \geq 2 \). We now discuss several important examples of all of these: in fact, of Pólya frequency functions (so, TN functions for all \( p \geq 1 \)). The first is the Gaussian family.

**Example 28.4.** For all \( \sigma > 0 \), the Gaussian function \( G_\sigma(x) := e^{-\sigma x^2} \) is a PF function, as shown in Lemma 6.8. For future use, we record its Laplace transform (discussed later):

\[
B(G_\sigma)(s) = e^{s^2/4\sigma} \sqrt{\pi/\sigma}, \quad s \in \mathbb{C}.
\]

**Example 28.5.** The kernel \( f(x) = e^{-e^x} \) is totally non-negative. Indeed, the ‘submatrix’

\[
T_f[x; y] = (\exp(\alpha_j \beta_k))_{j,k=1}^n, \quad n \geq 1, \ x, y \in \mathbb{R}^n,
\]
is a generalized Vandermonde matrix, where both $\alpha_j = e^{x_j}, \beta_k = -e^{-y_k}$ form increasing sequences. While $e^{-e^x}$ $\not\to$ 0 as $x \to -\infty$, by Lemma 27.3 $e^{x-e^x}$ is integrable, so a PF function.

The next example is an integrable $TN$ function that is pathological in nature, so not a PF function:

**Example 28.6.** The Dirac function $f(x) = 1_{x=c}$ can be easily verified to be $TN$, for $c \in \mathbb{R}$.

28.1. **Variation diminishing property for $TP$ and $TN$ matrices.** A widely-used property exhibited by $TP$ and $TN$ matrices is the variation diminishing property (this phrase – or ‘variationsvermindernd’ in German – was coined by Pólya; see Section 28.3 for more on its history). We now prove this property; it will be useful in Section 29 in explaining the real-rootedness of generating functions of finite Pólya frequency sequences. To proceed, we require some notation.

**Definition 28.7.** Given a vector $x \in \mathbb{R}^m$, let $S^-(x)$ denote the number of changes in sign, after removing all zero entries in $x$. Next, assign to each zero coordinate of $x$ a value of $\pm 1$, and let $S^+(x)$ denote the largest possible number of sign changes in the resulting sequence (running over all assignments of $\pm 1$). We also set $S^-(0) := 0$ and $S^+(0) := m$, for $0 \in \mathbb{R}^m$.

For instance, $S^-(1,0,0,-1) = 1$ and $S^+(1,0,0,-1) = 3$. In general, it is easy to see that $S^+(x), S^-(x)$ are the largest and smallest number of sign changes possible, when one keeps the nonzero coordinates of $x$ unchanged and modifies the zero coordinates.

We begin by characterizing $TP$ matrices in terms of the variation diminishing property.

**Theorem 28.8.** Suppose $m, n \geq 1$ are integers, and $A \in \mathbb{R}^{n \times m}$. The following are equivalent:

1. $A$ is totally positive.
2. For all $x \in \mathbb{R}^m \setminus \{0\}$, $S^+(Ax) \leq S^-(x)$. Moreover, if equality holds and $Ax \neq 0$, then the first (last) component of $Ax$ has the same sign as the first (last) nonzero component of $x$. (If either component in $Ax$ is zero, we replace it by the sign of the changed component in computing $S^+$.)

**Proof.** For this proof, define $d_p := (1, -1, \ldots, (-1)^{p-1})^T \in \mathbb{R}^p$, for any integer $p \geq 1$.

We first show (2) $\implies$ (1). Let $e_j \in \mathbb{R}^m$, $1 \leq j \leq m$ denote the standard basis; then $S^+(Ae_j) \leq S^-(e_j) = 0$, so $Ae_j$ cannot have zero entries – in fact all entries are positive by (2). Thus $A$ has positive entries.

We next claim that no minor of $A$ is zero. Indeed, let $2 \leq p \leq \min(m, n)$, and let $J \in [m]^{p\uparrow}$ and $K \in [n]^{p\uparrow}$ denote subsets of indices. We first claim $A_{J \times K}$ is non-singular. Indeed, otherwise there exists $0 \neq z \in \mathbb{R}^p$ such that $A_{J \times K} z = 0$. Extend $z$ to $\tilde{z} \in \mathbb{R}^m$ by placing the coordinates of $z$ in positions $K$, and padding by zeros elsewhere; then

$$S^+(A\tilde{z}) \leq S^-(\tilde{z}) \leq p - 1.$$ 

But $A\tilde{z}$ has $p$ coordinates zero, corresponding to $J$, so $S^+(\tilde{z}) \geq p$, a contradiction. This shows the claim.

Finally, we claim by induction on $1 \leq p \leq \min(m, n)$ that $A$ is $TP_p$, with the base case of $p = 1$ shown above. For the induction step, suppose $J \in [m]^{p\uparrow}$ and $K \in [n]^{p\uparrow}$ as above. Since $A_{J \times K}$ is non-singular, there exists a vector $z \in \mathbb{R}^p$ such that $A_{J \times K} z = d_p$ (defined at the start of the proof). By Cramer’s rule,

$$z_1 = \frac{1}{\det A_{J \times K}} \sum_{r=1}^p \det A_{J \setminus \{j_1\} \times K \setminus \{k_r\}},$$
28. Pólya frequency functions.

The variation diminishing and sign non-reversal properties.

and the numerator on the right is positive by assumption. Thus, to show the same for det $A_{J \times K}$, it suffices to show $z_1 > 0$. To see why, first extend $z$ to $\tilde{z} \in \mathbb{R}^m$ by embedding in positions $K$ and padding by zeros as above. Then $A\tilde{z}$ has $(-1)^{r-1}$ in position $j_r$, so $S^- (A\tilde{z}) \geq p - 1$. It follows that

$$p - 1 \leq S^- (A\tilde{z}) \leq S^+ (A\tilde{z}) \leq S^- (\tilde{z}) \leq p - 1.$$  

It follows that $S^+ (A\tilde{z}) = S^- (\tilde{z}) = p - 1$. From this we conclude: (i) the coordinates of $z$ alternate in sign; (ii) all coordinates of $A\tilde{z}$ in positions $1, 2, \ldots, j_1$ have the same sign; and (iii) this sign agrees with that of $z_1$ by (2). But in (ii) above, the sign is positive, since $(A\tilde{z})_{j_1} = 1$. It follows that $z_1 > 0$, and the proof of one implication is complete.

To show the converse, we first begin with a $TP$ matrix $A$ and draw some conclusions. Let $0 \neq x \in \mathbb{R}^m$ with $S^- (x) = p \leq m - 1$. Thus there exists $s \in [m]^{p, \uparrow}$ such that

$$(x_1, \ldots, x_1), (x_{s_1+1}, \ldots, x_{s_2}), \ldots, (x_{s_p+1}, \ldots, x_{m})$$

are components of $x^T$, with all nonzero coordinates in the $t$th component having the same sign, which we choose up to ‘orientation’ to be $(-1)^{t-1}$. Here we also have that not all coordinates in each sub-tuple $(x_{s_{t-1}+1}, \ldots, x_{s_t})$ are zero. Moreover, we set $s_0 := 0$ and $s_{p+1} = m$ for convenience.

Denote the columns of $A$ by $c_1, \ldots, c_m \in \mathbb{R}^n$, let

$$y_t := \sum_{i=s_{t-1}+1}^{s_t} |x_i| c_i \in \mathbb{R}^n, \quad t = 1, \ldots, p + 1,$$

and let $Y_{n \times (p+1)} := [y_1 | \cdots | y_{p+1}]$.

The first claim is that $Y$ is $TP$. Indeed, given $1 \leq r \leq \min(n, p+1)$ and subsets $J \in [n]^{r, \uparrow}$, $I \in [p+1]^{r, \uparrow}$, standard properties of determinants imply

$$\det Y_{J \times I} = \sum_{k_1 = s_{t_1-1}+1}^{s_{t_1}} \cdots \sum_{k_r = s_{t_r-1}+1}^{s_{t_r}} \prod_{t=1}^{r} |x_{k_t}| \cdots |x_{k_r}| \det A_{J \times K},$$

where $K = \{k_1, \ldots, k_r\}$ and we have $\det A_{J \times K} > 0$ for all $J, K$. Since there exist suitable indices $k_t \in [s_{t_t} - 1, s_{t_t}]$ such that $\prod_{t=1}^{r} |x_{k_t}| > 0$, it follows that $\det Y_{J \times I} > 0$ for all $I, J$ as desired. Hence $Y$ is $TP$.

With this analysis in hand, we now show (1) $\implies$ (2). The first claim is that $S^+ (Ax) \leq S^- (x)$; we consider two cases:

- Suppose $n \leq p + 1$. If $Ax \neq 0$, then $S^+ (Ax) \leq n - 1 \leq p = S^- (x)$. Otherwise $Ax = 0$. If $n = p + 1$ then $Y$ would be non-singular. But since $0 = Ax = Yd_{p+1}$, where $d_{p+1}$ was defined at the start of this proof, this would imply $d_{p+1} = 0$, a contradiction. Hence if $Ax = 0$ then $n \leq p$, whence $S^+ (Ax) = n \leq p = S^- (x)$.

- Otherwise, $n > p + 1$. Define $w := Ax = Yd_{p+1}$, and assume for contradiction that $S^+ (Ax) \geq p + 1 > p = S^- (x)$. Thus there exist indices $1 \leq j_1 < \cdots < j_{p+2} \leq n$ and a sign $\varepsilon = \pm 1$ such that $\varepsilon w_{j_t} (-1)^{t-1} \geq 0$ for $t \in [1, p + 2]$. Moreover, not all $w_{j_t}$ are zero, given the rank of $Y$. Now consider the matrix

$$M_{(p+2) \times (p+2)} = \begin{bmatrix} w_{j_t} & |Y_{J \times [p+1]}|, \end{bmatrix} \quad \text{where} \quad J = \{j_1, \ldots, j_{p+2}\}.$$
This is singular because the first column is an alternating sum of the rest. Expanding along the first column,

\[ 0 = \det M = \sum_{t=1}^{p+2} (-1)^{t-1} w_{jt} \det Y_{(J \setminus \{j\}) \times [p+1]} . \]

But all determinants in this sum are positive, all terms \((-1)^{t-1} w_{jt}\) have the same sign \(\varepsilon\), and not all \(w_{jt}\) are zero. This produces the desired contradiction.

Thus if \(A\) is \(TP\), then \(S^+(Ax) \leq S^-(x)\), and it remains to show the remainder of the assertion (2). Using the notation in the preceding analysis, it remains to show that if \(S^+(Ax) = S^-(x) = p\) with \(Ax \neq 0\), and if moreover \(\varepsilon w_{jt} (-1)^{t-1} \geq 0\) for \(t \in [1, p+1] - (1)\) as opposed to \([1, p + 2]\) in the second sub-case above – then \(\varepsilon = 1\) (given our original choice of ‘orientation’ above).

To show this, we use that \(Y\) is totally positive, so the submatrix \(Y_{J \times [p+1]}\) is non-singular, where \(J = \{j_1, \ldots, j_{p+1}\}\). In particular, \(Y_{J \times [p+1]} d_{p+1} = w_J\). Using Cramer’s rule to solve this system for the first coordinate of \(d_{p+1}\),

\[ 1 = d_1 = \frac{\det [w_J \{Y_{J \times [p+1]}{J\{1\}}\}]}{\det Y_{J \times [p+1]}} . \]

Multiplying both sides by \(\varepsilon \det Y_{J \times [p+1]}\), we have

\[ \varepsilon \det Y_{J \times [p+1]} = \sum_{t=1}^{p+1} \varepsilon (-1)^{t-1} w_{jt} \det Y_{(J \setminus \{j\}) \times ([p+1] \setminus \{1\})} . \]

Since each summand on the right is non-negative, and \(Y\) is \(TP\), it follows that \(\varepsilon = 1\). \(\square\)

A similar result holds for totally non-negative matrices.

**Theorem 28.10.** Suppose \(m, n \geq 1\) are integers, and \(A \in \mathbb{R}^{n \times m}\). The following are equivalent:

1. \(A\) is TN.
2. For all \(x \in \mathbb{R}^m \setminus \{0\}\), \(S^-(Ax) \leq S^-(x)\). Moreover, if equality holds and \(Ax \neq 0\), then the first (last) nonzero component of \(Ax\) has the same sign as the first (last) nonzero component of \(x\).

This proof requires a preliminary lemma on sign changes of limits of vectors:

**Lemma 28.11.** Given \(x = (x_1, \ldots, x_m) \in \mathbb{R}^m\), define \(\bar{x} := (x_1, -x_2, x_3, \ldots, (-1)^{m-1} x_m)\). Then

\[ S^+(x) + S^- (\bar{x}) = m - 1 . \]

Also, if \(x_k \to x \neq 0\) in \(\mathbb{R}^m\), then

\[ \lim \inf_{k \to \infty} S^- (x_k) \geq S^- (x), \quad \lim \sup_{k \to \infty} S^+ (x_k) \leq S^+ (x) . \]

**Proof.** We begin with the first identity. Note that both \(S^\pm\) are (a) invariant under the automorphism \(y \mapsto -y\) of \(\mathbb{R}^m\), and (b) additive over substrings intersecting at one nonzero number (where this common ‘endpoint’ is considered in both strings). Thus it suffices to prove the result for vectors \(x \in \mathbb{R}^m \setminus \{0\}\) with \(m \geq 2\) and \(x_2 = \cdots = x_{m-1} = 0\). This is a straightforward verification.

For the second part, by considering \(k\) large enough, we may assume that if the \(j\)th coordinate of \(x\) is nonzero, then it has the same sign as the \(j\)th coordinate of every \(x_k\) – in fact,
we may take these coordinates to all have the same value, since this does not affect $S^\pm(x_k)$.

Now by the observation in the paragraph following Definition [28.7]

$$S^-(x) \leq S^-(x_k) \leq S^+(x_k) \leq S^+(x),$$

and the result follows. \□

We can now prove the above characterization of $TN$ matrices.

**Proof of Theorem 28.10.** The proof of (1) $\implies$ (2) uses the two preceding results and Whitney’s density Theorem 6.7. Since $A$ is $TN$, there exists a sequence $A_k$ of $TP$ matrices converging entrywise to $A$. Now use Theorem 28.8 and Lemma 28.11 to compute:

$$S^-(Ax) \leq \liminf_{k \to \infty} S^-(A_kx) \leq \liminf_{k \to \infty} S^+(A_kx) \leq \liminf_{k \to \infty} S^-(x) = S^-(x).$$

Next, if equality holds and $Ax \neq 0$, then for all $k$ large enough, we have

$$p := S^-(Ax) \leq S^-(A_kx) \leq S^+(A_kx) \leq S^-(x) = p,$$

by Theorem 28.8 and Lemma 28.11. In particular, these are all equalities, whence $S^-(A_kx) = S^+(A_kx)$ for $k \gg 0$. This implies (for large $k$) that the sign changes/patterns in $A_kx$ have no dependence on zero entries. At the same time, both vectors $x$ and $A_kx$ admit ‘partitions’ of the form (28.9) with alternating signs, with precisely $p$ sign changes. Now Theorem 28.8 implies that these signs are in perfect agreement, for all large $k$. Hence the same holds for the sign patterns of $x$ and $Ax$.

That (2) $\implies$ (1) is shown similarly to the proof of Theorem 28.8 by induction on the size $1 \leq p \leq \min(m,n)$ of the minors of $A$. Again observing that $S^+(Ae_j) \leq S^-(e_j) = 0$, and the entries of $e_j$, whence of $Ae_j$, must be non-negative, we have that $A$ has all entries non-negative. For the induction step, suppose $J \in [m]^{p^\uparrow}$ and $K \in [n]^{p^\uparrow}$. If $A_{J \times K}$ is singular then we are done; else suppose henceforth that $\det A_{J \times K} \neq 0$. Hence there exists $z \in \mathbb{R}^p$ such that $A_{J \times K}z = d_p = (1, -1, \ldots, (-1)^{p-1})^T$. By Cramer’s rule,

$$z_1 = \frac{1}{\det A_{J \times K}} \sum_{r=1}^{p} \det A_{(J \setminus \{j_i\}) \times (K \setminus \{k_r\})},$$

and the numerator on the right is non-negative by assumption. Thus, to show $\det A_{J \times K} > 0$, it suffices to show $z_1 > 0$. For this we repeat the corresponding part of the proof of Theorem 28.8 using that $S^-(Ax) = S^-(x) = p - 1$. \□

**28.2. Variation diminishing property for Pólya frequency functions.** The above characterizations of $TN$ and $TP$ matrices have many applications in mathematics and other sciences; we do not expound on these, referring the reader to Karlin’s treatise [154] and numerous follow-up papers in the literature. Here we present continuous analogues of the above results on the variation diminishing property, albeit only in one direction. These are again found in Schoenberg’s 1951 paper in *J. d’Analyse Math.* We begin with the definition of the ‘variation’ that will diminish under the action of a $TN$ kernel.

**Definition 28.12.** Suppose $I \subset \mathbb{R}$ is an interval with positive measure, and a function $f : I \to \mathbb{R}$. The number $S^f_I$ of variations of sign of $f(x)$ on $I$ is defined as follows:

$$S^f_I := \sup\{S^-((f(x_1), \ldots, f(x_p)) : p \geq 1, x \in I^{p^\uparrow}\}.$$
Proposition 28.13 (Schoenberg, [255]). Suppose $\Lambda : \mathbb{R} \to [0, \infty)$ is a Pólya frequency function. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable on all finite intervals in $\mathbb{R}$, and such that the integral

$$g(x) := \int_{\mathbb{R}} \Lambda(x - t) f(t) \, dt$$

converges at all $x \in \mathbb{R}$. Then $S^-_R(g) \leq S^-_R(f)$.

Proof. We will write $S^-(\cdot)$ for $S^-_R(\cdot)$ in what follows. We may assume in the sequel that $S^-(g) > 0$ and $S^-(f) < \infty$. Now if $x \in \mathbb{R}^{m+1}$ satisfies: $g(x_1), g(x_2), \ldots, g(x_{m+1})$ are all nonzero and alternate in sign, then it suffices to show that $S^-(f) \geq m$, for one can now take the supremum over all such tuples $x$ to deduce $S^-(g) \leq S^-(f)$. Note that

$$g(x) = \lim_{a \to -\infty, \ b \to \infty} \int_{a}^{b} \Lambda(x - t) f(t) \, dt,$$

and this convergence is uniform when considered simultaneously at the $m + 1$ coordinates of $x$. Thus, select $-\infty < a < b < \infty$ such that the function

$$g_1(x) := \int_{a}^{b} \Lambda(x - t) f(t) \, dt$$

also alternates in sign at $x_1, \ldots, x_{m+1}$. Approximating this function by Riemann sums over $n > 0$ sub-intervals of $[a, b]$ of equal length, simultaneously at $x_1, \ldots, x_{m+1}$, it is possible to choose $n \gg 0$ (large enough) such that the sequence of Riemann sums

$$z_j^{(n)} := \frac{b - a}{n} \sum_{k=1}^{n} \Lambda \left( x_j - \frac{kb + (n-k)a}{n} \right) f \left( \frac{kb + (n-k)a}{n} \right), \quad j = 1, \ldots, m + 1 \quad (28.14)$$

also alternates in sign. In other words, $S^-(z^{(n)}) = m$, $\forall n \gg 0$.

We now invoke the total non-negativity of the kernel $T_\Lambda$, applied to $x \in \mathbb{R}^{m+1}$ as above, and $y \in \mathbb{R}^n$ given by $y_k = (kb + (n-k)a)/n$ for $k = 1, \ldots, n$. Thus the matrix

$$A_{(m+1) \times n} := T_\Lambda [x; y] = \left( \Lambda \left( x_j - \frac{kb + (n-k)a}{n} \right) \right)_{j \in [m+1], \ k \in [n]}$$

is totally non-negative. Now (28.14) and Theorem 28.10 imply that

$$m = S^-(z^{(n)}) = S^-(Av) \leq S^-(v), \quad \text{where} \quad v = (f(y_k))_{k=1}^n.$$

But then $m \leq S^-(f)$, as desired. \hfill $\Box$

Schoenberg goes on to prove an analogue of this variation diminishing property, when $f$ and hence $g$ (in the preceding result) are polynomials. The diminution is now in the number of real roots.

Proposition 28.15. Suppose $\Lambda$ is a Pólya frequency function, and $f \in \mathbb{R}[x]$ is a polynomial of degree $n$. Then $g(x) := \int_{\mathbb{R}} \Lambda(x - t) f(t) \, dt$ is also a polynomial of degree $n$, with $Z(g) \leq Z(f)$. Here $Z(f)$ denotes the number of real roots of $f$, counted with multiplicity.

Proof. By Proposition 28.3 (3)(c), $\Lambda$ has finite moments

$$\mu_j := \int_{\mathbb{R}} \Lambda(t) t^j \, dt < \infty, \quad j = 0, 1, \ldots \quad (28.16)$$
Proposition 28.19. Suppose \( \Lambda \) is a Pólya frequency function which vanishes on \( (-\infty,0) \).

1. Let \( f : (-\infty,0] \to \mathbb{R} \) be integrable on every finite interval, and such that \( g(x) := \int_{-\infty}^{x} \Lambda(x-t) f(t) \, dt \) converges for all \( x \leq 0 \). Then \( S_{(-\infty,0]}(g) \leq S_{(-\infty,0]}^{-}(f) \).

2. Suppose \( f \) is a real polynomial of degree \( n \), and let \( g(x) := \int_{0}^{\infty} \Lambda(t) f(x-t) \, dt \). Then \( g \) is also a real polynomial of degree \( n \), and \( Z_{-}(g) \leq Z_{-}(f) \), where \( Z_{-}(f) \) denotes the number of non-positive real roots of \( f \), counted with multiplicity.
28.3. Early results on total non-negativity and variation diminution. We conclude this section by discussing some of the origins of the variation diminishing property, and its connection to total non-negativity. First observe by Theorem 4.1 and Remark 4.4 for Hankel moment matrices, that total non-negativity is implicit in the solution of the Stieltjes moment problem – as also in the Routh–Hurwitz criterion for stability; see Theorem 29.20 – both from the 1800s.

Coming to the variation diminishing property: it shows up in the correspondence [86] between Fekete and Pólya (published in 1912) that has been mentioned above in the context of proving the ‘contiguous minor’ test for total positivity of a matrix. We now discuss a result of Fekete from this correspondence, which not only dealt with variation diminution but also gave rise to the notion of Pólya frequency sequences – discussed in the next section.

A common, historical theme underlying this section and the next, as well as Section 31 below (on the Laguerre–Pólya–Schur program), involves understanding real polynomials and their roots. In fact this theme dates back to Descartes, who in his 1637 work [72] proposed his ‘Rule of Signs’ – see e.g. Lemma 5.2. The question of understanding the roots remained popular (and does so to this day; see Section 31). As a notable example, we recall Laguerre’s 1883 paper [176], which deals with this theme, and opens by recalling Descartes’ rule of signs and proving it using Rolle’s theorem (the ‘standard’ proof these days). In [176], Laguerre used the word ‘variations’ to denote the sign changes in the Maclaurin coefficients of a polynomial or power series. Among the many results found in his memoir, we list two:

**Theorem 28.20** (Laguerre, 1883, [176]).

1. Given an interval \([a, b] \subset \mathbb{R}\) and an integrable function \(\Phi : [a, b] \to \mathbb{R}\), the number of zeros of the Fourier–Laplace transform \(\int_a^b e^{-sx} \Phi(x) \, dx\) is at most the number of zeros of the antiderivative \(\int_a^t \Phi(x) \, dx\) for \(t \in [a, b]\).

2. Suppose \(f(x)\) is a polynomial. Then the number of variations (sign changes in the Maclaurin coefficients) of the power series \(e^{sx} f(x)\) is a non-increasing function of \(s \in [0, \infty)\), and is uniformly bounded below by the number of positive roots of \(f\).

In particular, in part (2), the variation for any \(s > 0\) is ‘diminishing’, and always finite, since it is bounded above by the variation at \(s = 0\).

Especially this latter result was pursued by Fekete, who wrote to Pólya to the effect that “Laguerre did not fully justify” Theorem 28.20(2) in his work [176]. To address this, Fekete considered a formal power series with real coefficients \(c_0 + c_1 t + \cdots\), acting by multiplication on the space of such power series. With respect to the standard basis of monomials, this transformation is given by the triangular matrix

\[
T_c := \begin{pmatrix}
  c_0 & 0 & 0 & \cdots \\
  c_1 & c_0 & 0 & \cdots \\
  c_2 & c_1 & c_0 & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Fekete then stated the following result on variation diminution (a term coined by Pólya) – further asserting that it could be used to prove Theorem 28.20(2) (see Section 29.3):

**Proposition 28.21** (Fekete, 1912, [86]). Suppose for an integer \(p \geq 1\) that the matrix \(T_c\) (supported on \(\mathbb{Z} \times \mathbb{Z}\)) is \(TN_p\) – i.e., every finite submatrix is \(TN_p\). Given a vector \(\mathbf{x} = (x_0, x_1, \ldots, x_p, 0, 0, \ldots)^T\) (the coefficients of a polynomial), we have \(S^{-}(T_c \mathbf{x}) \leq S^{-}(\mathbf{x})\).
28. Pólya frequency functions.  
The variation diminishing and sign non-reversal properties.

Note here that $T_c \mathbf{x}$ represents the coefficients of a formal power series, and hence can form a sequence with infinitely many nonzero terms. Nevertheless, every such sequence has fewer sign changes than the finite sequence $\mathbf{x}$.

Thus, we have journeyed from Descartes (1637), to Laguerre (1883), to Fekete (1912), to Schoenberg (1930) and Motzkin (1936) – see Theorem 3.20 from Motzkin’s thesis and the preceding paragraph – to Schoenberg (1951), in studying the origins of variation diminution and subsequent developments. (This omits, with due apologies, the substantial contributions of Gantmacher, Krein, and others; as well as the 1915 paper [221] of Pólya, which proved a different variation-diminishing property of PF functions on polynomials and led Schoenberg to coin the phrase ‘Pólya frequency functions’ and which is briefly discussed in Section 32.) Certainly Fekete’s result and the aforementioned prior developments led Schoenberg and Gantmacher–Krein to develop the theories of total positivity, Pólya frequency functions, and variation diminution.

28.4. The sign non-reversal property. We now discuss another fundamental – and very recent – characterization of total positivity. This is based on the ‘second’ condition present in Theorems 28.8 and 28.10 – the first being the variation diminishing property. We require the following definition.

**Definition 28.22.** Fix integers $n \geq 1$.

1. A square real matrix $A_{n \times n}$ has the **sign non-reversal property** on a set of vectors $S \subset \mathbb{R}^n$, if for all nonzero vectors $x \in S$ there exists a coordinate $j \in [n]$ such that $x_j (Ax)_j > 0$. If the set $S$ is not specified, it is taken to be $\mathbb{R}^n$.

2. We also require the **non-strict sign non-reversal property** for a matrix $A \in \mathbb{R}^{n \times n}$, on a set of vectors $S \subset \mathbb{R}^n$. This says that for all $0 \neq x \in S$ there exists a coordinate $j \in [n]$ such that $x_j \neq 0$ and $x_j (Ax)_j \geq 0$.

3. Define $\mathbb{R}^n_{\text{alt}}$ to be the set of vectors in $\mathbb{R}^n$ whose coordinates are all nonzero and have alternating signs.

4. Finally, define $d_r := (1, -1, \ldots, (-1)^{n-1})^T \in \mathbb{R}^n_{\text{alt}}$.

With these notions defined, in the 2020 work [58], Choudhury, Kannan, and Khare have characterized TP/TN matrices purely in terms of their sign non-reversal property. More strongly:

**Theorem 28.23.** Fix integers $m, n \geq p \geq 1$ and a real matrix $A \in \mathbb{R}^{m \times n}$. The following are equivalent:

1. The matrix $A$ is TP$_p$.
2. Every square submatrix $B$ of $A$ of size $r \in [1, p]$ has the sign non-reversal property.
3. Every contiguous square submatrix $B$ of $A$ of size $r \in [1, p]$ has the sign non-reversal property on $\mathbb{R}^r_{\text{alt}} \subset \mathbb{R}^r$.

As is evident, (2) $\implies$ (3), so we will show (1) $\implies$ (2) and (3) $\implies$ (1). The latter implication can be weakened even further, to require the following two conditions:

- No alternating vector is killed by a contiguous submatrix (this is clearly weaker than Theorem [28.23(3)]);
- The non-strict sign non-reversal property holds for a single alternating vector. (This is weaker than the sign non-reversal property.)

More precisely, we have:

**Theorem 28.24.** The three conditions above are further equivalent to:
(4) For all contiguous square submatrices $B$ of $A$ of size $r \in [1,p]$: (i) $Bx \neq 0$ for all $x \in \mathbb{R}^p_{\text{alt}}$, and (ii) $B$ has the non-strict sign non-reversal property for $z = z^B := \det(B) \text{adj}(B) \mathbf{d}_r$, where $\text{adj}(B)$ denotes the adjugate matrix of $B$.

The reason for writing the condition (4)(ii) in this (a priori weaker) manner, is the corresponding characterization of $TN_p$ matrices. Namely, we will presently show that $A_{m \times n}$ is $TN_p$ if and only if every square submatrix of size $\leq p$ satisfies (4)(ii). See Theorem 28.27.

Proof. We first show that (1) implies the sign non-reversal property on vectors with non-negative coordinates – under the weaker characterization of working only with the positivity of the principal minors of $B$, of size $\leq r$. (Such matrices are called $P$-matrices; see e.g. [76, 88] for this argument.) The result is by induction on $r \geq 1$, with the $r = 1$ case obvious. Suppose the result holds for all $(r - 1) \times (r - 1)$ real matrices with positive principal minors. Now let $B_{r \times r}$ have the same property, with $x, -Bx \in \mathbb{R}^{r}$ having all non-negative coordinates. We need to prove $x = 0$.

By choice (and Cramer’s rule), $B^{-1}$ has positive diagonal entries. Let $b$ denote the first column of $B^{-1}$, and define

$$\theta := \min_{j \in [r], b_j > 0} \frac{x_j}{b_j},$$

noting that the minimum is taken over a non-empty set. Hence it is attained: $0 \leq \theta = x_{j_0}/b_{j_0}$ at some $j_0 \in [r]$. Then $y := x - \theta b$ has non-negative coordinates, by choice of $\theta$, and it has $j_0$th coordinate zero. But we also have

$$-By = -Bx + \theta Bb = -Bx + \theta \mathbf{e}_1,$$

and this has non-negative coordinates again.

We now claim $y = 0$. Indeed, if we obtain $y'$ by deleting the $j_0$th coordinate, and $B'$ by deleting the $j_0$th row and column, then an easy verification yields that $-B'y', y'$ have non-negative coordinates. By the induction hypothesis, $y' = 0$, whence $y = 0$. But then $Bx = \theta \mathbf{e}_1$ has non-negative coordinates. Since so does $-Bx$, we have $Bx = 0$. Since $\det(B) > 0$, we obtain $x = 0$. This completes the proof of (1) $\implies$ (2) by induction – for $x$ with non-negative coordinates.

Now suppose $x \in \mathbb{R}^r$ and $x_j(Bx)_j \leq 0$ for all $j$. Let $J \subset [r]$ index the negative coordinates of $x$, and define the diagonal matrix $D_J$ with $(k, k)$ entry $(-1)^{1(k \in J)}$. If now $x_j$ and $(Bx)_j$ have opposite signs for all $j$ (meaning their product is non-positive), then so do $(\mathbf{e}_j^T D)x = \mathbf{e}_j^T (Dx)$ and $(\mathbf{e}_j^T D)(Bx) = \mathbf{e}_j^T (DBD)(Dx)$. Thus $Dx, (DBD)(Dx)$ have corresponding coordinates of opposite signs. As $DBD$ also has all principal minors positive, the above analysis implies $Dx = 0$, and so $x = 0$. Hence (1) $\implies$ (2) for all vectors $x \in \mathbb{R}^r$.

(Though we do not require it, we also mention quickly why (2) $\implies$ (1). Let $B$ be any square submatrix of $A$; as the set of non-real eigenvalues is closed under conjugation, their product is strictly positive. It thus suffices to show that every real eigenvalue is positive. By the sign non-reversal property, if $Bx = \lambda x$ with $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^r$, and if $x_j \neq 0$, then $x_j(Bx)_j = \lambda x_j^2 > 0$, whence $\lambda > 0$.)

Next, that (2) $\implies$ (3) $\implies$ (4) is immediate. Finally, suppose (4) holds. By the Fekete–Schoenberg lemma 4.9, it suffices to show that all contiguous $r \times r$ minors are positive, for $1 \leq r \leq p$. The proof is by induction on $r \leq p$; the $r = 1$ case directly follows from (4) using $r = 1$ and $B_{1 \times 1} = (a_{jk})$. For the induction step, suppose $B_{r \times r}$ is a contiguous square submatrix of $A$, with $r \leq p$, and all contiguous minors of $B$ of size $\leq r - 1$ are positive. We first claim $B$ is non-singular. Indeed, suppose not and say $Bx = 0$ for some $x \neq 0$. As $Bx \neq 0$
for \( x \in \mathbb{R}_r^\text{alt} \) by (4)(i), one can decompose \( x \in \mathbb{R}^r \setminus \mathbb{R}_r^\text{alt} \) as in (28.9) and the subsequent lines, with \( p \) replaced by some \( q \leq r - 2 \). Let \( d_{q+1} := (1, -1, \ldots, (-1)^q)^T \in \mathbb{R}^{q+1} \). By the argument following (28.9), the matrix \( Y_{r \times (q+1)} \) is \( TP \) and \( Yd_{q+1} = Bx = 0 \). This is impossible since \( r > q + 1 \), so \( B \) is non-singular. Now set \( z := (\det B)(\text{adj } B) d_r \); by Cramer’s rule,

\[
j_z := (-1)^{j-1}(\det B) \sum_{k=1}^r \det B_{[r]\setminus(j) \times [r]\setminus(k)}, \quad \forall j = 1, \ldots, r. \tag{28.25}
\]

Note, since all proper minors of \( B \) are positive by the Fekete–Schoenberg lemma \( 4.9 \) \( z \in \mathbb{R}^r_\text{alt}. \) Hence by (4)(ii), there exists \( j \in [r] \) such that

\[
0 \leq z_j(Bz)_j = (\det B)^2(-1)^{j-1}z_j = (\det B)^3 \sum_{k=1}^r \det B_{[r]\setminus(j) \times [r]\setminus(k)}.
\tag{28.26}
\]

But the sum on the right is positive, so \( \det B > 0 \), and the proof is complete by induction. \( \square \)

The final result here, again by Choudhury–Kannan–Khare (2020) is the characterization of \( TN_p \) matrices through their sign non-reversal:

**Theorem 28.27.** Fix integers \( m, n \geq p \geq 1 \) and a real matrix \( A \in \mathbb{R}^{m \times n} \). The following are equivalent:

1. The matrix \( A \) is \( TN_p \).
2. Every square submatrix \( B \) of \( A \) of size \( r \in [1, p] \) has the non-strict sign non-reversal property on \( \mathbb{R}^r \).
3. Every square submatrix \( B \) of \( A \) of size \( r \in [1, p] \) has the non-strict sign non-reversal property on \( \mathbb{R}^r_\text{alt} \).

As in Theorem 28.23(4) above, this can be further weakened:

**Theorem 28.28.** The preceding three conditions are further equivalent to:

4. Every square submatrix \( B \) of \( A \) of size \( r \in [1, p] \), has the non-strict sign non-reversal property for \( z = z^B := \det(B)\text{adj}(B)d_r \).

**Proof.** First suppose (1) holds. By Whitney’s density theorem \( 6.7 \) there is a sequence \( A^{(l)} \) of \( TP_p \) matrices converging entrywise to \( A \). Now given \( B \), let \( B^{(l)}_{r \times r} \) be the submatrix of \( A^{(l)} \) indexed by the same rows and columns as \( B \). Now fix a vector \( x \neq 0 \in \mathbb{R}^r \), and index by \( J \subset [r] \) the nonzero entries in \( x \). Since \( B^{(l)} \) is \( TP \), by Theorem 28.23 there exists \( j_l \in [r] \) such that \( x_{j_l}B^{(l)}_{r \times r}j_l > 0 \); moreover, \( j_l \in J \forall l \). As \( J \) is finite, there exists \( j_0 \in [r] \) and an increasing subsequence of positive integers \( l_q \) such that \( j_{l_q} = j_0 \) for all \( q \geq 1 \). Now (2) follows:

\[
x_{j_0}(Bx)_{j_0} = \lim_{q \to \infty} x_{j_{l_q}}B^{(l_q)}_{r \times r}j_{l_q} \geq 0, \quad x_{j_0} \neq 0.
\]

(The proof of (2) \( \implies \) (1) is essentially the same as in the preceding proof, with \( \lambda x_j^2 \geq 0 \) now; once again, we do not require it.) That (2) \( \implies \) (3) \( \implies \) (4) is immediate. Finally, assume (4) and claim by induction on \( r \leq p \), that every \( r \times r \) minor of \( A \) is non-negative. The base case is immediate; for the induction step, let \( B_{r \times r} \) be a submatrix of \( A \), and assume that all \( (r-1) \times (r-1) \) minors of \( B \) are non-negative. If \( \det B = 0 \) then we are done; else assume \( B \) is invertible. Setting \( z := (\det B)(\text{adj } B)d_r \), Cramer’s rule again yields (28.25). But since \( B \) is invertible, no column of \( \text{adj } B \) can vanish. Hence at least one summand in (28.25), whence the sum, is strictly positive by the induction hypothesis. This in fact shows that \( z \in \mathbb{R}^r_\text{alt} \). Repeating the calculation in (28.26), we have \( \det B > 0 \), and so are done by induction. \( \square \)
29. Pólya frequency sequences and their generating functions.

In this section, we introduce and study Pólya frequency sequences, and take a look at root-location phenomena for their generating functions. Recall Fekete’s Proposition 28.21 (1912) from the preceding section, on the variation diminishing property of \( TN_p \) triangular Toeplitz matrices that are supported on \( \mathbb{Z} \times \mathbb{Z} \). Such matrices are now known as Pólya frequency sequences:

**Definition 29.1.**

1. A real sequence \( a = (a_n)_{n \in \mathbb{Z}} \) is a Pólya frequency (PF) sequence if the associated Toeplitz kernel \( T_a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \), sending \((x,y) \mapsto a_{x-y}, \) is \( TN \).
2. The PF sequence \( a \) is said to be one-sided if there exist \( n_+ \in \mathbb{Z} \) such that \( a_n = 0 \) either for all \( n < n_- \) or for all \( n > n_+ \) (or both).
3. More general is the notion of a \( p \) times (i.e., multiply) positive sequence, or a \( TN_p \) sequence \( a \), which corresponds to the matrix \( T_a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \) being \( TN_p \). (This too has a one-sided version, as above.)

For instance, one can specialize the results of Section 27.2 to \( X = Y = \mathbb{Z} \), to obtain characterizations of \( TN_p \) sequences or PF sequences.

**29.1. Examples.** We begin by studying examples of PF sequences, and of the subclass of one-sided PF sequences (\( TN \) or \( TN_p \)). Clearly, every constant (non-negative) sequence \( a_n \equiv c \) for \( c \geq 0 \) is a PF sequence. Our first source of non-constant PF sequences comes from Pólya frequency functions:

**Lemma 29.2.** If \( \Lambda : \mathbb{R} \to \mathbb{R} \) is a PF function, or even a \( TN \) function, then \( (\Lambda(an+b))_{n \in \mathbb{Z}} \) is a PF sequence for \( a, b \in \mathbb{R} \). If \( a \neq 0 \) and \( \Lambda \) is \( TP \), then so is \( (\Lambda(an+b))_{n \in \mathbb{Z}} \).

**Proof.** If \( a = 0 \) then the result is immediate. Now suppose \( a \neq 0 \), and \( x, y \in \mathbb{Z}^p \) for some \( p \geq 1 \). Then the \( p \times p \) matrix

\[
(\Lambda(a(x_j - y_k) + b))^p_{j,k=1} = \begin{cases} 
T_{\Lambda}[ax + b1; ay], & \text{if } a > 0, \\
T_{\Lambda}[a|y + b1; a|x]^T, & \text{if } a < 0,
\end{cases}
\]

and both choices have non-negative determinant. The final assertion is showed similarly. \( \square \)

Another elementary result is the closure of the set of \( TN_p \) sequences. The proof is easy.

**Lemma 29.3.** Suppose \( p \geq 1 \) and \( a^{(k)} = (a_n^{(k)})_{n \in \mathbb{Z}} \) is a \( TN_p \) sequence for all \( k \geq 1 \). If \( a^{(k)} \to a \) pointwise as \( k \to \infty \), then \( a \) is also \( TN_p \).

The above lemmas allow one to ‘draw’ PF sequences from PF functions, along any infinite arithmetic progression. Here are two examples:

**Example 29.4.** For a real number \( q > 0 \), the sequences \((q^n)_{n \in \mathbb{Z}}\) and \((q^n1_{n \geq 0})_{n \in \mathbb{Z}}\) are both PF sequences. The former is because \((q^{x_j-y_k})_{j,k=1}^{p} \) is a rank-one matrix with positive entries for any choice of integers \( x_j, y_k \), so \( TN \). If \( q \in (0, 1) \), then the latter kernel is drawn from \( \lambda_1(x) = e^{-x}1_{x \geq 0} \) (which is shown in Example ?? below to be a PF function), at the arguments \( x = -n \log(q) \). If \( q > 1 \) then it is drawn from \( f_c(x) = e^{cx}\lambda_1(x) \) at the arguments \( n \log(q) \) and with \( c = 2 \). (By Proposition 28.3.1, \( f_c \) is now a PF function as well.) Finally, if \( q = 1 \) then it is drawn from the Heaviside kernel

\[
H_1(x) = 1_{x \geq 0} = \lim_{c \to 1} f_c(x)
\]

at any infinite arithmetic progression. (Alternately, for all \( q \) one can use the analysis above 29.10, later in this section.)
Example 29.5. For a real number \( q \in (0, 1) \), the sequence \( (q^n)_n \) is a \( TP \) PF sequence. Indeed, this sequence is drawn from the Gaussian PF function \( G_\sigma \) for \( \sigma = -\log(q) > 0 \) (see Example 28.4).

29.2. Generating functions and representation theorem for one-sided PF sequences.

Given a real sequence \( a = (a_n)_{n \geq 0} \) of finite or infinite length, it is natural to encode it by the corresponding generating function

\[
\Psi_a(x) = \sum_{n \geq 0} a_n x^n.
\]

When the sequence \( (\ldots, 0, 0, a_0, a_1, a_2, \ldots) \) is a \( TN_p \) sequence, and only finitely many terms \( a_n \) are nonzero, one can deduce results about the locations of the roots of the corresponding generating polynomial \( \Psi_a(x) \) – and in turn, use this information to classify all finite \( TN \) sequences. We carry out this analysis in the present subsection and the next; it will be useful in the next part of this text, in classifying the preservers of total positivity for PF sequences. The present subsection ends by revealing the most general form of the generating function of one-sided PF sequences.

Proposition 29.6. Suppose \((a_n)_{n \in \mathbb{Z}}\) is \( TN_2 \), shifted and normalized such that \( a_0 = 1 \).

1. If \( a_k = 0 \) for some \( k > 0 \) (respectively \( k < 0 \)), then \( a_l = 0 \) for all \( l > k \) (respectively \( l < k \)).
2. If \( a_n = 0 \) for all \( n < 0 \), and the sequence \( a \) is \( TN_p \) for some \( p \geq 1 \), then its generating function \( \Psi_a \) has a nonzero radius of convergence.
3. If \( a_n = b_n = 0 \) for all \( n < 0 \), and \( a, b \) are \( TN_p \) sequences, then so is the sequence of Maclaurin coefficients of \( \Psi_a(x) \Psi_b(x) \).
4. If \( a_n = 0 \) for all \( n < 0 \), and \( a \) is \( TN \), then so is the sequence of Maclaurin coefficients of \( 1/\Psi_a(-x) \).

Note that the final assertion here does not go through if we assume \( a \) to be merely \( TN_p \) and not \( TN \); see e.g. [230]. Also note the similarity of the first assertion to Theorem 27.4.

Proof. The first part is easy to check: given integers \( k, m \) with \( a_k = 0 \), note that

\[
\det T_a[(k, k + m); (0, k)] = \det \begin{pmatrix} a_k & a_0 \\ a_{k+m} & a_m \end{pmatrix} = -a_{k+m}.
\]

Now if \( 0 < k, m \) then work with this matrix; if \( 0 > k, m \) then reverse its rows and columns. Either way, the non-negativity of the determinant implies \( a_{k+m} = 0 \) for all \( m > 0 \) (or all \( m < 0 \)), as desired.

For the second part, if only finitely many terms are nonzero then the result is obvious; otherwise by the first part, \( a_n > 0 \) for all \( n \geq 0 \). Now given \( n \geq 1 \), we have

\[
0 \leq \det T_a[(n, n + 1); (0, 1)] = \det \begin{pmatrix} a_n & a_{n-1} \\ a_{n+1} & a_n \end{pmatrix} = a_n^2 - a_{n+1}a_{n-1}.
\]

From this it follows that \( a_{n+1}/a_n \) is non-increasing in \( n \). Let \( \beta_1 \geq 0 \) be the infimum/limit of this sequence of ratios; then the power series \( \Psi_a(x) = \sum_{n \geq 0} a_n x^n \) has radius of convergence \( 1/\beta_1 \), by basic calculus.

To show the third part, first observe that the \( \mathbb{Z} \times \mathbb{Z} \) Toeplitz matrices \( T_a, T_b \) are both \( TN_p \) by definition. Moreover, their product is a well-defined, lower-triangular Toeplitz matrix by inspection, and corresponds precisely to ‘convolving’ the two sequences. But this ‘convolution’ also corresponds to multiplying the two generating functions. Thus, it suffices to show that \( T_a T_b \) is \( TN_p \). This follows by the Cauchy–Binet formula (see Theorem 5.5).
Finally, for the fourth part, the reciprocal of \( \Psi_\mathbf{a}(x) \) has a positive radius of convergence around 0, by basic calculus. Develop the reciprocal around 0, on the interval of convergence:

\[
\Psi_\mathbf{a}(x) = \frac{1}{\sum_{n \geq 0} b_n x^n}.
\]

Denote the tuple of coefficients by \( \mathbf{b} \). By the arguments used to prove the previous part, this yields the lower triangular Toeplitz matrix \( T_\mathbf{b} \) (indexed by \( \mathbb{Z} \times \mathbb{Z} \)) such that \( T_\mathbf{a} T_\mathbf{b} = T_\mathbf{b} T_\mathbf{a} = \text{Id}_{\mathbb{Z} \times \mathbb{Z}} \). Since \( T_\mathbf{a} \) is TN, we claim that so is the matrix with \((j,k)\) entry \((-1)^{j+k} b_{j-k} = (-1)^{-k} b_{j-k}\) – which is precisely \( T_\mathbf{c} \) for \( c_n = (-1)^n b_n \). But this would correspond to the desired generating function:

\[
\sum_{n \geq 0} (-1)^n b_n x^n = \frac{1}{\Psi_\mathbf{a}(-x)}.
\]

It thus remains to prove the above claim. This requires the well-known Jacobi complementary minor formula \(^{145}\): given integers \( 0 < p < n \), an invertible \( n \times n \) matrix \( \mathbf{A} \) (over a commutative ring), and equi-sized subsets \( J,K \in [n]^\oplus \),

\[
\det \mathbf{A} \cdot \det(A^{-1})_{K^c \times J^c} = (-1)^{j_1+k_1+\cdots+j_p+k_p} \det A_{J \setminus K}.
\]

where \( J^c : = [n] \setminus J \), and similarly for \( K^c \).

We first quickly sketch the proof of this result (via an argument found on the internet): by pre- and post- multiplying \( \mathbf{A} \) by suitable permutation matrices, one can reduce to the case of \( J = K = [p] \) – in which case the sign on the right is +1. Now let \( \mathbf{A}_j \) denote the \( j \)-th column of \( \mathbf{A} \), and \( A^{-1}_{j} \) of \( A^{-1} \) respectively. Writing \( e_j \) for the standard basis of \( \mathbb{R}^n \), we have:

\[
\mathbf{A} \begin{bmatrix} e_1 | \cdots | e_p | A^{-1}_{p+1} | \cdots | A^{-1}_n \end{bmatrix} = [A_1 | \cdots | A_p | e_{p+1} | \cdots | e_n].
\]

This can be rewritten as:

\[
\begin{pmatrix}
A_{J \times K} & A_{J \times K^c} \\
A_{J^c \times K} & A_{J^c \times K^c}
\end{pmatrix}
\begin{pmatrix}
\text{Id}_p & A^{-1}_{K \times J^c} \\
0 & A^{-1}_{K^c \times J^c}
\end{pmatrix}
= \begin{pmatrix}
A_{J \times K} & 0 \\
A_{J^c \times K} & \text{Id}_{n-p}
\end{pmatrix}.
\]

Taking the determinant of both sides proves Jacobi’s result for \( J = K = [p] \).

To conclude the proof of the fourth part, we now use Jacobi’s identity to prove the aforementioned claim (in bold). Let \( \mathbf{M} \) be a square submatrix of \( \mathbf{T}_\mathbf{c} \), with \( c_n = (-1)^n b_n \) as above. There exists a suitably large principal submatrix \( \mathbf{B}' \) of \( \mathbf{T}_\mathbf{c} \), indexed by contiguous rows and columns, of which \( \mathbf{M} \) is a \( p \times p \) submatrix for some \( p \). Let \( \mathbf{A} \) be the corresponding ‘contiguous’ principal submatrix of \( \mathbf{T}_\mathbf{a} \). Multiplying every row and column of \( \mathbf{B}' \) indexed by even numbers by \(-1\), we obtain a matrix \( \mathbf{B} \) such that \( \mathbf{A} \mathbf{B} = \text{Id} \). Finally, let \( \mathbf{N} \) be the ‘complementary’ submatrix of \( \mathbf{A} \), indexed by the rows and columns not indexing \( \mathbf{M} \) in \( \mathbf{B} \) (or in \( \mathbf{B}' \)). Applying Jacobi’s identity and carefully keeping track of signs shows that \( \det(\mathbf{M}) \geq 0 \), as desired. □

We now proceed toward the form of the generating series \( \Psi_\mathbf{a} \) for an arbitrary one-sided Pólya frequency sequence. First consider the case when \( \mathbf{a} \) contains only finitely many nonzero terms, which by Proposition \(^{29.4}\) must be ‘consecutive’. In other words,

\[
\mathbf{a} = (\ldots, 0, 0, a_0, \ldots, a_m, 0, 0, \ldots), \quad m \geq 0, \quad a_0, \ldots, a_m > 0.
\]

Lemma 29.8. With \( \mathbf{a} \) as above, if \( m = 0 \) or \( m = 1 \) (with \( a_0, a_m \) arbitrary positive scalars), then \( \mathbf{a} \) is a PF sequence. In particular, if \( \Psi_\mathbf{a}(x) \) is a polynomial with all roots in \((-\infty, 0)\), then \( \mathbf{a} \) is a PF sequence.
\textbf{Proof.} First let $m = 0, 1$, and suppose a square submatrix $M$ of $T_0$ has neither a zero initial/final row nor column. For $m = 0$, the only possible alternative is that $M$ is a diagonal matrix, with determinant a power of $a_0 > 0$. For $m = 1$, there are two alternatives: all entries along the main diagonal of $M$ are $a_0$ (whence the sub-diagonal has all entries $a_1$, and all other entries in $M$ are zero), or they are all $a_1$ (whence the super-diagonal has all entries $a_0$, and all other entries in $M$ are zero). Both cases yield $\det M > 0$.

Now suppose $\Psi_a(x)$ is a polynomial with all roots in $(-\infty, 0)$. Writing this as $a_m(x + \beta_1)(x + \beta_2)\cdots(x + \beta_m)$ (with $-\beta_j$ the roots of $\Psi_a$), we note by the preceding paragraph that $x + \beta_j = \Psi_{(\ldots, 0, \beta_j, 1, 0, \ldots)}$ is the generating polynomial of a PF sequence, as is $a_m = \Psi_{(\ldots, 0, a_m, 0, \ldots)}$. Therefore, so is their product $= \Psi_a(x)$, by Proposition 29.6(3).

\textbf{Example 29.9.} Lemmas 29.8 and 29.3 show that $a_n = 1_{n \geq 0} \delta^n/n!$ (where $\delta \geq 0$) forms a PF sequence, since its generating power series is $\Psi_a(x) = e^{\delta x} = \lim_{n \to \infty} (1 + \delta x/n)^n$.

We now make some deductions from the above results in this section. Choose scalars $\delta \geq 0$, as well as $\alpha_j, \beta_j \geq 0$ for integers $j \geq 1$ such that $\sum_j (\alpha_j + \beta_j) < \infty$. Then $\prod_{j=1}^n (1 + \alpha_j x)$ generates a PF sequence for all $n \geq 1$, by Lemma 29.8 and Proposition 29.6. Take the limit as $n \to \infty$; since $|1 + \alpha_j x| < e^{\alpha_j|x|}$ and $\sum_j \alpha_j < \infty$, the limit, given by

$$\prod_{j=1}^\infty (1 + \alpha_j x)$$

is an entire function, which generates a PF sequence by Lemma 29.3. Similarly, $1/(1 - \beta_j x)$ generates an infinite one-sided PF sequence for each $j \geq 1$, by Lemma 29.8 and Proposition 29.6. It follows as above that $1/\prod_{j=1}^\infty (1 - \beta_j x)$ also generates a PF sequence. Finally, $(1 + \delta x/n)^n$ generates a PF sequence for all $n \geq 1$, so once again its limit, $e^{\delta x}$ also generates a PF sequence. Now multiply the above three limiting entire functions, to obtain via Proposition 29.6(3) that

$$e^{\delta x} \prod_{j=1}^\infty \frac{1 + \alpha_j x}{1 - \beta_j x}, \quad \text{where } \alpha_j, \beta_j, \delta \geq 0, \quad \sum_{j=1}^\infty (\alpha_j + \beta_j) < \infty \quad (29.10)$$

is the generating function of a one-sided Pólya frequency sequence $a$, with $a_0 = 1$.

Remarkably, this form turns out to encompass all one-sided Pólya frequency sequences. This is a deep result, shown in a series of papers [2, 3, 77] by Aissen–Schoenberg–Whitney and Edrei – separately and together – and is stated here without proof.

\textbf{Theorem 29.11} (Aissen–Edrei–Schoenberg–Whitney, 1951–52). A function $\sum_{n=0}^\infty a_n x^n$ with $a_0 = 1$, is the generating function $\Psi_a(x)$ for a one-sided Pólya frequency sequence $a = (a_0, a_1, \ldots)$ if and only if (29.10) holds.

For proofs, the reader can either look into the aforementioned papers, or follow the treatment in Karlin [154], Chapter 8; one also finds there a representation theorem for the generating function of a two-sided PF sequence. The proof involves using ideas of Hadamard from his 1892 dissertation, as well as Nevanlinna’s refinement of Picard’s theorem.

\textbf{29.3. Applications: results of Fekete and Laguerre; dual Jacobi–Trudi identity.} We now complete the proof of a result by Laguerre in the preceding section, as promised there. The first step is to prove a strengthening of a weaker version of Fekete’s proposition:

\textbf{Proposition 29.12.} Suppose $T : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ is $TN_p$, for some integer $p \geq 2$. Given a real vector $x = (x_0, \ldots, x_{p-1}, 0, 0, \ldots)^T$, we have $S^-(T x) \leq S^-(x)$. 
29. Pólya frequency sequences and their generating functions.

Proof of Proposition 28.21. Suppose $S^\alpha(T\mathbf{x}) > S^\alpha(\mathbf{x})$ for some finite vector $\mathbf{x} = (x_0, x_1, \ldots)^T$ with $x_p = x_{p+1} = \cdots = 0$. Then there exists an initial segment $\mathbf{y}$ of $T\mathbf{x}$ of length $m + 1$ such that $S^\alpha(\mathbf{y}) > S^\alpha(\mathbf{x})$. Let $\mathbf{x}' := (x_0, \ldots, x_{p-1})^T$, and let the submatrix $T' := (t_{jk})_{j \leq m, k < p}$. Then $T'$ is $TN$ by assumption, and $\mathbf{y} = T'\mathbf{x}'$, so by the variation diminishing property (Theorem 28.10), $S^\alpha(\mathbf{y}) = S^\alpha(T'\mathbf{x}') \leq S^\alpha(\mathbf{x}') = S^\alpha(\mathbf{x})$, a contradiction. □

Using this result, we now prove Laguerre’s result as mentioned by Fekete in [86] – but using finite matrices instead of power series:

Proof of Theorem 28.20(2). For $s > 0$, let $T_s$ denote the infinite Toeplitz matrix with $(j, k)$ entry $s^{j-k}/(j-k)!$ if $j \geq k$, and 0 otherwise. This corresponds to the sequence with generating function $e^{sx}$, so by Example (29.9) the matrix $T_s$ is $TN$; moreover, the map $e^{sx} \mapsto T_s$, $s \geq 0$ is a homomorphism of monoids under multiplication.

We now turn to the proof. Applying Proposition 29.12, $S^\alpha(T_s\mathbf{x}) \leq S^\alpha(\mathbf{x})$ for any finite vector $\mathbf{x}$ (padded by infinitely many zeros). In particular, the integers $\{S^\alpha(T_s\mathbf{x}) : s > 0\}$ are uniformly bounded above by $S^\alpha(\mathbf{x}) < \infty$. Moreover, $S^\alpha(T_s\mathbf{x})$ and $S^\alpha(\mathbf{x})$ are precisely the number of variations in the functions $e^{sx}\Psi(\mathbf{x})(x)$ and $\Psi(\mathbf{x})(x)$, respectively.

We first show that $S^\alpha(T_s+t\mathbf{x}) \leq S^\alpha(T_s\mathbf{x})$ if $s, t > 0$. Given $m \geq 0$ and $s > 0$, let $T_s^{(m)}$ represent the leading principal $(m+1)\times(m+1)$ submatrix of $T_s$; and let $\mathbf{x}^{(m)} = (x_0, \ldots, x_m)^T$ as above. Since $S^\alpha(T_s\mathbf{x}) < \infty$ for all $s > 0$, there exists $m$ such that

$$S^\alpha(T_s+t\mathbf{x}) = S^\alpha(T_s^{(m)}\mathbf{x}^{(m)}), \quad S^\alpha(T_s\mathbf{x}) = S^\alpha(T_s^{(m)}\mathbf{x}^{(m)}).$$

Now compute, using for $T_s^{(m)}$ the variation diminishing property in Theorem 28.10

$$S^\alpha(T_s^{(m)}\mathbf{x}^{(m)}) = S^\alpha(T_s^{(m)}T_s^{(m)}\mathbf{x}^{(m)}) \leq S^\alpha(T_s^{(m)}\mathbf{x}^{(m)}) = S^\alpha(T_s\mathbf{x}).$$

Finally, we claim that $S^\alpha(T_s\mathbf{x}) < \infty$ is at least the (finite) number of positive zeros of $e^{sx}\Psi(\mathbf{x})(x)$, which clearly equals the number of zeros of $\Psi(\mathbf{x})(x)$ and hence would show the result. This follows from a variant of the (stronger) Descartes’ Rule of Signs – see Theorem 10.7. □

The next application of the above analysis on generating functions of one-sided Pólya sequences is to algebraic combinatorics. Suppose $\Psi_a(x) = (1 + \alpha_1 x) \cdots (1 + \alpha_m x)$, with $\alpha_1, \ldots, \alpha_m > 0$ (so $a_0 = 1$). Then $a_0, a_1, \ldots, a_m$ are precisely the \textit{elementary symmetric polynomials} $a_j = e_j(\alpha_1, \ldots, \alpha_m)$ in the roots $\alpha_j$. Taking limits as above, if $\alpha_j \geq 0$ and $\sum_{j=1}^\infty \alpha_j < \infty$, then $a_j = e_j(\alpha_1, \alpha_2, \ldots)$ for all $j \geq 0$. Here, the $m$th elementary symmetric polynomial in (in)finitely many variables $u_1, u_2, \ldots$ is defined to be

$$e_j(u) := \sum_{1 \leq k_1 < k_2 < \cdots < k_j} u_{k_1} \cdots u_{k_j},$$

with $e_0(u) := 1$ and $e_j(u) := 0$ if $u$ has fewer than $j$ entries.

In fact a stronger phenomenon occurs: if we replace the $\alpha_j$ by variables $u = (u_1, u_2, \ldots)$, then every minor of the infinite triangular Toeplitz matrix

$$
geq\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
\end{pmatrix}
$$

is \textit{monomial positive}, meaning that any minor drawn from it is a non-negative sum of monomials in the $u_j$. In fact an even stronger result is true: the above matrix is a (skew) Schur
polynomial in the $u_j$. In particular, it is (skew) Schur positive (i.e., a non-negative sum of Schur polynomials). This phenomenon is known as the dual Jacobi–Trudi identity, and is in a sense, the ‘original’ case of numerical positivity being monomial positivity – in fact, being Schur positivity. See Appendix E for more on this. (We will see another, more recent such instance in Theorem 42.6 which follows from a more general Schur positivity phenomenon shown by Lam–Postnikov–Pylyavskyy [178].)

29.4. Location of the roots, for generating functions of finite $TN_p$ sequences. Recall by Lemma [29.8] that if a polynomial $f$ has only negative (real) roots, then it generates a PF sequence. In the rest of this section, we prove the converse result: namely, if a real sequence $a$ has only finitely many nonzero entries, then it is a PF sequence only if the polynomial $\Psi_a(x) = a_0 + \cdots + a_m z^m$ has all negative zeros.

To do so, we will deduce necessary (and sufficient) conditions for a finite sequence to be $TN_p$. The aforementioned conclusion will then follow by considering all $p \geq 1$. We begin with a result by Schoenberg [256] in Ann. of Math. 1955, which says: in order to check whether or not $(\ldots, 0, a_0, a_1, \ldots, a_m, 0, \ldots)$ is a $TN_p$ sequence, we do not need to check infinitely many minors of unbounded size.

Theorem 29.14. Suppose $(\ldots, 0, a_0, a_1, \ldots, a_m, 0, \ldots)$ is a real sequence with $a_0, a_1, \ldots, a_m > 0$ for some $m \geq 0$. This sequence is $TN_p$ (for some integer $p \geq 1$) if and only if the matrix

$$A_p := \begin{pmatrix}
a_0 & a_1 & \cdots & \cdots & a_m & 0 & 0 & \cdots & 0 \\
0 & a_0 & \cdots & \cdots & a_{m-1} & a_m & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & a_0 & \cdots & a_{m-p+1} & a_{m-p+2} & a_{m-p+3} & \cdots & a_m
\end{pmatrix}_{p \times (m+p)}$$

is $TN$. Moreover, in this case the generating polynomial $\Psi_a(z) = a_0 + \cdots + a_m z^m$ has no zeros in the sector $|\arg z| < p\pi/(m+p-1)$.

To show this result, Schoenberg first proved a couple of preliminary lemmas.

Lemma 29.15. Suppose $(a_n)_{n \in \mathbb{Z}}$ is a summable sequence of non-negative numbers, such that the $p \times \mathbb{Z}$ ‘matrix’

$$A'_p := (a_{k-j})_{0 \leq j < p, k \in \mathbb{Z}}$$

has rank $p$, i.e., an invertible $p \times p$ submatrix. Then $(a_n)_n$ is $TN_p$ if and only if the matrix $A'_p$ is $TN$.

Proof. One implication is immediate. Conversely, suppose $A'_p$ is $TN$, and for every $\sigma > 0$, draw a $\mathbb{Z} \times \mathbb{Z}$ matrix from a Gaussian kernel, say $M_\sigma := (e^{-\sigma(j-k)^2})_{j,k \in \mathbb{Z}}$. By Lemma [6.8] this matrix is symmetric and totally positive, and goes entrywise to $\text{Id}_\mathbb{Z}$ as $\sigma \to \infty$. Now since $\sum_{n \in \mathbb{Z}} a_n$ is finite, one can form the matrix

$$B_\sigma := T_\alpha M_\sigma, \quad (B_\sigma)_{j,k} = \sum_{n \in \mathbb{Z}} a_{n-j} e^{-\sigma(n-k)^2}, \quad j, k \in \mathbb{Z}.$$

We claim that $B_\sigma$ is $TP_p$. To see why, it suffices to check that any contiguous submatrix of $B_\sigma$ is $TP_p$; in turn, for this it suffices to check for all $1 \leq r \leq p$ that all contiguous $r \times r$ minors of $B_\sigma$ are positive, by the Fekete–Schoenberg Lemma [4.9]. By the (generalized) Cauchy–Binet formula, each such minor is the product of an $r \times r$ minor of a contiguous submatrix $A'_{p,n}^{\alpha}$ of $T_\alpha$ and an $r \times r$ minor of a contiguous submatrix $M_\sigma^{\alpha}$ of the totally positive matrix $M_\sigma$.

Since all such matrices $A^\ast$ are identical up to a horizontal shift, $A^\ast$ has at least one invertible minor by assumption (for all $1 \leq r \leq p$), and all others are non-negative, being minors of the
\( p \times \mathbb{Z} \) matrix \( A_p' \). Since \( M^* \) is totally positive, we obtain that all \( r \times r \) contiguous minors of \( B_\sigma \) are positive, for each \( \sigma > 0 \). It is not hard to see that these converge to the corresponding \( r \times r \) minor of \( T_a M_\infty = T_a \), as \( \sigma \to \infty \), and the proof is complete.

The next lemma is interesting in its own right, hence isolated into a standalone result.

**Lemma 29.16.** Given matrices \( A, B \), define their ‘concatenation’ \( A \boxplus B \) to be the ‘block diagonal’ matrix

\[
\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},
\]

but with a horizontal shift such that the final column of \( A \) is directly above the initial column of \( B \). Now \( A \boxplus B \) is \( TN \) if and only if \( A, B \) are \( TN \).

**Proof.** One implication is immediate. Conversely, suppose \( A, B \) are \( TN \), and the column common to them is numbered \( n \) in \( A \boxplus B \). Choose a square submatrix \( M \) of \( A \boxplus B \). If \( M \) does not have a column of \( A \boxplus B \) from before the \( n \)th column and one from after the \( n \)th, then either \( M \) has a zero row or \( M \) is a submatrix of \( A \) or of \( B \), whence \( \det(M) \geq 0 \).

Otherwise \( M \) has two columns indexed in \( A \boxplus B \) by \( n_1, n_2 \) with \( n_1 < n < n_2 \). If \( M \) does not include the \( n \)th column, then \( M \) is a submatrix of \( A \boxplus B \) with the \( n \)th column removed, which is a block diagonal matrix, and hence \( \det(M) \geq 0 \) by the hypotheses. Finally, if \( M \) also includes the \( n \)th column of \( A \boxplus B \), then removing this column from \( M \) again yields a block diagonal matrix \( \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \). Hence \( M \) is of the form

\[
M = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} + \begin{pmatrix} v_A & 0 \\ 0 & v_B \end{pmatrix}.
\]

Expanding \( \det(M) \) along the ‘common’ column, it is easy to see that

\[
\det(M) = \det(B') \det(A'[v_A] + \det(A') \det[v_B][B'],
\]

and as all four determinants are non-negative by assumption, we are done. □

With these preliminaries at hand, we can proceed.

**Proof of Theorem 29.14.** The first part follows from Lemma 29.15, since now most of the columns of the matrix \( A_p' \) defined in that lemma are zero. Here we use not \( A_p \), but the matrix obtained by reversing the rows and columns of \( A_p \), which is \( TN \) if and only if \( A_p \) is.

Now suppose \( A_p \) is \( TN \). If \( m = 0 \) then the result is obvious, so we assume \( m > 0 \) henceforth. By Lemma 29.16, so is the \( np \times (nm + np - n + 1) \) matrix

\[
M_n = A_p^{\boxplus n} := A_p \boxplus \cdots \boxplus A_p, \quad n \geq 1,
\]

where the ‘sum’ is \( n \) times, and we use that \( \boxplus \) is associative.

The first observation is that \( M_n = A_p^{\boxplus n} \) has full rank \( np \). Indeed, we may consider from each component \( A_p \) the initial \( p \) columns, which yields an upper triangular \( np \times np \) submatrix with all diagonal entries \( a_0 > 0 \). For future use, denote the set of these \( np \) columns by \( J \).

Let \( \alpha = \rho e^{i\theta} \) be a root of \( \Psi_a(z) \). If \( \theta = \pi \) then we are done, and if \( \theta = 0 \) then \( \Psi_a(z) > 0 \). Since \( \alpha \) is also a root, we may therefore assume that \( \theta \in (0, \pi) \). Define \( x_j := \Im(\alpha^j) = \rho^j \sin(j \theta) \) for \( 0 \leq j \leq n(m + p - 1) \). We now count the number of sign changes \( S^-(\{x_0, \ldots, x_t\}) \) for \( t \geq 0 \) : this equals the number of times \( \alpha^j \) crosses the \( X \)-axis in \( \mathbb{C} \), so

\[
S^-(\{x_0, \ldots, x_t\}) = [t \theta / \pi] + \varepsilon, \quad \text{where} \ \varepsilon \in \{0, -1\}. \quad (29.17)
\]

At the same time, \( \sum_{j=0}^m a_j \alpha^j = 0 \), so taking the imaginary parts yields:

\[
\sum_{j=0}^m a_j \alpha^j \sin((\nu + j)\theta) = 0, \quad \forall \nu \in \mathbb{Z}^\geq 0.
\]
In other words, $M_n \mathbf{x} = 0$, where $\mathbf{x} := (x_0, \ldots, x_n)^T$. Also note that if some $x_j = 0$ for $0 < j < n(m + p - 1)$, then $x_{j-1}x_{j+1} < 0$, since $\theta \in (0, \pi)$ by assumption. Hence small enough perturbations to $\mathbf{x}$ will not change the number of sign-changes except at most at the extremal coordinates.

Now let $\mathbf{y} := (-1, 1, \ldots, (-1)^m)^T$ and let $\mathbf{x}_0 := (M_n)^{-1} \mathbf{y}$, with $J \subset [nm + np - n + 1]$ the set of columns chosen above, containing $a_0$. Let $\tilde{\mathbf{x}}_0 \in \mathbb{R}^{nm + np - n + 1} \setminus \{0\}$ denote the vector with the coordinates of $\mathbf{x}_0$ in the positions indexed by $J$, and padded by zeros otherwise. Then $M_n \tilde{\mathbf{x}}_0 = \mathbf{y}$. Now by the end of the preceding paragraph, choose $\epsilon > 0$ small enough such that $S^-(\mathbf{x} + \epsilon \tilde{\mathbf{x}}_0) \leq S^-(\mathbf{x}) + 2$. Recalling that $M_n \mathbf{x} = 0$, we have:

$$np - 1 = S^-(\mathbf{y}) = S^-((M_n(x + \epsilon \tilde{\mathbf{x}}_0)) \leq S^-((\mathbf{x} + \epsilon \tilde{\mathbf{x}}_0)) \leq S^-((\mathbf{x}) + 2,$$

where the first inequality is by the variation diminishing property of the $TN$ matrix $M_n = A_p^m$ (see Theorem 28.10). Hence by (29.17) with $t = n(m + p - 1)$,

$$np - 1 \leq |n(m + p - 1)\theta/\pi| + 2 \leq \frac{n(m + p - 1)\theta}{\pi} + 2.$$

Since $n$ was arbitrary, letting $n \to \infty$ finishes the proof. \hfill $\square$

We can now deduce the desired corollary about finite Pólya frequency sequences.

**Corollary 29.18.** Suppose $\mathbf{a} = (\ldots, 0, a_0, \ldots, a_m, 0, \ldots)$ is a real sequence, with $a_0, \ldots, a_m > 0$. Then $\mathbf{a}$ is a Pólya frequency sequence if and only if $\Psi_\mathbf{a}(x) = \sum_{j=0}^{m} a_jx^j$ has $m$ negative real roots, counted with multiplicity.

**Proof.** The ‘if’ part follows from Lemma 29.8, and the ‘only if’ part follows from the final assertion of Theorem 29.14, letting $p \to \infty$. \hfill $\square$

### 29.5. Complements: Hurwitz and Jacobi matrices

We conclude this section by discussing some well-known matrices associated to finite PF sequences and their generating polynomials. First, Corollary 29.18 has received recent attention: instead of working with the matrix $A_p$ as in Theorem 29.14 one can study other variants. A few such results without proof are now presented along these lines. Given a polynomial with real coefficients

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n, \quad n, a_0, \ldots, a_n > 0,$$

and an integer $1 \leq M \leq n$, define the $M$th **generalized Hurwitz matrix** to be the $\mathbb{Z} \times \mathbb{Z}$ matrix $H_M(f)$, given by

$$H_M(f)_{j,k} := a_{M-k-j}, \quad j, k \in \mathbb{Z}. \quad (29.19)$$

We now present five results (without proof, and perhaps not all of them ‘best possible’) in the literature, which are similar to each other, and one of which is the Aissen–Edrei–Schoenberg–Whitney corollary above:

**Theorem 29.20.** Let $n, a_0, \ldots, a_n > 0$ and define $f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$. Also fix an integer $1 \leq M \leq n$.

1. ($M = 2$, Routh–Hurwitz, see e.g. 143.) The polynomial $f$ is ‘stable’, i.e. has no zeros $z$ with $|\arg z| \leq \pi/2$, if and only if all leading principal minors of the Hurwitz matrix $H_2(f)$, of order $\leq n$, are positive.
2. ($M = 2$, Holtz–Tsaglov 137.) The polynomial $f$ has no zeros $z$ with $|\arg z| < \pi/2$ if and only if the Hurwitz matrix $H_2(f)$ is TN.
3. ($M = 1$, Aissen–Edrei–Schoenberg–Whitney.) The polynomial $f$ has no zeros $z$ with $|\arg z| < \pi$ if and only if the Toeplitz matrix $H_1(f)$ is TN.
4. ($M = n$, Cowling–Thron 166.) The polynomial $f$ has no zeros $z$ with $|\arg z| \leq \pi/n$. 


29. Pólya frequency sequences and their generating functions.

(5) \((M \in [1, n], \text{Holtz–Khrushchev–Kushel [136].})\) The polynomial \(f\) has no roots \(z\) with \(|\arg z| < \pi/M\) if the generalized Hurwitz matrix \(H_M(f)\) is \(TN\).

We show next that if the generating polynomial above has no roots in the sector \(|\arg(z) - \pi| \leq \pi/(p + 1)\), then the finite sequence \((\ldots, 0, a_0, \ldots, a_m, 0, \ldots)\) is \(TN_p\). To do so, we first study when ‘infinite Toeplitz tri-diagonal (Jacobi) matrices’ are \(TN_p\). This was carried out by Karlin in Trans. Amer. Math. Soc. (1964), and he showed the following lemma.

Lemma 29.21 (Karlin, [153]). Given \(a, b, c \in (0, \infty)\), define the corresponding Jacobi matrix \(J(a, b, c)\) where \(J(a, b, c)_{j,k} = a_j b_{j-k}\), and elements of \(K\) is a product of principal minors and elements of \(K\). Thus, to check if \(J(a, b, c)\) is \(TN_p\), we need to examine the principal minors of size at most \(p\).

For the ensuing discussion, fix \(a, c > 0\). Given \(\lambda \in \mathbb{C}, b \in (0, \infty)\), and an integer \(r \geq 1\), define \(J_{r,b}(\lambda)\) to be the leading principal submatrix of \(J(a, b, c) - \lambda Id_Z\). Expanding along an initial/terminal row/column, it is easy to observe that the determinants

\[
d_{r,b}(\lambda) := \det J_{r,b}(\lambda), \quad r = 1, 2, \ldots
\]

are polynomials in \(\lambda\), which satisfy the recurrence:

\[
d_{r+1,b}(x) = (b - x)d_{r,b}(x) - acd_{r-1,b}(x), \quad r \geq 0
\]

with the initial conditions \(d_{0,b}(x) = 1, d_{-1,b}(x) = 0\). Reformulate this in terms of the ‘shifted’ polynomials

\[
e_{r,b}(x) = \sqrt{ac}^{-r}d_{r,b}(b - 2\sqrt{ac}x), \quad r \geq 1,
\]

to obtain the recurrence:

\[
x e_{r,b}(x) = \frac{e_{r-1,b}(x) + e_{r+1,b}(x)}{2}, \quad r \geq 1; \quad e_{0,b}(x) = 1, \quad e_{-1,b}(x) = 0.
\]

Restricting to \(x = \lambda \in (-1, 1)\), so that \(x = \cos(\theta)\) for some \(\theta \in (0, \pi)\), one can show by induction that \(e_r(x) = \sin((r + 1)\theta)/\sin(\theta)\) satisfies the initial conditions and the recurrence. In fact, this corresponds to the Chebyshev polynomials of the second kind, given by

\[
e_r(\cos \theta) = \frac{\sin((r + 1)\theta)}{\sin \theta}, \quad \theta \in (0, \pi), \quad r \geq -1.
\]

We now fix \(a, c > 0\) and first vary \(b \in (0, 2\sqrt{ac})\), so that \(\frac{b}{2\sqrt{ac}} = \cos(\theta) \in (0, 1)\), whence \(\theta \in (0, \pi/2)\). Our desired value of interest is the determinant

\[
d_r(0) = \sqrt{ac}^r e_{r,b}(b/2\sqrt{ac}) = \sqrt{ac}^r e_{r,b}(\cos \theta) = \sqrt{ac}^r \frac{\sin((r + 1)\theta)}{\sin \theta}.
\]

This is non-negative if and only if \((r + 1)\theta \leq \pi\), if and only if \(1 > \frac{b}{2\sqrt{ac}} \leq \cos(\pi/(r + 1))\). This completes the classification if \(0 < b < 2\sqrt{ac}\), since the sequence \(\cos(\pi/(r + 1))\) is increasing in \(1 \leq r \leq p\).
Returning to our original notation of \( J_{r,b}(a,c) \), note by the multilinearity of the determinant in the rows/columns that for any square \( r \times r \) matrix \( B \) over a unital commutative ring,

\[
\det(B + \lambda \text{Id}) = \lambda^r + \sum_{\emptyset \neq I \subset [r]} \lambda^{-|I|} \det B_{I \times I}.
\]  

(29.22)

We now apply this fact to the matrices \( J_{r,b}(a,c) \), if \( b \geq 2\sqrt{ac} \). Set

\[
b_r := \cos(\pi/(r + 1)), \quad B := J_{r,b_r}(a,c), \quad \lambda := \frac{b}{2\sqrt{ac}} - b_r \geq 0.
\]

Then \( b_r > b_1, \ldots, b_{r-1} \), so \( \det B_{I \times I} \geq 0 \) from the above analysis. It follows from (29.22) that \( \det J_{r,b}(a,c) \geq 0 \). Since this holds for all \( 1 \leq r \leq p \), the proof is complete. \( \square \)

With Lemma [29.21] at hand, we present the final proof in this section: that of a sufficient condition for a polynomial to generate a finite Pólya frequency sequence.

**Theorem 29.23.** Fix an integer \( p \geq 1 \), and suppose a polynomial \( f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n \) as above, with real coefficients and such that all zeros of \( f \) lie in the sector

\[|\arg(z) - \pi| \leq \frac{\pi}{p + 1}.
\]

Then the sequence \( a = (\ldots, 0, a_0, \ldots, a_n, 0, \ldots) \) is \( TN_p \).

**Proof.** If \( p = 1 \) then the result is clear, so we suppose below that \( p \geq 2 \). Decompose \( f \) into linear and complex-conjugate factors:

\[f(z) = a_0 \prod_j (z - \alpha_j) \prod_k (z - \rho_k e^{-i\theta_k})(z + \rho_k e^{i\theta_k}),\]

with all \( \rho_k > 0 \). The hypotheses imply that \( \alpha_j < 0 \) and \( |\theta_k - \pi| \leq \pi/(p + 1) \). Now \( z - \alpha_j \) generates a PF sequence by Lemma [29.8] and each irreducible quadratic

\[z - \rho_k e^{-i\theta_k})(z + \rho_k e^{i\theta_k}) = z^2 - 2z\rho_k \cos(\theta_k) + \rho_k^2\]

does the same by Lemma [29.21] since the condition \( -\cos \theta_k \geq \cos \pi/(p + 1) \) is equivalent to \( |\theta_k - \pi| \leq \pi/(p + 1) \) (with \( p \geq 2 \)). Hence their product \( f(z) \) also generates a PF sequence by Proposition [29.6] (3). \( \square \)

Finally, we present a (standalone) sufficient condition for a square matrix to be \( TP \):

**Theorem 29.24.** Let \( A = (a_{jk})_{j,k \geq 1} \) with all \( a_{jk} \in (0, \infty) \). If

\[a_{jk}a_{j+1,k+1} > a_{j,k+1}a_{k+1,j} \cdot 4 \cos^2(\pi/(n + 1)), \quad \forall j, k \geq 1,
\]

then \( A \) is \( TP_n \); moreover, the constant \( 4 \cos^2(\pi/(n + 1)) \) cannot be reduced.

This result only uses the positivity of the entries and a growth condition on the \( 2 \times 2 \) minors. It was conjectured by Dimitrov–Péna in 2005 [73], and proved (independently) by Katkova–Vishnyakova in 2006 [159]. That the constant is best possible is revealed via Lemma [29.21] as follows: given \( 0 \leq c < 4 \cos^2(\pi/(n + 1)) \), choose \( \theta \in \left(\frac{\pi}{n+1}, \frac{2\pi}{n+1}\right) \) such that \( c < 4 \cos^2 \theta \). Now consider the Jacobi matrix \( J(1, 2 \cos \theta, 1)_{n \times n} \).
29. Pólya frequency sequences and their generating functions.

29.6. **Further results on root-location.** Theorems 29.14, 29.20, and 29.23 showed the connection between Pólya frequency sequences, totally non-negative matrices, and root-location results. We conclude this section with additional results on these connections, as well as classical and well-known results on zeros of polynomials. The first is the celebrated Gauss–Lucas theorem, found in Lucas’s 1874 work [188].

**Theorem 29.25** (Gauss–Lucas). If \( p(z) \) is a non-constant polynomial, then the roots of \( p'(z) \) in \( \mathbb{C} \) are contained in the convex hull of the set of roots of \( p(z) \).

**Proof.** Let \( p(z) = p_n \prod_{j=1}^{n} (z - \xi_j) \). If \( \xi_j \) is a root of \( p' \) as well as \( p \), then \( \xi_j = 1 \cdot \xi_j + \sum_{k \neq j} 0 \cdot \xi_k \). If \( \xi \) is a root of \( p' \) but not of \( p \), then we compute:

\[
\frac{p'(z)}{p(z)} = \sum_{j=1}^{n} \frac{1}{z - \xi_j} \quad \Rightarrow \quad 0 = \frac{p'(\xi)}{p(\xi)} = \sum_{j=1}^{n} \frac{\bar{\xi} - \bar{\xi}_j}{|\xi - \xi_j|^2}.
\]

Setting \( A := \sum_{j=1}^{n} |\xi - \xi_j|^{-2} > 0 \), we obtain via simplifying and conjugating:

\[
A \bar{\xi} = \sum_{j=1}^{n} |\xi - \xi_j|^{-2} \xi_j \quad \Rightarrow \quad \xi = \sum_{j=1}^{n} \frac{|\xi - \xi_j|^{-2}}{A} \xi_j,
\]

and so \( \xi \) is in the convex hull of the \( \xi_j \). \( \square \)
30. Examples of Pólya frequency functions: Laplace transform, convolution.

In Section 27 we saw some characterizations of $T_N^p$ functions, and also studied the exponential decay of $T_N^2$ (whence all $T_N^p$) functions. We also saw several examples of $T_N$ (in fact, Pólya frequency) functions in Section 28. We now return to Pólya frequency functions, and discuss additional examples as well as a recipe to generate new examples of $T_N^p$ or PF functions from old ones.

30.1. The bilateral Laplace transform of a totally non-negative function.

We begin by defining and studying the Laplace transform more generally – for $T_N^2$ functions.

**Definition 30.1.** The bilateral Laplace transform of a (measurable) function $f : \mathbb{R} \to \mathbb{R}$ is denoted by $B(f)$, and defined at a complex argument $s \in \mathbb{C}$ to be

$$B(f)(s) := \int_{\mathbb{R}} e^{-sx} f(x) \, dx.$$ 

This expression is defined if the following integrals both converge as $R \to \infty$:

$$\int_{0}^{R} e^{-sx} f(x) \, dx, \quad \int_{-R}^{0} e^{-sx} f(x) \, dx,$$

in which case the sum of their limits is taken to be $B(f)(s)$.

The following result uses the characterization of $T_N^2$ functions in Theorem 27.4 above, to show the existence of the Laplace transform:

**Lemma 30.2.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is $T_N^2$ and not an exponential. Then $B(f)$ exists in the open vertical strip in $\mathbb{C}$, given by

$$-\infty \leq \alpha := \inf_{x \in I} (\log f)'(x) < \beta := \sup_{x \in I} (\log f)'(x) \leq \infty,$$

where $f'$ exists on a co-countable set, and we set $\alpha := -\infty$ (resp. $\beta := \infty$) if $f(x) \equiv 0$ for large $x > 0$ (resp. small $x < 0$). If $f$ is integrable then this strip contains the imaginary axis.

**Proof.** Let $I$ denote the interval of support of $f$, as in Theorem 27.4. There are three possibilities: (a) $I$ is bounded, in which case the result is easy; (b) $I$ is unbounded only on one side (in which case $f$ can be an exponential function on $I$); or (c) $I = \mathbb{R}$, in which case $f(x)$ is not an exponential.

We will work in the third case, as the cases for $I \subseteq \mathbb{R}$ are simpler. It suffices to show that the integral in $B(f)$ is absolutely convergent on a vertical strip. We now appeal to Proposition 27.7 and its proof, used henceforth without further reference. First by Lemma 26.3 $(\log f)'$ is defined on a co-countable subset of $I$ and is non-increasing there. Thus the following limits make sense, and equal the asserted formulae:

$$\alpha := \lim_{x \to \infty} \frac{f'(x)}{f(x)}, \quad \beta := \lim_{x \to -\infty} \frac{f'(x)}{f(x)};$$

moreover, $-\infty \leq \alpha < \beta \leq \infty$, since $f$ is not an exponential on $I = \mathbb{R}$.

We claim that the integral in $B(f)(s)$ is absolutely convergent for $\Re(s) \in (\alpha, \beta)$, whence convergent as desired. Indeed:

$$|B(f)(s)| \leq \int_{\mathbb{R}} |e^{-sx}| f(x) \, dx = \int_{0}^{\infty} e^{-x\Re(s)} f(x) \, dx + \int_{-\infty}^{0} e^{-x\Re(s)} f(x) \, dx.$$
Since $\alpha < \Re(s) < \beta$, choose points $x_1 < x_2$ in $I$ such that $\alpha < g'(x_2) < \Re(s) < g'(x_1) < \beta$. Then convergence follows because $f$ shrinks to zero faster than the exponential bounds in the proof of Proposition 27.7.

Finally, suppose $f$ is integrable on $I$, where we once again assume $I = \Re$. Then $f(x) \to 0$ as $|x| \to \infty$, whence $\log f(x) \to -\infty$ as $|x| \to \infty$. It follows that $\alpha < 0 < \beta$. \hfill $\Box$

We next bring the limits of the vertical strip $\alpha, \beta$ into the form found in the literature:

**Proposition 30.4.** The limits in Lemma 30.3 can also be written as

$$
\alpha := \lim_{x \to \infty} \frac{\log f(x)}{x}, \quad \beta := \lim_{x \to -\infty} \frac{\log f(x)}{x},
$$

where once again, we set $\alpha := -\infty$ (resp. $\beta := \infty$) if $f(x) \equiv 0$ for sufficiently large $x > 0$ (resp. small $x < 0$).

**Proof.** The result is nontrivial only for $I$ unbounded on one or both sides of the origin; we show it here only for the case $I = \Re$. Since $-\log f$ is convex, the result follows from a more general fact about arbitrary convex functions:

Suppose $g : \Re \to \Re$ is convex, so that $g'$ is defined on a co-countable set. Then,

$$
\sup_{x \in \Re} g'(x) = \lim_{x \to \infty} g'(x) = \lim_{x \to \infty} \frac{g(x)}{x},
$$

$$
\inf_{x \in \Re} g'(x) = \lim_{x \to -\infty} g'(x) = \lim_{x \to -\infty} \frac{g(x)}{x}.
$$

We only show the first part; the second is similar. By Lemma 26.3(1), the divided difference

$$
h(x, y) := \frac{g(y) - g(x)}{y - x}, \quad x \neq y
$$

is coordinatewise non-decreasing. Choose any $x_0$ at which $g$ is differentiable. Then for $y > x_0$,

$$
h(x_0, y) \geq \lim_{y \to x_0^+} h(x_0, y) = g'(x_0);
$$

but now taking $y \to \infty$,

$$
g'(x_0) \leq \lim_{y \to \infty} \frac{g(y) - g(x_0)}{y - x_0} = \lim_{y \to \infty} \frac{g(y)}{y}.
$$

Taking the supremum over $x_0 \in \Re$ (or the limit as $x_0 \to \infty$) yields one inequality. For the other, let $g$ be differentiable at $x_0$ and let $y < x_0$. Then

$$
h(y, x_0) \leq \lim_{y \to x_0^-} h(y, x_0) = g'(x_0) \leq \sup_{x \in \Re} g'(x).
$$

Since this holds for all $x_0 > y$, now taking $x_0 \to \infty$ yields the desired result:

$$
\sup_{x \in \Re} g'(x) \geq \lim_{x_0 \to \infty} \frac{g(x_0) - g(y)}{x_0 - y} = \lim_{x_0 \to \infty} \frac{g(x_0)}{x_0}.
$$

\hfill $\Box$

### 30.2. Examples of Pólya frequency functions; convolution

Having discussed the Laplace transform, we next discuss a recipe to generate new examples of $T_N^p$ functions (or Pólya frequency functions) from old ones, for all $p \geq 1$.

**Definition 30.5.** Given Lebesgue measurable functions $f, g : \Re \to \Re$, define their convolution, denoted by $f \ast g$, to be the function given by the following integral, wherever defined:

$$
(f \ast g)(x) := \int_\Re f(y)g(x - y) \, dy.
$$
The recipe can now be stated: if \( f, g \) are integrable \( TN_p \) functions for any \( 1 \leq p \leq \infty \), then so is \( f \ast g \). To show this, we require some basic properties of convolutions, which are now collected together for ease of future reference.

**Lemma 30.6** (Convolution properties). Suppose \( f, g : \mathbb{R} \to \mathbb{R} \) are both in \( L^1(\mathbb{R}) \).

1. For almost all \( x \in \mathbb{R} \), the function \( y \mapsto f(y)g(x-y) \) is Lebesgue measurable and integrable, so that \( f \ast g \) is defined for almost every \( x \in \mathbb{R} \).
2. \( f \ast g \in L^1(\mathbb{R}) \), and \( \| f \ast g \|_1 \leq \|f\|_1 \|g\|_1 \).
3. If \( f, g \geq 0 \) on \( \mathbb{R} \), then so is \( f \ast g \).
4. The Laplace transform is an ‘algebra homomorphism’ for addition and convolution:
   \[
   \mathcal{B}(f \ast g)(s) = \mathcal{B}(f)(s)\mathcal{B}(g)(s),
   \]
   whenever the two terms on the right converge absolutely at a common point \( s \in \mathbb{R} \).

Regarding the last part, we leave to the reader the verification that \( L^1(\mathbb{R}) \) under addition and convolution forms a commutative \( \mathbb{R} \)-algebra.

**Proof.**

1. First note that \( A(x, y) := f(x)g(y) \) is Lebesgue measurable, since \( f, g \) are. Moreover, \( L(x, y) := (y, x-y) \) is an invertible linear transformation of \( \mathbb{R}^2 \), whence measurable. Thus \( (A \circ L)(x, y) := f(y)g(x-y) \) is measurable. But now \( A \circ L \) is also integrable:
   \[
   \int_{\mathbb{R}^2} |(A \circ L)(x, y)| \, dx \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |g(x-y)| \, dx \right) |f(y)| \, dy = \|g\|_1 \|f\|_1 < \infty.
   \]
   Hence the assertion follows by Fubini’s theorem.
2. This is now straightforward:
   \[
   \| f \ast g \|_1 = \int_{\mathbb{R}} |(f \ast g)(x)| \, dx \leq \int_{\mathbb{R}^2} |g(x-y)f(y)| \, dx \, dy,
   \]
   and as in the preceding part, the right-hand side equals \( \|g\|_1 \|f\|_1 \).
3. This is immediate from the definition of \( f \ast g \).
4. We compute:
   \[
   \mathcal{B}(f)(s)\mathcal{B}(g)(s) = \int_{\mathbb{R}} f(y) \, dy \int_{\mathbb{R}} e^{-s(y+u)}g(u) \, du = \int_{\mathbb{R}} f(y) \, dy \int_{\mathbb{R}} e^{-sx}g(x-y) \, dx
   \]
   \[
   = \int_{\mathbb{R}} e^{-sx} \left( \int_{\mathbb{R}} f(y)g(x-y) \, dy \right) \, dx = \int_{\mathbb{R}} e^{-sx} (f \ast g)(x) \, dx = \mathcal{B}(f \ast g)(s),
   \]
   where \( f \ast g \) is defined almost everywhere from above. The interchange of integrals in the first equality on the second line is justified by Fubini’s theorem – which applies here because
   \[
   \int_{\mathbb{R}^2} |f(y)e^{-sx}g(x-y)| \, dy \, dx = \int_{\mathbb{R}} e^{-sy}|f(y)| \, dy \cdot \int_{\mathbb{R}} e^{-su}|g(u)| \, du,
   \]
   and both integrals are finite by assumption. □

As an immediate consequence, the above recipe follows:

**Corollary 30.7.** Suppose \( f, g : \mathbb{R} \to \mathbb{R} \) are integrable \( TN_p \) functions for some \( p \geq 1 \) (or both \( TN \) functions). Then so is \( f \ast g \).

**Proof.** That \( f \ast g \) is integrable follows from Lemma 30.6. That it is \( TN_p \) follows from the Basic Composition Formula (see (5.14)) and Corollary 6.1. □
Two applications of this corollary will be provided presently.

Having studied the Laplace transform and convolution, we now come to additional examples of $TN$ and Pólya frequency functions.

**Example 30.8.** The Heaviside function $H_1(x) := 1_{x \geq 0}$ is $TN$. This can be shown using direct computations, see e.g. Chapters 1, 3 of Karlin’s book \[154\], where it is shown that the ‘transpose’ kernel $K(x,y) = 1_{x \leq y}$ satisfies:

$$
\det K[x;y] = 1(x_1 \leq y_1 < x_2 < y_2 < \cdots < x_p \leq y_p),
$$

for all integers $p \geq 1$ and tuples $x, y \in \mathbb{R}^n$.

**Example 30.9.** We are now interested in convolving the previous example with itself several times. However, the function $H_1$ is not integrable. Thus, first use Lemma 27.3 to define

$$
\lambda_1(x) := e^{-x}H_1(x) = 1_{x \geq 0}e^{-x}.
$$

Note, this is an integrable $TN$ function on $\mathbb{R}$, which is discontinuous at the origin. We now claim that the $n$-fold convolution $f_n$ of $\lambda_1$ with itself is the function \[\frac{x^n}{(n-1)!}\lambda_1(x),\] for all $n \geq 1$.

The verification is by induction on $n \geq 1$, with the base case immediate. To show the induction step, use that $f_n(x) := x^{n-1}\lambda_1(x)/(n-1)!$, and compute:

$$
f_{n+1}(x) = \int_{\mathbb{R}} f_n(y)f_1(x-y) \, dy.
$$

By the induction hypothesis, the integrand vanishes unless $y, x-y$ are both non-negative, whence so is $x$. Thus $f_{n+1}(x) = 0$ for $x < 0$, and for $x \geq 0$, we compute:

$$
f_{n+1}(x) = \int_0^x f_n(y)f_1(x-y) \, dy = \int_0^x \frac{y^{n-1}}{(n-1)!}e^{-y} \cdot e^{-(x-y)} \, dy = e^{-x}x^n/n!.
$$

In particular, $x^n\lambda_1(x)$ is an integrable $TN$ function, by Corollary 30.7. Similar to the Gaussian, we record the Laplace transform of these functions, for future use. More generally, given a non-negative power $\alpha \geq 0$ and a scalar $\beta > 0$, let $g_{\alpha,\beta}(x) := x^\alpha\lambda_1(\beta x)$. Then,

$$
\mathcal{B}(g_{\alpha,\beta}) = \int_0^\infty e^{-xs}x^\alpha e^{-\beta x} \, dx = \int_0^\infty e^{-x(s+\beta)}x^\alpha \, dx = \frac{\Gamma(\alpha + 1)}{(s+\beta)^{\alpha+1}}, \quad s > -\beta. \tag{30.10}
$$

These examples will play a role below, in classifying the total-positivity preservers on arbitrary domains. For now we present a final example, again obtained via convolution:

**Example 30.11.** Let $f(x) := \lambda_1(x)$ and $g(x) := \lambda_1(-x)$. As these are integrable $TN$ functions, so is their convolution, which one verifies is $e^{-|x|}/2$. Hence $e^{(\alpha-\beta)|x|/2}$ is $TN$ for all $\alpha < \beta$. Multiplying by $e^{(\alpha+\beta)x/2}$, it follows by Lemma 27.3 that the function

$$
f(x) = \begin{cases} 
    ce^{\beta(x-x_0)}, & \text{if } x \leq x_0, \\
    ce^{\alpha(x-x_0)}, & \text{if } x > x_0
\end{cases} \tag{30.12}
$$

is $TN$ for $c \geq 0$ and $x_0 \in \mathbb{R}$ and integrable when $\alpha < 0 < \beta$, as above. Notice also that the limiting cases of $\alpha = -\infty$, $\beta = +\infty$ (both leading to $f$ vanishing on a semi-axis), are integrable $TN$ functions; while if $\alpha = \beta$ then $f$ is an exponential, whence also $TN$. 
30.3. Pólya frequency functions and the Laguerre–Pólya class. It is rewarding to place the theory of Pólya frequency functions (and more generally, TN functions) in its historical context before proceeding further. Begin with a scalar δ ≥ 0 and a summable positive sequence:

\[ \alpha_j > 0, \quad j = 1, 2, \ldots, \quad \sum_{j \geq 1} \alpha_j < \infty, \]

so that the terms 1/α_j are bounded below by a positive number. Then the convolution

\[ f_n(x) := (\varphi_{\alpha_1} \ast \cdots \ast \varphi_{\alpha_n})(x), \quad \varphi_{\alpha}(x) := \frac{1}{a} \lambda_1(x/a) \]

is a Pólya frequency function with Laplace transform \( \prod_{j=1}^{n} (1 + \alpha_j s)^{-1} \). Given δ ≥ 0, the PF function \( \Lambda_n(x) := f_n(x - \delta) \) therefore satisfies:

\[ \mathcal{B}(\Lambda_n)(s) = \Phi_n^*(s), \quad \forall n \geq 1, \quad \Re(s) > \max_{j \geq n} (-1/\alpha_j), \quad \text{where} \quad \Phi_n^*(s) := \frac{e^{-\delta s}}{\prod_{j=1}^{n} (1 + \alpha_j s)}. \]

Notice for \( n \geq 2 \) that the \( \Phi_n^* \) are dominated uniformly on the imaginary axis by an integrable function:

\[ |\Phi_n^*(ix)| \leq \frac{1}{|(1 + \alpha_1 ix)(1 + \alpha_2 ix)|}, \quad \forall x \in \mathbb{R}, \quad n \geq 2. \]  \tag{30.13}

Hence for \( n \geq 2 \), the Laplace inversion formula recovers \( \Lambda_n \) from \( \Phi_n^* \) via the Fourier–Mellin integral, which converges absolutely:

\[ \Lambda_n(x) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-iT}^{iT} e^{sx} \Phi_n^*(s) \, ds, \quad \forall x \in \mathbb{R}. \]

Now since \( \sum_{j \geq 1} \alpha_j < \infty \), we have the convergence of the functions

\[ \Phi_n^*(s) \to \Phi^*(s) := \frac{e^{-\delta s}}{\prod_{j=1}^{\infty} (1 + \alpha_j s)}, \]

where the infinite product in \( \Phi^*(s) \) is defined in the vertical strip \( \max_j (-1/\alpha_j) < \Re(s) < \infty \), and converges there since

\[ \prod_{j=1}^{\infty} |1 + \alpha_j s| \leq \prod_{j=1}^{\infty} |1 + \alpha_j| \leq \exp \sum_{j=1}^{\infty} (\alpha_j |s|) < \infty. \]

Hence by Lebesgue’s dominated convergence theorem, the integrals \( \Lambda_n \) also converge to a function:

\[ \lim_{n \to \infty} \Lambda_n(x) = \lim_{n \to \infty} \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-iT}^{iT} e^{sx} \Phi_n^*(s) \, ds = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-iT}^{iT} e^{sx} \Phi^*(s) \, ds, \]

and this holds for all real \( x \). Denote the function on the right by \( \Lambda(x) \); then one can show that \( \Lambda \) is also a Pólya frequency function, which vanishes on \( (-\infty, 0) \) and is such that \( \mathcal{B}(\Lambda)(s) = \Phi^*(s) \) for \( \Re(s) > \max_j (-1/\alpha_j) \). Moreover, the reciprocal of this bilateral Laplace transform is the restriction to the strip \( \Re(s) > \max_j (-1/\alpha_j) \) of an entire function with only (real) negative zeros:

\[ \frac{1}{\mathcal{B}(\Lambda)(s)} = \frac{1}{\Phi^*(s)} = e^{\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s). \]  \tag{30.14}

A similar phenomenon occurs when one considers Pólya frequency functions that need not vanish on a semi-axis. In this case one can convolve functions of the form \( \varphi_{\alpha}(\pm x) \) for
Examples of Pólya frequency functions: Laplace transform, convolution.

0 ≠ a ∈ ℝ, as well as the Gaussian kernel (and shifted variants of these). Further taking limits produces Pólya frequency functions whose Laplace transforms are of the form

\[ \Phi^*(s) = \frac{e^{\gamma s^2 - \delta s}}{\prod_{j=1}^{\infty} (1 + \alpha_j s) e^{-\alpha_j s}}, \]  

(30.15)

where

\[ \gamma \in [0, \infty), \quad \alpha_j, \delta \in \mathbb{R}, \quad 0 < \gamma + \sum_{j} \alpha_j^2 < \infty, \quad \max_{\alpha_j > 0} (-1/\alpha_j) < \Re(s) < \min_{\alpha_j < 0} (-1/\alpha_j). \]

Schoenberg showed in *J. d’Analyse Math.* (1951) the following remarkable result: the above toy examples (30.14) and (30.15) (created by convolving variants of \( \lambda_1 \) and the Gaussian) are in fact representative of all Pólya frequency (PF) functions \( \Lambda \) satisfying \( \int_{\mathbb{R}} \Lambda(x) \, dx = 1 \) – with the first toy example a prototype for all PF functions that vanish on \( (-\infty, 0) \). E.g. the PF functions \( \Lambda \) as in (30.14) are characterized by the fact that \( \frac{1}{\Phi(\Lambda)(s)} \) is (the restriction to a vertical strip, of) an entire function \( ce^{\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s) \), where \( c \in (0, \infty), \delta, \alpha_j \geq 0, \) and \( \sum_{j} \alpha_j < \infty. \)

In fact, such entire functions were the subject of a beautiful theory built up around the turn of the 20th century, by experts before Schoenberg – including Laguerre, Pólya, and Schur. In the next section, we provide a brief detour into this rich area, before returning to its connections to Pólya frequency functions and \( TN \) functions.

This section undertakes a brief historical journey through one of the most longstanding and mathematically active areas in analysis, with a rich history as well as modern activity: the study of the zeros of (complex) polynomials and entire functions. To name a dozen experts with related work before 1930: Descartes (1637), Budan, Gauss, Fourier, Sturm, Cauchy (1800–1840), Chebyshev, Hermite, Poulain, Weierstrass, Routh, Biehler, Lucas (1840–1880), Laguerre, Hadamard, Maló, Markov, Hurwitz, Grace, Van Vleck (1880–1910), Fekete, Kakeya, Pólya, Jensen, Schur, Cohn, Szegő, Walsh, Obrechkoff (1910–1930).

For these and many other classical contributions, the reader is referred to e.g. the 1929 survey [277] by Van Vleck in Bull. Amer. Math. Soc. In the subsequent nine decades, activity in this area has continued, including papers, surveys, and books; in this section, we briefly allude to the works [67, 68] by Craven–Csordas (and Smith) and the classic text of Levin [182]. (We already saw some modern results along these lines, in Theorem 29.20; also see the modern monograph [231] by Rahman–Schmeisser for more on this area.)

The section concludes by alluding to a few important contributions to this area, all taken from this millennium.

We begin with notation.

Definition 31.1.

(1) Given a region $S \subset \mathbb{C}$, let $\pi(s)$ denote the class of polynomials with all zeros in $S$, and coefficients in $\mathbb{R}$ (sometimes this is replaced by $\mathbb{C}$). Given an integer $n \geq 1$, let $\pi_n(S) \subset \pi(S)$ denote the subset of polynomials with degree at most $n$.

(2) Given a complex polynomial $p$, let $Z_{nr}(p)$ denote the number of non-real roots of $p$.

A question that has interested analysts for more than a century is to understand operations – even linear ones – under which $\pi_n(S)$ is stable. This is an old question for which not many nontrivial answers were known – especially until 2004; see the discussion preceding Theorem 31.38 below. Certainly, some easy answers have long been known. For example, if $p(x) \in \pi(\mathbb{C}) = \mathbb{R}[x]$ is real-rooted, then so are:

1. its product $p(x)q(x)$ with a real-rooted polynomial $q(x)$.
2. the ‘shift’ $ap(bx + c)$ for scalars $a, b, c \in \mathbb{R}$, $a \neq 0$.
3. the derivative $p'(x)$ – this is Rolle’s theorem. Notice, the derivative operator commutes with all additive shifts/translations.
4. ‘multiplicative differentiation’ $xp'(x)$, again by Rolle’s theorem. In contrast to the preceding operation, this operator commutes with all multiplicative shifts/dilations.
5. the ‘inversion’ $x^{\deg(x)}p(1/x)$, whose roots are 0 if $p(0) = 0$ and the reciprocals of the nonzero real roots of $p$.

All but the first of these operations also are answers to the more general question, of understanding linear transformations on $\pi_n(\mathbb{C})$ that do not increase the number of non-real roots $Z_{nr}(\cdot)$. In particular, they answer our first question, of understanding linear operators preserving real-rootedness. A related, second question involves understanding which (linear) operations $T$ preserve real-rootedness, now only on real polynomials with all non-positive roots (or all non-negative roots). Notice that the operations above are also (positive) examples of such linear operations (e.g. assuming $q(x)$ also has one-sided roots).

Often, one assumes a further restriction on the linear map $T$ in order to have more structure to work with. Two such conditions are that $T$ commutes with the usual/additive derivative $\partial$, or with the multiplicative derivative $x\partial$. It is easily seen that the latter operators are of
the form
\[ T(x^k) = \gamma_k x^k, \quad \gamma_0, \gamma_1, \ldots \in \mathbb{R} \]
(or \( \gamma_k \in \mathbb{C} \) if one considers the analogous problem). Similarly, one can show that the former operators each have a ‘power series expansion’
\[ T = \sum_{k=0}^{\infty} \hat{T}(k) \partial^k, \quad \hat{T}(k) \in \mathbb{R}. \]

Notice here that applying such an operator to a polynomial only requires finitely many terms, so the sequence \( \hat{T}(k) \) can be arbitrary. One associates to this operator its symbol:
\[ G_T(z) := \sum_{k=0}^{\infty} \hat{T}(k) z^k. \]  

In this section we present the characterizations of both classes of operators – commuting with additive and multiplicative differentiation \( \partial, x \partial \) respectively – and they both bear the name of Pólya (with Benz and Schur, respectively). We will mostly focus on the latter case, in which the scalars \( \gamma_k \) are called multipliers. Thus, the Pólya–Schur theorem classifies all multiplier sequences (which preserve real-rooted polynomials).

For now, we return to the opening discussion of preserving real-rootedness, or more generally, diminishing the number of non-real roots. Our journey begins with a classical result due to Poulain [229] in 1867, answering a question of Hermite [125] from the previous year:

**Theorem 31.3** (Hermite–Poulain). Suppose \( q(x) = \sum_{k=0}^{m} q_k x^k \) is a polynomial with \( q_0, q_m \neq 0 \) and all real roots.

1. If \( p(x) \in \mathbb{R}[x] \), then \( Z_{nr}(q(\partial)p) \leq Z_{nr}(p) \); here the differential operator \( q(\partial) \) acts via:
   \[ (q(\partial)p)(x) := \sum_{k=0}^{m} q_k p^{(k)}(x). \]

2. If \( q \) has only positive (respectively, negative) zeros, and \( A \in \mathbb{R} \), then the number of zeros in \( [A, \infty) \) (respectively, \( (-\infty, A] \)) of \( q(\partial)p \) exceeds that of \( q \).

Thus, the Hermite–Poulain theorem extends Rolle’s theorem – i.e., that differentiation diminishes the number of non-real roots of a polynomial – which is the special case \( q(x) = x \).

**Proof.** To show (1), write \( q(x) = q_m \prod_j (x - \alpha_j) \), where no \( \alpha_j \) is zero since \( q_0 \neq 0 \). Since \( q(\partial) = q_m \prod_j (\partial - \alpha_j) \), it suffices to show that \( (\partial - \alpha)p(x) \) has at least as many real roots as \( q \). We now present Poulain’s proof of this, in a sense ‘differential-equation theoretic’:

\[ (\partial - \alpha)p(x) = e^{\alpha x} \partial (e^{-\alpha x} p(x)), \]

it suffices to show \( p(x) \) has at most as many real roots as \( (e^{-\alpha x} p(x))' \), where \( \alpha \neq 0 \). This follows by Rolle’s theorem, since \( e^{-\alpha x} p(x) \) vanishes at the roots of \( p \) as well as \( \alpha x \to \infty \).

(This is the trick that was used in proving the weak and strong versions of Descartes’ rule of signs, in Lemma 5.2 and Theorem 10.3 respectively.)

This shows the first part, but also the second: the preceding sentence suggests how to proceed if all roots of \( q \) are nonzero, with a common sign. Indeed, a small refinement of the preceding proof now works on \([A, \infty)\) (respectively, \( (-\infty, A] \)). \[\square\]

Notice that the operator \( q(\partial) \) commutes with ‘additive’ differentiation \( \partial \), a notion discussed above – and it preserves real-rootedness. A complete characterization of such linear preservers was carried out by Benz, in *Comment. Math. Helv.* in 1934:
Theorem 31.4 (Pólya–Benz theorem, [25]). Suppose $T$ is a linear operator on the space of complex polynomials, which commutes with differentiation $\partial$. Then $T$ preserves real-rootedness if and only if its symbol $G_T(z) := \sum_{k=0}^{\infty} \hat{T}(k) z^k$, defined in (31.2), is an entire function that is either zero or in the Laguerre–Pólya class $\mathcal{LP}_2$ (with $C = G_T(0)$ now a nonzero complex number, not just in $\mathbb{R}$).

31.1. Multiplier sequences and early results. In the remainder of this section, we focus on the linear transformations which preserve real-rootedness, commute with ‘multiplicative differentiation’ $x\partial$, and turn out to be intimately linked to our main objects of focus: Pólya frequency functions. These are the so-called ‘diagonal transforms’ of $\pi$ differentiation $x\partial$ on the linear transformations which preserve real-rootedness, commute with ‘multiplicative $\partial$.

Example 31.6 (Laguerre, 1884). Given a real number $a > 0$ and an integer $k > 0$, the sequence
\[ a(a + 1) \cdots (a + k - 1), \quad (a + 1)(a + 2) \cdots (a + k), \quad \ldots \]
is a multiplier sequence which preserves the real-rootedness/one-sidedness of roots – and more generally, diminishes the number of non-real roots. This follows from a more general result by Lagrange – see Theorem 31.8(2), specialized here to $q(x) = (x+a)(x+a+1) \cdots (x+a+k-1)$.

In their famous 1914 work [224] in $J. \text{reine angew. Math.}$, Pólya and Schur provided ‘algebraic’ and ‘transcendental’ characterizations of the above two classes of multipliers; these are explained below. That said, the study of such multipliers had begun well before. We present here a quintet of well-known results in this direction – these are shown presently:

- by Laguerre, in 1882 in $C. \text{R. Acad.}$ and in 1884 in $Acta Math.$;
- by Maló in 1895 in $J. \text{Math. Spéc.}$ (this is briefly used in the next part of the text);
- by Jensen in 1913 in $Acta Math.$; and
- by Schur in 1914, in $J. \text{reine angew. Math.}$.

To state and prove these, define the Schur composition $\odot$ of two polynomials/power series:
\[ p(x) = \sum_{j \geq 0} p_j x^j, \quad q(x) = \sum_{j \geq 0} q_j x^j, \quad \implies \quad (p \odot q)(x) := \sum_{j \geq 0} j!p_j q_j x^j. \quad (31.7) \]

Theorem 31.8. Suppose $p(x) = p_0 + p_1 x + \cdots + p_n t^n$ and $q(x) = q_0 + q_1 x + \cdots + q_m x^m$ are polynomials in $\mathbb{R}[x]$, with $m, n \geq 0$ and $q$ real-rooted.

1. (Laguerre [175], 1882.) The polynomial $\sum_{k \geq 0} (q_k/k!) x^k$ is real-rooted. In other words, $1, 1, 1/2!, 1/3!, \ldots$ is a multiplier sequence of the first kind.

2. (Laguerre [177], 1884.) Suppose $q \neq 0$ on $[0, n]$. Then the polynomial
\[ \hat{q}(x) := q(0)p_0 + q(1)p_1 x + \cdots + q(n)p_n x^n \]
has at least as many real roots as $p$, that is, $Z_{nr}(\hat{q}) \leq Z_{nr}(p)$.
For an aperture $0 \leq \alpha \leq \pi$, let $S_\alpha$ denote an open sector in the complex plane, given by
\begin{equation}
S_\alpha := \{ z \in \mathbb{C} : \arg(z) \in (\theta_\alpha, \theta_\alpha + \alpha) \}
\end{equation}
for a fixed `initial angle' $\theta_\alpha$. Similarly, let $S_\beta$ denote an open sector for $\beta \in [0, \pi]$ and fixed initial angle $\theta_\beta$. Now suppose $p(z) = \sum_{k=0}^{n} p_k z^k$ and $q(z) = \sum_{k=0}^{m} q_k z^k$ are complex polynomials, with $p_n, q_m \neq 0$. If $p(z), q(z)$ have all roots in the sectors $S_\alpha, S_\beta$ respectively, then their Schur composition $p \odot q$ has all roots in the open sector
\begin{equation}
-S_\alpha S_\beta := \{ -z : z \in S_\alpha, z \in S_\beta \}.
\end{equation}

Proof. First suppose $(p \odot q)(z) = 0$ for some $z \not\in -S_\alpha S_\beta$. For any such $z$, (by abuse of notation) $-zS_\alpha^{-1}$ is then disjoint from $S_\beta$, so we can embed both of these in open half-planes (i.e. open sectors of aperture $\pi$) $-zS_\alpha^{-1} \subset S_1$ and $S_\beta \subset S_2$ such that $S_1 \cap S_2 = \emptyset$. In particular, it suffices to show the result for $\alpha = \beta = \pi$. Now for a second reduction: the polynomials
\begin{align*}
p_1(z) := p(-iz e^{i \theta_\alpha}) &= \sum_{j=0}^{n} p_{1,j} z^j, \\
q_1(z) := q(-iz e^{i \theta_\beta}) &= \sum_{j=0}^{m} q_{1,j} z^j
\end{align*}
can be verified to have all of their roots in the left half-plane in $\mathbb{C}$, i.e. in
\begin{equation}
L := \{ z \in \mathbb{C} : \Re(z) < 0 \}.
\end{equation}

Hence in this ‘reduction’ case, if we show the following claim – that the polynomial
\begin{equation}
(p_1 \odot q_1)(z) := (p \odot q)(-ze^{i(\theta_\alpha + \theta_\beta)})
\end{equation}
has no roots in $(0, \infty)$ – then $(p \odot q)(z)$ has no roots in $-S_\alpha S_\beta$, as desired.

While this claim can be shown using Grace’s Apolarity Theorem (see e.g. [291]), we mention de Bruijn’s direct argument. First claim that if $\lambda \in (0, \infty)$, $\eta \in L$, and $P(z)$ is any nonzero polynomial with degree $n \geq 0$, leading coefficient $P_n \neq 0$, and all roots in $L$, then $(\lambda \partial - \eta)P$ is not identically zero and its roots still lie in $L$. Indeed, write $P(z) = P_n \prod_{j=1}^{n} (z - \xi_j)$; then
\begin{equation}
(\lambda \partial - \eta)P(z) = P(z) \left( \sum_{j=1}^{n} \frac{\lambda}{z - \xi_j} - \eta \right), \quad \forall z \in \mathbb{C} \setminus L.
\end{equation}

The first factor on the right is nonzero by assumption, while each summand and $-\eta$ both have positive real part, so that the second factor on the right is in $-L$. It follows that the
nonzero polynomial \((\lambda \partial - \eta)P(z)\) again has all roots in \(L\). Now start with
\[
P(z) = p_1(z) = p_{1,n} \prod_{j=1}^{n}(z - \xi_j), \quad q_1(z) = q_{1,m} \prod_{j=1}^{m}(z - \eta_j),
\]
and apply the above reasoning inductively for each \(\eta = \eta_j\), to conclude that
\[
(q_1(\lambda \partial)p_1)(z) = q_{1,0}p_1(z) + q_{1,1}\lambda p_1'(z) + \cdots + q_n\lambda^n p_1^{(n)}(z)
\]
has all its roots in \(L\). In particular, it does not vanish at 0, so
\[
0 \neq q_{1,0}p_{1,0} + 1!q_{1,1}p_{1,1}\lambda + \cdots + l!q_{1,l}p_{1,l}\lambda^l, \quad l = \min\{m, n\}.
\]
This precisely says \(0 \neq (p_1 \circ q_1)(\lambda)\); as \(\lambda \in (0, \infty)\) was arbitrary, the proof is complete. \(\square\)

This result led de Bruijn to derive a host of corollaries, which we discuss before returning
to the proof of Theorem 31.8. The first is the ‘closed’ sector version of Theorem 31.9

**Corollary 31.12** (de Bruijn, [53]). Suppose \(p, q \neq 0\) are complex polynomials, whose roots all lie in closed sectors \(\overline{S}_\alpha, \overline{S}_\beta\) respectively. If both \(\alpha, \beta \in [0, \pi]\), then either \(p \circ q \equiv 0\) or it has all roots in the sector \(-\overline{S}_\alpha, \overline{S}_\beta\). (If an aperture is 0, that closed sector is a half-line.)

**Proof.** This is obtained from Theorem 31.9 via limiting arguments (and the continuity of roots, e.g. by Hurwitz’s theorem), when the apertures of both sectors are positive. If either aperture is zero, write the corresponding half-line as an intersection of a sequence of nested closed sectors with positive apertures, and apply Corollary 31.12 for each of these. \(\square\)

The next corollary shows two results of Weisner, from his 1942 paper in *Amer. J. Math.*:

**Corollary 31.13** (Weisner, [291] Theorem 1 and its Corollary]). Suppose \(p, q\) are polynomials in \(\mathbb{R}[x]\) with \(q(x)\) having all real roots.

(1) If the roots of \(p(x)\) lie in a closed sector \(\overline{S}_\alpha\) with aperture \(\leq \pi\), then either \(p \circ q \equiv 0\) or it has all roots in the ‘double sector’ \(\pm \overline{S}_\alpha\).

(2) If the roots of \(q\) are moreover negative, and a closed sector with aperture in \([0, \pi]\) contains the roots of \(p\), it also contains the roots of \(p \circ q\).

**Proof.** For the first part: if \(p \circ q \equiv 0\) or \(S_\alpha\) has aperture \(\pi\) then the result is immediate; now suppose neither condition holds. Apply Corollary 31.12 twice: with \(S_\beta\) the lower and upper half-planes \(\pm iL\), where \(L\) is the left half-plane \((31.11)\). It follows that the roots of \(p \circ q\) lie in
\[
(-iL \cdot \overline{S}_\alpha) \cap (iL \cdot \overline{S}_\alpha) = \overline{S}_\alpha \cup -\overline{S}_\alpha.
\]
This shows the first part; for the second, apply Corollary 31.12 with \(\overline{S}_\beta = (-\infty, 0]\). \(\square\)

The next corollary is a mild strengthening of another result by de Bruijn [53]; his proof works for the following as well.

**Corollary 31.14** (de Bruijn, [53]). Fix scalars \(-\infty < \delta < 0 \leq \Delta \leq +\infty\), and suppose the polynomials \(p, q \in \mathbb{C}[z] \setminus \{0\}\) both have all roots in the strip \(\Im(z) \geq -\Delta\). Then the polynomial
\[
\varphi_\delta(z) := \sum_{k=0}^{\infty} \frac{\delta^k}{k!} p^{(k)}(z)q^{(k)}(z)
\]
has all roots also in the same strip \(\Im(z) \geq -\Delta\).

Notice that \(\varphi_\delta(z)\) has the same form as the Schur composition, but in the ‘other’ parameter. Namely, \(\varphi_\varphi(0) = (p \circ q)(z)\). This is used in the proof.
Proof. Fix a non-real number $w$ with $\Im(w) < -\Delta$. We need to show that $\varphi_\delta(w) \neq 0$. To do so, consider the polynomials

$$P(z) := \sum_{k=0}^{\infty} z^k \frac{p_k(w)}{k!} = p(z + w), \quad Q(z) := \sum_{k=0}^{\infty} z^k \frac{q_k(w)}{k!} = q(z + w).$$

By assumption, the zeros $z$ of $P,Q$ lie in the upper half-plane, which is an open sector with aperture $\pi$. Hence by Theorem 31.9 the roots $z$ (not $w$) of their Schur composition

$$\varphi_z(w) = (P \odot Q)(z)$$

all lie in $\mathbb{C} \setminus (-\infty, 0]$. In particular, $\varphi_\delta(w) = (P \odot Q)(\delta) \neq 0$, by choice of $\delta$. \hfill \Box

Remark 31.15. If instead $p,q$ have all roots in the strip $\Im(z) \leq \Delta$, a similar argument shows that so does $\varphi_\delta$. Intersecting these two results yields the version in de Bruijn’s paper [53], i.e. for the strip $|\Im(z)| \leq \Delta$. In particular, if $\Delta = 0$, this also shows that if $p,q$ are real-rooted, then so is $\varphi_\delta$. However, the above approach has the advantage that de Bruijn’s results also yield root-location results in asymmetric strips $\Im(Z) \in [-\Delta, \Delta']$ for $0 \leq \Delta \neq \Delta'$.

Finally, we prove the classical results stated above.

Proof of Theorem 31.8. We show the five parts in a non-sequential fashion, beginning with part (2) by Laguerre. It turns out this is precisely the counterpart of the Hermite–Poulain theorem 31.3, now for the ‘multiplicative differential’ operator $x\partial$ instead of the usual derivative $\partial$. Indeed, write $q(x) = q_m \prod_{j=1}^{m} (x - \alpha_j)$ with $\alpha_j \in \mathbb{R}$, and compute at each monomial:

$$q(x\partial)(x^k) = q_m \prod_{j=1}^{m} (x\partial - \alpha_j)(x^k) = x^k \cdot q_m \prod_{j=1}^{m} (k - \alpha_j) = q(k)x^k.$$ 

Since the factors $(x\partial - \alpha_j)$ pairwise commute, it again suffices to show $(x\partial - \alpha)p(x)$ has at least as many real roots as $p$, if $\alpha \in \mathbb{R} \setminus [0, \deg(p)]$ (so we may assume $\deg(p) = n$). We now study the order of the root at 0, and the positive/negative roots of both polynomials $p$ and $(x\partial - \alpha)p$. The orders of the root at zero agree. Coming to positive roots, write using Poulain’s idea:

$$(x\partial - \alpha)p(x) = x^{\alpha+1} \partial(x^{-\alpha}p(x)).$$

Now if $\alpha < 0$ then $x^{-\alpha}p(x)$ has an additional zero at $x = 0$; while if $\alpha > n$, $x^{-\alpha}p(x)$ has an additional ‘zero’ at $x = +\infty$. In both cases, one argues as in the proof of the Hermite–Poulain theorem 31.3 above, using Rolle’s theorem to obtain that the positive roots of $(x\partial - \alpha)p(x)$ are at least as many as that of $p(x)$ – if $x > 0$. A similar argument holds for $x < 0$.

Next, part (5), i.e. the Schur composition theorem, is a special case of Corollary 31.13(1), in which the roles of $p$ and $q$ are reversed, and one takes $S_\alpha$ to be a closed semi-axis in $\mathbb{R}$. To now obtain the result of Laguerre (part (1)), first apply ‘inversion’ to observe that if

$$q(x) = q_0 + q_1x + \cdots + q_rx^r$$

is real-rooted, then so is its ‘inversion’ (possibly up to a power of $x$),

$$Q(x) := q_m + q_{m-1}x + \cdots + q_0x^m.$$ 

Now let $P(x) := (1 + x)^m$; then by part (5) the polynomial

$$(P \odot Q)(x) = q_m + q_{m-1} \frac{m!}{(m-1)!} x + q_{m-2} \frac{m!}{(m-2)!} x^2 + \cdots + q_0 m! x^m$$

is real-rooted, then so is its ‘inversion’ (possibly up to a power of $x$),

$$Q(x) := q_m + q_{m-1} x + \cdots + q_0 x^m.$$ 

Finally, we prove the classical results stated above.
is also real-rooted. Again invert this (and multiply by powers of \( x \) if necessary), and then divide throughout by \( m! \) to obtain Laguerre’s result, i.e. part (1).

The proof of Jensen’s result (part (4)) is similar to that of part (1). Start with a real-rooted polynomial \( p(x) \) and take the Schur composition with \( (1 + x)^n \). Hence the polynomial

\[
\sum_{k \geq 0} p_k \cdot n(n-1) \cdots (n-k+1) \cdot x^k
\]

is real-rooted. Replacing \( x \) by \( x/n \), so is the polynomial

\[
p_0 + p_1 x + (1 - \frac{1}{n})p_2 x^2 + \cdots
\]

Since this holds for all polynomials, Jensen’s result follows. Finally, Maló’s result (part (3)) immediately follows by combining parts (1) and (5).

\[\square\]

**Remark 31.16.** For additional variants of Theorem 31.8(2) due to Laguerre – involving the multiplicative differentiation operator \( x \partial \), see Pinkus’s paper [217].

**Remark 31.17.** By using inversion twice, and Theorem 31.8(1) in between, we obtain yet another result of Laguerre: if \( q(x) = q_0 + \cdots + q_m x^m \) is real-rooted, then so is the polynomial

\[
\frac{q_0}{m!} + \frac{q_1 x}{(m-1)!} + \cdots + \frac{q_m x^m}{0!}.
\]

Some concluding remarks: the study of root-location of real/complex polynomials remains evergreen; see e.g. the very recent work [55] on “zero-sector reducing” linear operators on \( \mathbb{R}[x] \) (in addition to related highlights of modern mathematics, described presently). Finally, we present without proof, a result of Schur [259] and Szegő [276] – as well as one by Pólya – which involve a different kind of ‘composition’:

**Theorem 31.18.** Let \( n \geq 1 \) be an integer, and \( p(z), q(z) \) be polynomials given by

\[
p(z) := \sum_{k=0}^{n} \binom{n}{k} p_k z^k, \quad q(z) := \sum_{k=0}^{n} \binom{n}{k} q_k z^k.
\]

(1) (Schur–Szegő composition theorem.) If \( q \) is real-rooted with all roots in \((-1,0)\), and the roots of \( p \) lie in a convex region \( K \) containing the origin, then all roots of the following ‘composition’ of theirs lie in \( K \):

\[
h(x) := \sum_{k=0}^{n} \binom{n}{k} p_k q_k z^k.
\]

(2) (Pólya – see [291] Theorem 3.) If \( q \) is real-rooted, and the roots of \( p \) lie in a sector \( S \), then the roots of \( h(x) \) lie in the double sector \( \pm S \).

31.2. **Laguerre–Pólya entire functions.** We now proceed toward Pólya and Schur’s characterizations of multiplier sequences. Definition 31.5 implies that the multipliers of the first kind form a sub-class of the second kind. It turns out that these characterizations are related to the polynomials being acted upon, and we begin by understanding these polynomials.

Recall that the Hermite–Poulain theorem 31.3(1) says that if \( p, q \) are real-rooted polynomials, then \( q(\partial)p \) is also thus. (As a special case, Rolle’s theorem says that the class of real-rooted polynomials is closed under differentiation.) If we now take limits of such polynomials – in a suitable sense – then we may expect that such properties hold as well. This does turn out to be true in several cases:
(1) As just mentioned, Theorem 31.3(1) says that if \( q \) is a real-rooted polynomial, then the differential operator \( q(\partial) \) preserves real-rootedness on polynomials.

(2) More generally, the same turns out to hold if one considers functions that are the limits, uniform on every compact subset of \( \mathbb{C} \), of real-rooted polynomials.

(3) Notice for the exponential function \( e^{\alpha x} \) that \( q(\partial) e^{\alpha x} = q(\alpha) e^{\alpha x} \). ‘Dually’, it turns out that differential operator \( e^{\alpha \partial} \) (for \( \alpha \in \mathbb{R} \)) preserves real-rootedness on polynomials. Indeed, this clearly holds if \( e^{\alpha \partial} \) acts on linear polynomials, so we need to show that the same property is preserved under products. But this is a straightforward calculation:

\[
e^{\alpha \partial} (p(x) q(x)) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sum_{k=0}^{\infty} \binom{n}{k} p^{(k)}(x) q^{(n-k)}(x) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} p^{(k)}(x) \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} q^{(j)}(x)
\]

(31.19)

where all sums are finite, since \( p, q \) are polynomials. In fact, since the product rule says that the locally nilpotent operator \( \partial \) is a derivation of the algebra \( \mathbb{R}[x] \) of polynomials, (31.19) is simply an instance of the fact that a nilpotent derivation of an algebra \( R \) exponentiates to an algebra automorphism of \( R \).

In the spirit of the two questions following Definition 31.1, notice that the final case (and others) works even if \( \alpha > 0 \) and \( p, q \) have non-positive roots, since it works for each linear factor. Thus, one can also consider limits – again uniform on every compact subset of \( \mathbb{C} \) – of polynomials with real roots which lie on one semi-axis (i.e., are all non-positive or non-negative). It turns out that such limiting functions are always entire, and have been completely characterized, in famous works of Laguerre [175] (1882) and Pólya [220] (1913). These characterizations, which are now presented, preceded – and motivated – Pólya and Schur’s work on multipliers and their classification.

**Definition 31.20.** An entire function \( \Psi(z) \neq 0 \) is in the first Laguerre–Pólya class, if it admits a Hadamard–Weierstrass factorization

\[
\Psi(s) = C s^m e^{\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s), \quad \text{with } C \in \mathbb{R}, \quad m \in \mathbb{Z}_{\geq 0}, \quad \delta, \alpha_j \geq 0, \quad \sum_j \alpha_j < \infty. \tag{31.21}
\]

Similarly, \( \Psi \neq 0 \) is in the second Laguerre–Pólya class, if it admits a Hadamard–Weierstrass factorization

\[
\Psi(s) = C s^m e^{-\gamma s^2 + \delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s)e^{-\alpha_j s}, \quad \text{with } C \in \mathbb{R}, \quad m \in \mathbb{Z}_{\geq 0}, \quad \gamma \geq 0, \quad \delta, \alpha_j \in \mathbb{R},
\]

and

\[
\sum_j \alpha_j^2 < \infty. \tag{31.22}
\]

We will denote these classes of functions by \( \mathcal{LP}_1 \) and \( \mathcal{LP}_2 \), respectively.

A few observations are in order: the first is the inclusion between these classes: \( \mathcal{LP}_1 \subset \mathcal{LP}_2 \). Second, all functions in \( \mathcal{LP}_2 \), whence in \( \mathcal{LP}_1 \), have real roots. Third, a function \( \Psi(s) \in \mathcal{LP}_1 \) has all non-positive roots, so \( \Psi(-s) \) is also entire, with all non-negative roots.

We now have the following relationship between the Laguerre–Pólya classes \( \mathcal{LP}_1, \mathcal{LP}_2 \), and the discussion on uniform limits of real-rooted polynomials previous to Definition 31.20.

**Theorem 31.23.**

(1) (Laguerre, [175].) Suppose an entire function \( \Psi(s) \) lies in the class \( \mathcal{L}P_1 \) (resp., in \( \mathcal{L}P_2 \)). Then there exists a sequence \( \psi_n(s) \) of polynomials with all roots in \( (-\infty, 0] \) (resp., in \( \mathbb{R} \)), that converges locally uniformly on \( \mathbb{C} \) (i.e. on every compact subset).

(2) (Pólya, [220].) Conversely, fix a neighborhood \( U \subset \mathbb{C} \) of the origin. Suppose a sequence \( \psi_n(s) \) of polynomials, each of which has roots in \( (-\infty, 0] \) (resp., in \( \mathbb{R} \)), converges uniformly on \( U \) to a function not identically zero. Then the \( \psi_n \) converge locally uniformly on \( \mathbb{C} \) to an entire function \( \Psi \) in the class \( \mathcal{L}P_1 \) (resp., \( \mathcal{L}P_2 \)).

In fact, one can write down explicit families of polynomials converging to Laguerre–Pólya entire functions: the sequences of polynomials

\[
\psi_{1,n}(s) := C s^m (1 + \frac{\delta s}{n})^n \prod_{j=1}^{n} (1 + \alpha_j s), \\
\psi_{2,n}(s) := C s^m (1 - \frac{\gamma s^2}{n})^n (1 + \frac{\delta s}{n})^n \prod_{j=1}^{n} (1 + \alpha_j s)(1 - \frac{\alpha_j s}{n})^n,
\]

converge to the functions \( \Psi(x) \) in Definition 31.20.

Theorem 31.23(2) was originally obtained in a weaker form by Laguerre, who needed to assume that \( U = \mathbb{C} \) to conclude that \( \Psi \in \mathcal{L}P_1 \) or \( \mathcal{L}P_2 \). The stronger version above is by Pólya in [220], and is required to classify the multiplier sequences (presently). Also note: if instead we consider locally uniform limits of polynomials with non-negative zeros, we obtain functions \( \Psi(-s) \), with \( \Psi(s) \in \mathcal{L}P_1 \).

Remark 31.25. Lindwart–Pólya showed [185] that in the above cases and more general ones, the uniform convergence of polynomials \( \psi_n(s) \) on some disk \( D(0, r) \subset \mathbb{C} \) implies uniform convergence on any compact subset of \( \mathbb{C} \). (The reader may recall here the ‘convergence extension theorems’ of Stieltjes and Vitali.) This has since been extended to smaller sets than \( D(0, r) \), e.g. by Korevaar–Loewner, Levin, and others.

Theorem 31.23 has seen several generalizations in the literature; see the works of Korevaar and Obrechkoff among others, e.g. [171, 172, 209, 210, 211, 212]. We present here a sample result, taken from Levin [182], on uniform limits of polynomials with zeros in a sector in \( \mathbb{C} \):

**Theorem 31.26.** Fix a neighborhood \( U \subset \mathbb{C} \) of the origin. Suppose a sequence \( \psi_n(s) \) of polynomials, each of which has roots in a closed sector \( S_\theta \) with aperture \( \theta < \pi \), converges uniformly on \( U \) to a function not identically zero. Then the \( \psi_n \) converge locally uniformly on \( \mathbb{C} \) to a function of the form

\[
\Psi_\theta(s) = C s^m e^{\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s),
\]

where \( C \in \mathbb{C}^\times \), \( m \in \mathbb{Z}^\geq 0 \), \( \alpha_j, \delta \) are either 0 or lie in \( -\overline{S}_\theta^{-1} := \{ -1/s : 0 \neq s \in \overline{S}_\theta \} \), and \( \sum_j |\alpha_j| < \infty \). Moreover, an entire function \( \Psi \) can be locally uniformly approximated by a sequence \( \psi_n \) with the above properties (over \( \overline{S}_\theta \)) if and only if \( \Psi \) is of the above form \( \Psi_\theta \).

Note, the same polynomials as in (31.24) (with suitable parameter values) converge to \( \Psi_\theta \).

31.3. From Laguerre–Pólya to Pólya–Schur. Finally, we come to Pólya and Schur’s classification of multiplier sequences (see Definition 31.5). We begin with some basic properties.

**Lemma 31.27.** Suppose \( \Gamma = (\gamma_0, \gamma_1, \ldots) \) is a multiplier sequence of the first kind.
Theorem 31.29 (Pólya–Schur, [224]). Given real $\Gamma = (\gamma_k)_{k=0}^{\infty}$, the following are equivalent:

1. $\Gamma$ is a multiplier sequence of the first kind.
2. (Algebraic characterization.) For all $n \geq 0$, the polynomial $\Gamma[(1 + x)^n]$ is real-rooted, with all zeros of the same sign, i.e., in the Laguerre–Pólya class $LP_1$.
3. (Transcendental characterization.) The generating series

$$\Psi_{\Gamma}(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = \Gamma[e^x]$$

belongs to the Laguerre–Pólya class $LP_1$, or else $\Psi_{\Gamma}(-x)$ does so.

We outline why this result holds, modulo Theorem 31.23

Proof-sketch of Theorem 31.29. Suppose $\Gamma[-\cdot]$ is a multiplier of the first kind, and all $\gamma_k \geq 0$. (The case of $\gamma_k$ alternating, via Lemma 31.27, is handled by considering $\Psi_T(-x)\). In this case, one shows that $\Gamma[-\cdot]$ stabilizes the class of $H$-polynomials (ones with all zeros satisfying $\Im(\Gamma(z)) \geq 0$). From this it is easily deduced that if $p(x)$ has all roots in a given sector $S_\theta$ with aperture $0 \leq \theta \leq \pi$, then so does $\Gamma[p(x)]$. This also holds upon replacing $S_\theta$ by any ray in $\mathbb{C}$, by considering decreasing open sectors containing it.

Apply this analysis to $p_n(x) = (1 + x/n)^n$, to obtain that all roots of

$$
\Gamma[p_n(x)] = \sum_{k=0}^{n} \gamma_k \left( \frac{n}{k} \right) (x/n)^k = \sum_{k=0}^{n} \gamma_k \frac{k!}{k!} \prod_{j=1}^{k-1} \left( 1 - \frac{j}{n} \right) x^k
$$

are real and negative. One now shows that this sequence of polynomials forms a normal family, hence converges locally uniformly as $n \to \infty$ to an entire function, which is clearly given by $\Psi_T(x)$. Moreover, $\Psi_T(x) = \Gamma[e^x]$ must be in $\mathcal{LP}_1$.

Conversely, if $\Psi_T(x) \in \mathcal{LP}_1$, then it can be approximated locally uniformly by a sequence of polynomials $\psi_n(x) = \sum_{k \geq 0} \psi_{n,k} x^k$, all of whose roots are in $(-\infty, 0]$. Now suppose $p(x) = \sum_{k \geq 0} p_k x^k$ is a polynomial with all real roots. By Corollary 31.13(1), the polynomial $(\psi_n \circ p)(x)$ is real-rooted. Taking $n \to \infty$, the same holds for $\Gamma[p(x)]$, as desired. □

There is a similar characterization of multipliers of the second kind (see also Theorem 32.5):

Theorem 31.30 (Pólya–Schur, [224]). Given real $\Gamma = (\gamma_k)_{k=0}^\infty$, the following are equivalent:

(1) $\Gamma$ is a multiplier sequence of the second kind.
(2) (Algebraic characterization.) For all $n \geq 0$, the polynomial $\Gamma[(1 + x)^n]$ is real-rooted, i.e., in the Laguerre–Pólya class $\mathcal{LP}_2$.
(3) (Transcendental characterization.) The generating series $\Psi_T(x) = \Gamma[e^x]$ is an entire function, and belongs to the Laguerre–Pólya class $\mathcal{LP}_2$.

Clearly (1) $\implies$ (2). The proof of (2) $\implies$ (3) $\implies$ (1) resembles the corresponding proof for multiplier sequences of the first kind.

Remark 31.31. Notice the ‘reversal’, in a rough sense: $\Psi_T$ in $\mathcal{LP}_1$ acts on – and preserves real-rootedness on – functions in $\mathcal{LP}_2$, and vice versa. This is because acting on larger test sets imposes more constraints and reduces the available functions / generating series.

We close this part with two connections to Pólya frequency sequences – specifically, to the representations of one-sided PF sequences in Theorem 29.11 following the discussion prior to 29.10. First, the latter theorem implies the following 1951 observation:

Theorem 31.32 (Aissen–Edrei–Schoenberg–Whitney, [2]). Suppose $a = (a_n)_{n \geq 0}$ is a real sequence, with $a_0 = 1$, such that its generating function $\Psi_a(x) := \sum_{n=0}^{\infty} a_n s^n$ is entire. Then the following are equivalent:

(1) $a$ is a Pólya frequency sequence, i.e., the bi-infinite matrix $(a_{j-k})_{j,k \in \mathbb{Z}}$ is totally non-negative (where we set $a_n := 0$ for $n < 0$).
(2) The function $\Psi_a(s) = e^{\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s)$, for some $\delta, \alpha_j \in [0, \infty)$ such that $\sum_j \alpha_j < \infty$.
(3) The function $\Psi_a(s)$ belongs to the Laguerre–Pólya class $\mathcal{LP}_1$.
(4) The sequence $(n! a_n)_{n \geq 0}$ is a multiplier sequence of the first kind.

Proof. That (2) $\iff$ (3) and (3) $\iff$ (4) follow from Theorems 31.23 and 31.29 respectively. Finally, that (1) $\iff$ (2) follows from Theorem 29.11 since $\Psi_a$ is entire. □
Remark 31.33. The other connection is that Aissen et al’s representation theorem \[29.11\] implies the Laguerre–Pólya theorem \[31.23\](2) for \(LP_1\). Indeed, as the authors observe in \[2\], suppose \(p_n(s)\) is an infinite sequence of polynomials with \(p_n(0) = 1\) and all roots negative, and suppose \(p_n \to \Psi\). Then the Maclaurin coefficients of each \(p_n\) generate a ‘finite’ Pólya frequency sequence by Corollary \[29.18\] whence so do the coefficients of \(\Psi(s)\). Therefore \(\Psi\) is in \(LP_1\), by Corollary \[31.32\].

31.4. The Laguerre–Pólya class, the Riemann hypothesis, and modern applications of real-rootedness. While at first glance, the definitions do not reveal a connection between the Laguerre–Pólya class and Pólya frequency functions, we saw at the end of the preceding section that there is at least a ‘one-way’ connection. In fact the (remarkable) connection goes both ways – parallel to the theory of Pólya–Schur multipliers – and will be precisely described in the next section, via the bilateral Laplace transform.

To conclude this section, we start with this transform and conduct a very quick tour of some of the gems of modern mathematics – starting with the (not modern) Riemann hypothesis. In 1927 in \textit{J. reine angew. Math.}, Pólya initiated the study of functions \(\Lambda(t)\) such that \(B(\Lambda)(s)\) is real-rooted. Pólya’s work \[222\] was motivated by the Riemann hypothesis, conjectured in 1859 by Riemann \[234\]. It says that the analytic continuation of the Riemann zeta function

\[
\zeta(s) := \sum_{n \geq 1} n^{-s}, \quad \Re(s) > 1
\]

has (trivial zeros at \(s = -2, -4, \ldots\) and) nontrivial zeros all on the critical line \(\Re(s) = 1/2\). An equivalent formulation is via the \textit{Riemann \(xi\)-functions}

\[
\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \Xi(s) = \xi(1/2 + iz),
\]

where \(\Gamma\) is Euler’s gamma function. Note that \(\xi(s) = \xi(1-s)\) and \(\Xi(s) = \Xi(-s)\). Now the Riemann hypothesis is equivalent to the fact that \(\Xi\) has only real zeros. (In fact this was how Riemann stated his conjecture.) Since the function \(\Xi\) is entire of order one, this leads to a folklore result, which can be found in e.g. Pólya’s 1927 work \[223\]:

**Theorem 31.34.** The Riemann hypothesis is equivalent to the statement: \(\Xi \in LP_2\).

Thus, the Laguerre–Pólya class occupies a special place in analytic number theory.

The Riemann hypothesis is one of the most studied problems in modern mathematics. It was originally formulated in the context of the distribution of prime numbers, and has far-reaching consequences. Most of the work toward settling this conjecture has employed methods from complex analysis and analytic number theory.

We now present three mutually inter-related reformulations of the Riemann hypothesis from analysis, very recently presented by Gröchenig, and ‘orthogonal’ to the aforementioned methods. The main ingredients are Theorem \[31.34\] and Schoenberg’s Theorem \[32.9\] below, which asserts that a function is a Pólya frequency function if and only if its bilateral Laplace transform is the reciprocal of a function \(\Psi\) in the Laguerre–Pólya class with \(\Psi(0) > 0\):

**Theorem 31.35** (Gröchenig, 2020, \[109\]). Let \(1/2 + it_0\) be the first zero of the zeta function on the critical line \(\Re(s) = 1/2\). The Riemann hypothesis holds, if and only if there exists a
Folksy frequency function \( \Lambda \) satisfying:

\[
\frac{1}{\Xi(s)} = B(\Lambda)(s), \quad |\Re(s)| < t_0.
\]

Taking the Fourier transform instead of the Laplace transform, this yields the Riemann hypothesis if and only if the function

\[
\Lambda(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\xi(u+1/2)} e^{-ixu} \, du
\]

is a Pólya frequency function. Equivalently, write \( \Xi(s) = \Xi_1(-s^2) \) (since \( \Xi \) is even); thus \( \Xi_1 \) is entire of order \( 1/2 \). Now the Riemann hypothesis holds, if and only if there exists a one-sided Pólya frequency function \( \Lambda \) with support in \([0, \infty)\), and a scalar \( \alpha < 0 \), satisfying:

\[
\frac{1}{\Xi_1(s)} = B(\Lambda)(s), \quad \Re(s) > \alpha.
\]

We follow the above discussion with a disparate development, from mathematical physics: the Lee–Yang program. In material science, it has been observed that certain magnetic materials lose magnetism at a critical temperature. This phase transition is called the Curie point/temperature. Such phenomena in statistical physics led to work on the Ising and other models, by Ising, Onsager, and several others. In the 1950s, Lee and Yang related this study to locating zeros of the ‘partition function’ associated with the model (and the underlying Lee–Yang measure). As a result, they were able to compute the phase transition for the Ising model. (This was part of their body of work that earned them the 1957 Nobel Prize in Physics.) Lee–Yang showed in [180, 299] that under desirable conditions, all zeros of the partition function are purely imaginary, or under a specialization, all on the circle:

**Theorem 31.36** (Lee–Yang, 1952). Given an integer \( n \geq 1 \), a matrix \( J \in [0, \infty)^{n \times n} \) (the ‘ferromagnetic coupling constants’), and magnetic fields \( h_1, \ldots, h_n \in \mathbb{C} \), define the corresponding ‘partition function’

\[
Z_J(h) := \sum_{\sigma \in \{-1, 1\}^n} \exp(\sigma^T J \sigma + \sigma^T h).
\]

Here, the \( \sigma_j \) are the ‘spins’. Then \( Z_J(h) \) is nonzero if \( \Re(h_j) > 0 \forall j \), and all zeros \( h \in \mathbb{C} \) of \( Z_J(h, h, \ldots, h) \) are purely imaginary.

This result leads to the so-called Lee–Yang circle theorem (see also [242] for its connections to the original work of Lee–Yang):

**Theorem 31.37.** Let \( A = A^* \in \mathbb{C}^{n \times n} \) be Hermitian, with all \( a_{jk} \) in the closed unit disk. Then the multi-affine Lee–Yang polynomial

\[
f(z_1, \ldots, z_n) := \sum_{S \subseteq [n]} \prod_{j \in S} z_j \prod_{k \in S} a_{jk}
\]

has no zeros in \( \mathbb{D}^n \), where \( \mathbb{D} \) is the open unit disk. In particular, the map \( f(z, \ldots, z) \) has all zeros on the unit circle \( S^1 \).

An essential part of the ensuing analysis in the program initiated by these results, involves understanding operators preserving spaces of polynomials with roots lying in / avoiding a prescribed domain in \( \mathbb{C} \). More precisely, we are back to understanding linear operators on spaces of (multivariate) polynomials, preserving (higher dimensional) versions of stability, real-rootedness, and hyperbolicity. This includes higher-dimensional versions of Pólya–Schur
Theorem 31.38 ([15]). Let \( T : \pi_n(\mathbb{C}) \rightarrow \pi(\mathbb{C}) \) be a linear operator on polynomials. The following are equivalent:

1. \( T \) preserves real-rootedness, i.e., \( T : \pi_n(\mathbb{R}) \rightarrow \pi(\mathbb{R}) \).
2. The linear map \( T \) has rank at most 2, and is of the form \( T(p) = \alpha(p)f + \beta(p)g \), where \( \alpha, \beta : \mathbb{R}[x] \rightarrow \mathbb{R} \) are linear functionals, and \( f, g \in \pi(\mathbb{R}) \) have ‘interlacing’ roots, i.e. between any two (possibly equal) roots of \( f \) lies a root of \( g \), and vice versa.
3. The symbol of \( T(x,y) \), given by \( G_T(x,y) := T((x+y)^n) = \sum_{k=0}^{n} \binom{n}{k} T(x^k)y^{n-k} \), is stable. In other words, it has no root \((x,y)\) with \( \Im(x), \Im(y) > 0 \).
4. The symbol of \( T(x,-y) \), given by \( G_T(x,-y) := T((x-y)^n) \), is stable.

Here Borcea–Brändén define stability in keeping with Levin’s notion of \( H \)-stability; note that a univariate real polynomial \( q(x) \) is stable (i.e., has no roots in the upper half-plane) if and only if it is real-rooted: \( Z_{nr}(q) = 0 \).

Returning to the historical account, in [44, 45, 46, 47, 48] Borcea–Brändén also

- characterized linear operators preserving \( S \)-stability for other prescribed subsets \( S \subset \mathbb{C} \) (including – in [45] – \( S \) a line, a circle, a closed half-plane, a closed disk, and the complement of an open disk);
- developed a multivariable Szász principle and multidimensional Jensen multipliers;
- proved three conjectures of C.R. Johnson;
- presented a framework that incorporated a vast number of (proofs of) Lee–Yang and Heilmann–Lieb type theorems;

among other achievements. See also a detailed listing of the modern literature in the field, in [45]. Together with Liggett in [49], they also developed the theory of negative dependence for “strongly Rayleigh (probability) measures”, enabling them to prove various conjectures of Liggett, Pemantle, and Wagner, and to construct counterexamples to other conjectures on log-concave sequences. See also the survey of Wagner [283] for more details and connections.

The above works, originating from the Laguerre–Pólya–Schur program on the location of roots of polynomials, were taken forward very recently, by Marcus, Spielman, and Srivastava. In another series of striking papers that used interlacing families of polynomials, real stability, and hyperbolicity (among many other ingredients), the authors proved the longstanding Kadison–Singer conjecture [152], and also showed the existence of bipartite Ramanujan (expander) graphs of every degree and every order (settling conjectures of Lubotzky and Bilu–Linial). See [194, 195, 196]. These contributions are only a small part of a larger and very active current area of research, referring to the geometry of roots of polynomials. Stability of dynamical systems, global optimization, and in particular control theory were and are immediate beneficiaries of the theoretical advances. Under the covers of the newly founded SIAM Journal of Applied Algebra and Geometry many exciting discoveries touching the subject have appeared, with key concepts such as linear matrix inequalities, hyperbolic polynomials, spectrahedra, and semi-definite programming.
32. Schoenberg’s results on Pólya frequency functions.

Having discussed the Laguerre–Pólya entire functions and the Pólya–Schur multiplier sequences, we return to our primary objects of interest: Pólya frequency functions. The goal in this section is to discuss some of the foundational results on these functions.

We first recall the definition of these functions and two closely related classes of functions:

**Definition 32.1.** Suppose \( \Lambda : \mathbb{R} \to [0, \infty) \) is Lebesgue measurable.

1. \( \Lambda \) is said to be **totally non-negative** (TN) if given an integer \( p \geq 1 \) and tuples \( x, y \in \mathbb{R}^p \), \( \det T_{\Lambda}[x ; y] := \det (\Lambda(x_j - y_k))_{j,k=1}^p \geq 0 \).
2. We will say \( \Lambda \) is **non-Dirac** if \( \Lambda \) does not vanish at least at two points.
3. \( \Lambda \) is a **Pólya frequency (PF) function** if \( \Lambda \) is non-monotone – equivalently, integrable with mass a positive real number.

The equivalence in the third assertion was proved in Proposition 28.3(3).

**Remark 32.2** (Non-Dirac TN functions). The functions in the Definition 32.1(2) were termed **totally positive functions** by Schoenberg – recall that in his papers and Karlin’s book, and even later, TN and TP matrices/functions were termed TP and STP, i.e., (strictly) totally positive. Finally, by Theorem 27.4, a non-Dirac TN function is strictly positive on an interval of positive length, and continuous on its interior.

**32.1. Precursors by Pólya and Hamburger.** We now discuss the origins of Pólya frequency functions. Recall that in his 1951 paper \[255\], Schoenberg proved the variation diminishing property for Pólya frequency functions \( \Lambda \), in terms of the values of functions as well as for real zeros of polynomials. (See Propositions 28.13 and 28.15, respectively.) To the collection of prior results proved about the variation diminishing property – including for power series by Fekete in 1912 \[86\] and for matrices by Schoenberg in 1930 \[246\], Motzkin in 1936 in his thesis \[202\], and others including prominently by Gantmacher–Krein – we now add a result shown in 1915 by Pólya, in connection with the Laguerre–Pólya class \( \mathcal{LP}_2 \).

Specifically, Pólya studied the reciprocal of a function \( \Psi_k(s) \in \mathcal{LP}_k \) for \( k = 1, 2 \), where \( \Psi_k(0) > 0 \). He expanded the meromorphic function \( 1/\Psi_k \) as:

\[
\frac{1}{\Psi_k(s)} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!} \mu_j s^j
\]  

(32.3)

Pólya then applied the differential operator \( 1/\Psi_2(\partial) \) to a real polynomial \( p(x) \), via:

\[
\frac{1}{\Psi_2(\partial)} p(x) = \mu_0 p(x) - \mu_1 p'(x) + \mu_2 \frac{p''(x)}{2!} - \cdots \pm \mu_n \frac{p^{(n)}(x)}{n!},
\]  

(32.4)

in the spirit of the preceding section, and where \( n = \deg(p) \geq 0 \). Notice that since \( \partial \) is locally nilpotent on \( \mathbb{C}[x] \), every formal power series in \( \partial \) yields a well-defined operator on \( \mathbb{C}[x] \), yielding an algebra homomorphism \( T \) from \( \mathbb{C}[[s]] \) to linear operators on \( \mathbb{C}[x] \).

With these preliminaries, as a first result Pólya showed another condition equivalent to being in the second Laguerre–Pólya class, i.e. a multiplier sequence of the second kind:

**Theorem 32.5** (\[221\]). Given a formal power series \( \psi(s) := \sum_{j \geq 0} \mu_j (-s)^j/j! \) with \( \mu_0 > 0 \), the following are equivalent:

1. Given a polynomial \( p \in \mathbb{R}[x] \), \( \psi(\partial)(p(x)) \) has no more real roots than \( p(x) \):
\[
Z_{nr}(\psi(\partial)(p(x))) \geq Z_{nr}(p(x)).
\]

2. \( \Psi_2(s) := 1/\psi(s) \) is in the Laguerre–Pólya class \( \mathcal{LP}_2 \).
Proof-sketch. First, given the algebra homomorphism $T$ sending a power series $\psi(s)$ to $\psi(τ)$, and since $\psi(0) > 0$, the assertion $(1')$ can be rephrased as:

$(1')$ Let $Ψ_2(σ) := 1/ψ(σ)$. If $p ∈ \mathbb{R}[x]$, then $Ψ_2(σ)p$ has at least as many real roots as $p$.

We now show that $(2) \implies (1')$. If $Ψ_2$ is constant then $(1')$ is immediate; else approximate $Ψ_2(s)$ locally uniformly by a sequence of real-rooted polynomials $ψ_0(s)$; since $μ_0 = Ψ_2(0) > 0$ and $Ψ_2$ is non-constant, the same holds for $ψ_n$ for $n ≫ 0$. In particular, for large $n$, the Hermite–Poulain theorem yields $Z_{nr}(ψ_n(σ)p) ≤ Z_{nr}(p)$; and we also have $\deg(ψ_n(σ)p) = \deg(p) = \deg(Ψ_2(σ)p)$. Hence $Z_{nr}(Ψ_2(σ)p) ≤ Z_{nr}(p)$ by the continuity of roots.

Conversely, we assume $(1')$ and show $(2)$. Let $g(x) := x^n$ for some $n ≥ 0$. If $Ψ_2(s) = ∑_{j≥0} ν_j s^j/j!$, then by $(1')$, the polynomial

$$Ψ_2(σ)(x^n) = ∑_{j=0}^n \binom{n}{j} ν_j x^{n−j}$$

has at least $n$ roots, whence is real-rooted for $n ≥ 0$. Define the real sequence $N := (ν_0, ν_1, \ldots)$; thus $Ψ_2(σ)(x^n) = N[(1 + x)^n]$ is real-rooted for all $n ≥ 0$. It follows by the Pólya–Schur characterizations of multiplier sequences of the second kind (see Theorem [31.30]) that $Ψ_N(x) = N[e^x] = Ψ_2(x)$ is in the class $LP_2$, as desired. □

Thus, Pólya showed that $Ψ_2(σ)$ weakly increases (i.e., does not decrease) the number of real roots. We return to this result and proof presently; first we continue with the account of Pólya’s work [221]. Another result dealt with the reciprocal functions $1/Ψ_k(s)$, $k = 1, 2$:

**Theorem 32.6** (Pólya, [221]). Suppose $Ψ_k(s) ∈ LP_k$ for $k ∈ \{1, 2\}$ is such that $Ψ_k(0) > 0$ and $Ψ_k(x) ≠ Ce^δx$. Let $1/Ψ_k(s) = ∑_{j≥0} μ_j(−s)^j/j!$, and define the Hankel matrices

$$H_{µ,j,n} := \begin{pmatrix} μ_j & μ_{j+1} & \cdots & μ_{j+n} \\ μ_{j+1} & μ_{j+2} & \cdots & μ_{j+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ μ_{j+n} & μ_{j+n+1} & \cdots & μ_{j+2n} \end{pmatrix}, \quad j ≥ 0.$$  

Then $det H_{µ,0,n} > 0 \; ∀n ≥ 0$. If $k = 1$ then moreover $det H_{µ,1,n} > 0 \; ∀n ≥ 0$.

Proof-sketch. Pólya first showed in [221] Satz II and Satz III the following claim:

“Suppose $k = 1$ or $2$, and a function $Ψ_k$ is as in the assumptions. If a nonzero real polynomial is non-negative on $R$ and $k = 2$ (resp., non-negative on $[a, ∞)$ and $k = 1$ for some $a ∈ R$), then the polynomial $1/Ψ_k(σ)$ is always positive on $R$ (resp. on $[a, ∞)$).”

Now to prove $det H_{µ,0,n} > 0$, choose a nonzero vector $u = (u_0, \ldots, u_n)^T ∈ \mathbb{R}^{n+1}$, and set $p(x) := (∑_{j=0}^n u_j x^j)^2$. By the previous paragraph and a straightforward computation,

$$0 < \frac{1}{Ψ_2(σ)} | p \bigg|_{x=0} = µ_0 u_0^2 − µ_1 (u_0 u_1 + u_1 u_0) + µ_2 (u_0 u_2 + u_1^2 + u_2 u_0) − \cdots = ∑_{j,k=0}^n (−1)^{j+k} µ_{j+k} u_j u_k.$$  

Since this inequality holds for all nonzero vectors $u$, the matrix $((-1)^{j+k} µ_{j+k})_{j,k=0}^n$ has positive determinant by Sylvester’s criterion (Theorem [2.7]). But $H_{µ,0,n}$ is obtained from this matrix by pre- and post-multiplication by the matrix $\text{diag}(1, −1, 1, \ldots, (−1)^{n−1})$. 
This shows the result for \( k = 2 \); similarly for \( k = 1 \), choose a nonzero vector \( u \) as above, and set \( p(x) := x(\sum_{j=0}^{n} u_j x^j)^2 \geq 0 \) on \([0, \infty)\). By the opening paragraph, we compute:

\[
0 < \frac{1}{\Psi_1(-\partial)} \left| \frac{\partial}{\partial x} \right|_{x=0} = \sum_{j,k=0}^{n} \mu_{j+k+1} u_j u_k,
\]
similar to the \( k = 2 \) case. Since this holds for all \( u \neq 0 \), it follows that \( \det H_{\mu,1,n} > 0 \).

These positive determinants were taken up at the turn of that decade by Hamburger, who in 1920–21 published his solution to the Hamburger moment problem \([122]\). Around the same time, he applied this solution to Pólya’s theorem \(32.6\), and showed that the positivity of the Hankel determinants in it is not sufficient to recover the Laguerre–Pólya class. Hamburger also showed, however, that the functions \( 1/\Psi_k \) are Laplace transforms:

**Theorem 32.7** (Hamburger, 1920, \([121]\)). Fix an entire function \( \Psi(s) \) with \( \Psi(0) > 0 \) and \( 1/\Psi(s) = \sum_{j \geq 0} \mu_j (-s)^j/j! \).

1. If \( \det H_{\mu,0,n} > 0 \) for all \( n \geq 0 \), then the reciprocal \( 1/\Psi(s) \) is the bilateral Laplace transform of a certain function \( \Lambda(x) \geq 0 \), in the maximal strip \( \Re(s) \in (\alpha, \beta) \) containing the origin where \( 1/\Psi(s) \) is regular.
2. If \( \det H_{\mu,0,n} , \det H_{\mu,1,n} \) are positive for all \( n \geq 0 \), then the reciprocal \( 1/\Psi(s) \) is the (bilateral) Laplace transform of a certain function \( \Lambda(x) \geq 0 \), with \( \Lambda \equiv 0 \) on \((-\infty,0)\), in the maximal half-plane \( \Re(s) \in (\alpha, \infty) \) of regularity of \( 1/\Psi(s) \).

From these results, one sees that the \( \mu_j \) are precisely the moments of \( \Lambda \) (see \((28.16)\)), or of the non-negative measure \( \Lambda(t)dt \):

\[
\mu_j := \int_{\mathbb{R}} \Lambda(t) t^j \ dt < \infty, \quad j = 0, 1, \ldots.
\]

Combined with the Hamburger and Stieltjes moment problems – see Remarks \((2.22)\) and \((4.4)\) respectively – and since \( \Lambda(t)dt \) has infinite support, this explains why the moment-matrices \( H_{\mu,0,n} , H_{\mu,1,n} \) have positive determinants, i.e. are positive definite by Sylvester’s criterion.

### 32.2. Schoenberg’s characterizations of PF functions.

Schoenberg built upon the above results, by (a) understanding the nature of the functions \( \Lambda \) in Hamburger’s theorem \(32.7\) and (b) characterizing the functions \( \Lambda(x) \in L^1(\mathbb{R}) \) that satisfy the variation diminishing property:

\[
S^{-}(g_f) \leq S^{-}(f) \ \forall f : \mathbb{R} \to \mathbb{R}, \quad \text{where} \quad g_f(x) := \int_{\mathbb{R}} \Lambda(x-t)f(t) \ dt. \quad (32.8)
\]

Remarkably, the \( \Lambda \) in both cases are essentially one and the same.

**Theorem 32.9** (Schoenberg, \([252, 255]\)). Suppose \( \Lambda : \mathbb{R} \to \mathbb{R} \) is Lebesgue measurable. The following are equivalent:

1. \( \Lambda \) is a non-Dirac TN function (see Definition \((32.1)\), not of the form \( e^{ax+b} \) for \( a,b \in \mathbb{R} \).
2. The bilateral Laplace transform of \( \Lambda \) is \( 1/\Psi(s) \), with \( \Psi(0) > 0 \) and \( \Psi(s) \) in the Laguerre–Pólya class \( L_2, \) i.e. of the form

\[
\Psi(s) = Ce^{-\gamma s^2+\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s) e^{-\alpha_j s},
\]

where \( C > 0, \gamma \geq 0, \delta, \alpha_j \in \mathbb{R}, 0 < \gamma + \sum_j \alpha_j^2 < \infty, \) and the equality \( B(\Lambda)(s) = 1/\Psi(s) \) holds on a maximal strip \( \Re(s) \in (\alpha, \beta) \). Here \(-\infty \leq \alpha < \beta \leq \infty, \) and if \( \alpha \) and/or \( \beta \) is finite then it is a zero of \( \Psi(\cdot) \).
In particular, in assertion (2), the condition \(0 < \gamma + \sum_j \alpha_j^2 < \infty\) implies that \(\Psi(s)\) is not of the form \(Ce^{\delta s}\). Moreover, in light of Proposition 28.3, one can translate the maximal strip by multiplying by an exponential factor:

**Corollary 32.10.** Given a non-Dirac TN function \(\Lambda : \mathbb{R} \to \mathbb{R}\), the following are equivalent:

1. \(\Lambda\) is a Pólya frequency function.
2. The maximal strip in the preceding theorem contains the imaginary axis.
3. \(\Lambda\) is non-monotone.
4. \(\Lambda\) is integrable.

The above results were for general Pólya frequency functions (or non-Dirac TN functions) vis-a-vis the Laguerre–Pólya class \(LP_2\). The corresponding equivalence for one-sided TN/PF functions was also shown by Schoenberg, in the same works [252, 255].

**Theorem 32.11.** Given \(\Lambda : \mathbb{R} \to \mathbb{R}\) Lebesgue measurable, the following are equivalent:

1. \(\Lambda\) is a non-Dirac TN function, vanishing on \((-\infty, 0)\), and such that \(B(\Lambda)(s)\) converges for \(\Re(s) > 0\). (In particular, \(\Lambda\) is not an exponential \(e^{ax+b}\) for \(a > 0\).)
2. The bilateral Laplace transform of \(\Lambda\) is \(1/\Psi(s)\), with \(\Psi(s) > 0\) for \(s > 0\) and \(\Psi(s)\) in the Laguerre–Pólya class \(LP_1\), i.e. of the form

\[
\Psi(s) = Ce^{\delta s} \prod_{j=1}^{\infty} (1 + \alpha_j s),
\]

where \(C > 0\), \(\delta, \alpha_j \geq 0\), \(0 < \sum_j \alpha_j < \infty\), and the equality \(B(\Lambda)(s) = 1/\Psi(s)\) holds on a maximal strip \(\Re(s) \in (\alpha, \beta)\). Here \(\alpha\) denotes the first zero of \(\Psi(\cdot)\).

Moreover, such a function \(\Lambda\) is a Pólya frequency function if and only if \(\Psi(0) > 0\).

Note that \(\Psi(0) > 0\) if and only if \(\Lambda\) is integrable; thus it cannot be constant on \((0, \infty)\).

Schoenberg’s theorems [32.9] and [32.11] characterize non-Dirac TN functions and Pólya frequency functions, both one- and two-sided, in terms of the Laguerre–Pólya class.

**Proof-sketch of Theorem 32.9** In the concluding portion of Section 30 (see the discussion around (30.14) and (30.15)) we saw an outline of why for \(k = 1, 2\), the function \(1/\Psi_k(s)\) for any \(\Psi_k \in LP_k\) is the Laplace transform of a Pólya frequency function (one-sided or general, for \(k = 1, 2\) respectively). We outline here a proof of why the reverse implication holds for \(k = 2\). The outline opens with Schoenberg’s words [252]:

“A proof of Theorem 32.9 is essentially based on the results and methods developed by Pólya and Schur. The only additional element required is a set of sufficient conditions insuring that a linear transformation be variation diminishing.”

This last sentence of Schoenberg refers to his 1930 paper, in which he showed that TN matrices are variation diminishing. Using this property, he showed the same for Pólya frequency functions, whence for polynomials (and then did the same for one-sided PF functions); see Section 28.2. Now suppose \(\Lambda\) is a PF function; we proceed to outline the proof of why \(B(\Lambda)\), which converges on a maximal strip \(\Re(s) \in (\alpha, \beta)\) with \(\alpha < 0 < \beta\), is of the form \(1/\Psi_2(s)\) for \(\Psi_2(s)\) in the Laguerre–Pólya class \(LP_2\). Following Pólya, write

\[
B(\Lambda)(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \mu_j s^j, \quad \Re(s) \in (\alpha, \beta),
\]
where \( \mu_j = \int_\mathbb{R} \Lambda(t)t^j \, dt \) are the moments, as discussed after Theorem 32.7. Also note that \( \mathcal{B}(\Lambda)(0) = \mu_0 > 0 \), so we write the reciprocal power series:

\[
\Psi_2(s) := \frac{1}{\mathcal{B}(\Lambda)(s)} = \sum_{j=0}^\infty \nu_j s^j.
\]

Now return to the integral transformation \( f \mapsto g_f \) as in (32.8). By Proposition 28.15, \( Z(g_f) \leq Z(f) \), where \( Z(\cdot) \) denotes the number of real roots. Here we recall Schoenberg's computation (28.17), perhaps inspired by Pólya's trick (32.4):

\[
g_f(x) = \int_\mathbb{R} \Lambda(t)f(x-t) \, dt = \sum_{j=0}^\infty (-1)^j \frac{\mu_j(\partial^j f)(x)}{j!} = \mathcal{B}(\Lambda)(\partial) f(x).
\]

We use here that both sides are polynomials, so that only finitely many terms \( \mu_j \partial^j \) act nontrivially. Now take the reciprocal power series, e.g. via the map \( T \) following (32.4), and obtain:

\[
f(x) = \int_\mathbb{R} \Lambda(t)g_f(x-t) \, dt = \sum_{j=0}^\infty \frac{(-1)^j \mu_j(\partial^j g_f)(x)}{j!} = \mathcal{B}(\Lambda)(\partial) g_f(x).
\]

Thus, Schoenberg showed in [253, 255] the connection between his characterization of PF functions via the Laguerre–Pólya class, Pólya–Schur multipliers, and Pólya's theorem 32.5 proving the variation diminishing property over polynomials. This explains why Schoenberg proposed in [253] the name \textit{Pólya frequency functions} for this family of functions.

We conclude with another characterization of Pólya frequency functions, which Schoenberg announced in [253] and showed in [254]. This occurs via the variation diminishing property:

\textbf{Theorem 32.13} (Schoenberg, [253, 254]). Given \( \Lambda : \mathbb{R} \to \mathbb{R} \) Lebesgue integrable, let the kernel \( f \mapsto g_f \) as in (32.8), for continuous, bounded \( f : \mathbb{R} \to \mathbb{R} \). The following are equivalent:

1. \( \Lambda \) is variation diminishing: \( S^-_{\mathbb{R}}(g_f) \leq S^-_{\mathbb{R}}(f) \) for all continuous, bounded \( f : \mathbb{R} \to \mathbb{R} \).
2. Either \( \Lambda \) or \( -\Lambda \) is a Pólya frequency function, or \( \Lambda \) is a Dirac function \( C\delta_{x=a} \) for some \( C,a \in \mathbb{R} \).

Thus, the converse to Proposition 28.13 holds as well.

\textbf{32.3. Support of PF functions; (strict) total positivity.} We now discuss two consequences of the above results, which were also proved by Schoenberg. The first is that a Pólya frequency function necessarily cannot be compactly supported:

\textbf{Proposition 32.14.} The support of a Pólya frequency function \( \Lambda : \mathbb{R} \to \mathbb{R} \) is unbounded.

Recall by Theorem 27.4 that the support of a \( TN_2 \) function is an interval, so in fact the support of a PF function now must be of the form \((a,b)\) where at least one of \(|a|, |b|\) is infinite.

\textbf{Proof.} We already know the support of a Pólya frequency function is not a singleton. Let \( I \) be a bounded interval with endpoints \(-\infty < a < b < \infty\). The claim is that for any function \( \Lambda : \mathbb{R} \to \mathbb{R} \) supported on \( I \) and continuous in its interior, the Laplace transform

\[
\mathcal{B}(\Lambda)(s) := \int_a^b e^{-sx} \Lambda(x) \, dx, \quad s \in \mathbb{C}
\]

is entire, with \( k \)-th derivative

\[
\sum_{n=0}^\infty \frac{(-s)^n}{n!} \int_a^b x^n \Lambda(x) \, dx.
\]

Indeed, one shows that the series
is uniformly convergent for $s$ in any bounded domain in $\mathbb{C}$. Hence its sum is entire, and equals $\mathcal{B}(\Lambda)(s)$; moreover, one can differentiate term by term to compute its $k$th derivative:

$$\sum_{n=k}^{\infty} \frac{s^{n-k}}{(n-k)!} \int_{a}^{b} (-x)^{n} \Lambda(x) \, dx = \int_{a}^{b} e^{-sx}(-x)^{k} \Lambda(x) \, dx.$$

Having shown the claim, we proceed to the proof. By Theorem 32.9,\[ \mathcal{B}(\Lambda)(s) = C^{-1} e^{\gamma s^{2} - \delta s} \prod_{j=1}^{\infty} \left( 1 + \alpha_{j}s \right), \] for appropriate values of the parameters. Now if $\Lambda$ has bounded support, then from above the right-hand side is entire. It follows that $\alpha_{j} = 0 \forall j$. But then $\mathcal{B}(\Lambda)(s) = C^{-1} e^{\gamma s^{2} - \delta s}$. Now by Laplace inversion and Example 28.4, $\Lambda$ itself is a (shifted) Gaussian density, hence is positive on an interval, so we can choose $t$ such that $\gamma > 0$. By Theorem 32.9, $\mathcal{B}(\Lambda)(s)$ is non-singular, whence has positive determinant since $\Lambda$ is non-singular, whence has positive determinant since $\Lambda$ itself is a (shifted) Gaussian density, hence has unbounded support. This provides the necessary contradiction.

The final result here, due to Schoenberg and Whitney in Trans. Amer. Math. Soc. (1953), provides a sufficient condition when a Pólya frequency function is (strictly) totally positive. We mention only a part of their results, and show only the sub-part relevant for our purposes.

**Theorem 32.16** (Schoenberg–Whitney).

1. If $\Lambda_{+}, \Lambda_{0} : \mathbb{R} \to \mathbb{R}$ are PF functions, and $\Lambda_{+}$ yields a TP kernel, then so does $\Lambda_{+} \ast \Lambda_{0}$.

2. Suppose $\Lambda(x)$ has bilateral Laplace transform $\mathcal{B}(\Lambda(x))$. If $\gamma > 0$, or $\gamma = 0$ and $\sum_{j} |\alpha_{j}|$ diverges, then the kernel $T_{\Lambda}$ is TP (of all orders).

**Proof.** For the first part: that $\Lambda_{+} \ast \Lambda_{0}$ is a PF function follows from Corollary 30.7 We now claim that given $p \geq 1$ and $y \in \mathbb{R}^{p+1}$, there exists $t = (t_{1}, \ldots, t_{p}) \in \mathbb{R}^{p+1}$ such that $T_{\Lambda_{0}}[t; y]$ is non-singular. The proof is by induction on $p \geq 1$; for the base case, since $\int_{\mathbb{R}} \Lambda_{0}(x) \, dx \in (0, \infty)$, it is positive on an interval, so we can choose $t_{1}$ as desired.

For the induction step, let $p \geq 2$ and $y \in \mathbb{R}^{p+1}$. Choose $t := (t_{1}, \ldots, t_{p-1}) \in \mathbb{R}^{p-1+1}$ such that the matrix $[(\Lambda_{0}(t_{j} - y_{k}))_{j,k=1}^{p-1}]$ is non-singular, whence has positive determinant since $\Lambda_{0}$ is TN. Now expand the determinant of $T_{\Lambda_{0}}[t; y]$ along the last row; if this vanishes for all $t \in \mathbb{R}$, we obtain an equation $\sum_{k=1}^{p} a_{k} \Lambda_{0}(t - y_{k}) = 0$, where all $a_{k} \in \mathbb{R}$ and $a_{p} > 0$. Taking the bilateral Laplace transform of both sides,

$$\psi(s) \sum_{k=1}^{p} a_{k} e^{-sy_{k}},$$

must vanish for $s$ in some interval, where $\psi \neq 0$ by the Schoenberg representation theorems. Hence the sum vanishes identically on an interval – which is false by Descartes’ rule of signs (Theorem 10.3). Thus, $T_{\Lambda_{0}}[t; y]$ is non-singular for some $t \in \mathbb{R}$. Clearly $t \neq t_{j} \forall j$, so by suitably permuting the rows (and relabelling the $t, t_{j}$ if needed), the induction step is proved.

Having shown the claim, and using the continuity on an interval of both $\Lambda_{+}, \Lambda_{0}$, one checks using the Basic Composition Formula (5.14) that $\Lambda_{+} \ast \Lambda_{0}$ is TP, for each $p \geq 1$:

$$\det((\Lambda_{+} \ast \Lambda_{0})(x_{i} - y_{k}))_{i,k=1}^{p} = \int_{t_{1} < t_{2} < \cdots < t_{m}} \cdots \int_{t_{1} < t_{2} < \cdots < t_{m}} \det(\Lambda_{+}(x_{i} - t_{j}))_{i,j=1}^{p} \det(\Lambda_{0}(t_{j} - y_{k}))_{j,k=1}^{p} \prod_{j=1}^{p} d\mu(y_{k}).$$

We only (require, hence) show the second part for $\gamma > 0$. By Theorem 32.9, $e^{-\gamma s^{2}/2} \mathcal{B}(\Lambda)(s)$ is the bilateral Laplace transform of a PF function, say $\Lambda_{0}$; and $e^{-\gamma s^{2}/2}$ of a TP (Gaussian) PF function $\Lambda_{+}$, say by Example 28.4 By the preceding part, $\Lambda = \Lambda_{+} \ast \Lambda_{0}$ is TP. □
Discontinuous PF functions.

33. FURTHER ONE-SIDED EXAMPLES: THE HIRSCHMAN–WIDDER DENSITIES.
DISCONTINUOUS PF FUNCTIONS.

We conclude this part of the text – on Pólya frequency (PF) and related functions – with some one-sided examples, which vanish on the negative semi-axis. Apart from their intrinsic interest, they will be useful in the next part, as test kernels that help to classify the composition operators preserving (one-sided) PF functions.
BIBLIOGRAPHIC NOTES AND REFERENCES

Section 27 on Pólya frequency functions, and totally non-negative/positive functions, is essentially from the monumental work by Schoenberg [255]. See also the accounts in Karlin’s book [154] and in the survey [76]. The modification of Theorem 27.4 (characterizing the $TN_2$ functions) to arbitrary domains $J - J$ is taken from Khare [162]. Section 27.2 containing the characterization of $TN_p$ functions for all $p \geq 3$ and related results, is by Khare [162], following Lemma 27.8 which is taken from Förster–Kieburg–Kösters [93].

Proposition 28.3 collecting basic properties of Pólya frequency functions, is from Schoenberg [255] – as are the results in Section 28.2 on the variation diminishing properties of these functions. The precursor to these facts is the variation diminishing property for $TN$ and $TP$ matrices. This is the focus of Section 28.1 – the treatment here can essentially be found in Pinkus’ book [218, Chapter 3]. Section 28.4 on the characterization of $TN$ and $TP$ matrices through sign non-reversal, is taken from Choudhury–Kannan–Khare [58].

Pólya frequency sequences were introduced by Fekete and Pólya [86], in an attempt to prove a result by Laguerre [176]. The treatment in Section 29.2 of generating functions of one-sided Pólya frequency sequences, is taken from the works [2, 3, 77, 256] by Aissen–Schoenberg–Whitney and Edrei, separately and together. Theorem 29.14 on the root-location properties of such generating functions, as well as the two subsequent lemmas used in its proof, are from Schoenberg’s 1955 paper [256]. The connection to elementary symmetric polynomials is classical; see e.g. Macdonald’s monograph [191]. Lemma 29.21 on the total non-negativity of Jacobi matrices, is from Karlin’s work [153]; and Theorem 29.23 is at least from Karlin’s book [154].

The contents of Section 30 (apart from the basics of convolution and the Laplace transform) are from Schoenberg’s 1951 paper [255] – including the examples of $TN$ functions: the discontinuous functions $H_1(x)$ and $\lambda_1(x)$, their shifted variants, convolutions of these, and the concluding connection to the Laguerre–Pólya class.

The exposition of the Laguerre–Pólya class and results preceding its development are taken from numerous sources – we mention the surveys [67] by Craven–Csordas and [76] by Dym–Katzenelson, the book by Karlin [154], and the paper by Aleman–Beliaev–Hedenmalm [8] for the ‘general flavor’. Coming to specific proofs: the treatment of the Hermite–Poulain Theorem 31.3 and Laguerre’s Theorem 31.5 (2) (on the multiplier sequence $q(k)$) are taken from Pinkus’ article [217]. The remaining parts of Theorem 31.8 – involving Laguerre’s multipliers $1/k!$ and the composition theorems by Maló and Schur – are proved using the approach in Levin’s monograph [182]. These proofs also draw from de Bruijn’s short note [53], where he showed Theorem 31.9 (on ‘roots lying in sectors’) and its numerous corollaries listed out above, including the ones by Weisner [291]. (For historical completeness: the Hadamard–Weierstrass factorization emerges primarily out of [119, 289]; see also the monograph [182].)

Theorems 32.5 and 32.6 involving early findings of Pólya about functions $1/\Psi(s)$ for $\Psi$ in the Laguerre–Pólya class, are taken from [221]. Theorem 32.7 showing that $1/\Psi(s)$ has a Laplace transform representation, is taken from Hamburger’s paper [121]. Section 32.2 containing characterizations of Pólya frequency functions (both general or one-sided) in terms of the Laguerre–Pólya class or variation diminution, is taken from the two announcements [252, 253] and the subsequent ‘full papers’ [254, 255] by Schoenberg. Proposition 32.14 on the unbounded support of a Pólya frequency function is from [255], and Theorem 32.16 on the total positivity of certain PF functions is from Schoenberg and Whitney’s paper [257].
Part 5:
Composition operators preserving totally positive kernels

Part 5: Composition operators preserving totally positive kernels


Having classified – in a previous part of this text – the preservers of Loewner positivity, monotonicity, and convexity on infinite domains, we now turn to preservers of total non-negativity and total positivity. This section is concerned with preservers of $TN_p$ for $p$ finite, with emphasis on power functions. In particular, we will see the occurrence of a critical exponent phenomenon in total positivity – this time for powers of one-sided $TN_p$ functions.

34.1. Connections to representation theory and probability. We end this second look at critical exponents (the first was in Part 2 of this text) by providing connections to other areas of mathematics – specifically, via the Wallach set (or Gindikin ensemble, or Berezin–Wallach set). The following is a very brief account of these topics, and the references here should provide the reader with starting points for further exploration into this rich area, at the intersection of representation theory, complex analysis, and probability.

Suppose $n \geq 1$ and $D \subset \mathbb{C}^n$ is a tube domain, i.e. of the form $D = \mathbb{R}^n + i\Omega$, where $\Omega$ is a homogeneous irreducible self-dual convex cone in $\mathbb{R}^n$. Denote by $H_1$ the associated Bergman space, consisting of holomorphic functions $F$ on $D$ satisfying:

$$\int_D |F(x + iy)|^2 \, dx\,dy < \infty,$$

and let $H_2$ denote the Hardy space, consisting of holomorphic functions $F$ on $D$ satisfying:

$$\sup_{y \in \Omega} \int_{\mathbb{R}^n} |F(x + iy)|^2 \, dx < \infty.$$

Let $P(z - \overline{w})$ denote the Bergman kernel on $D$; thus $F(w) = \langle F, P_w \rangle_{H_1}$ for all $F \in H_1$, where $P_w(z) := P(z - \overline{w})$. Then the Hardy space $H_2$ has a reproducing kernel of the form $P^p$, for some power $p < 1$ of the Bergman kernel. In Acta Math. (1976), Rossi and Vergne classified the powers of $P$ which are reproducing kernels for some Hilbert space of holomorphic functions on $D$. They called the set of such powers $p$ the Wallach set, and showed in [239] that it consists of an arithmetic progression and a half-line: $\{0, c/r, 2c/r, \ldots, c\} \sqcup (c, \infty)$ for some $c > 0$ and $r \geq 0$. The exact meaning of $p, c, r$ can be found in [239].

Rossi–Vergne named the aforementioned set after Wallach, who was studying it at the time (note, Wallach’s papers [281] appeared in print later, in 1979 in Trans. Amer. Math. Soc.). Wallach, following up on work of Harish-Chandra, was studying the holomorphic discrete series of connected, simply connected Lie groups $G$. Specifically, he classified the set of twist-parameters $p$ of the center of $K$ (a maximal compact reductive subgroup of $G$) for which the corresponding $K$-finite highest weight module over $\mathfrak{g} = \text{Lie}(G)$ (complexified) is irreducible and unitarizable, or it is reducible and its radical is unitarizable. In [239], Rossi and Vergne obtained the same (Wallach) set of parameters $p$, with the sufficiently large $p$ leading to the holomorphic discrete series of weighted Bergman spaces. See also [84], where Faraut and Korányi worked over symmetric domains $D$, and studied Hilbert spaces of holomorphic functions on $D$.

The Wallach set also appears in at least two other settings, both again in the 1970s:

- Berezin [27] had encountered such a set while studying Kähler potentials of Siegel domains, in the context of quantization. (See also [208] for a recent avatar of this set of exponents, defined in the context of positive semidefinite kernels and recalling various results discussed above).
• Gindikin [105] had shown that given a symmetric cone, and a Riesz distribution $R_\mu$ associated to it, $R_\mu$ is a positive measure if and only if $\mu \in \mathbb{C}$ lies in an associated Wallach set. A simple proof of this result (and more) was given in 2011 by Sokal [271].

On a conceptual note: the work of Gindikin on the Wallach set arises through ratios of gamma functions on symmetric cones, which are of the form $G/K$ for a group $G$ and a maximal compact subgroup $K$. Now, the ‘usual’ gamma function can be defined via the Laplace transform on the cone $\mathbb{R}^+$. In fact, this can be done over arbitrary symmetric cones in Euclidean Jordan algebras – for instance, in the cone $\mathbb{P}_n(\mathbb{R}) \cong GL_n(\mathbb{R})/O_n(\mathbb{R})$, where $G = GL_n, K = O_n$ stand for the general linear and orthogonal groups of $n \times n$ real matrices, respectively.

Via the Iwasawa decomposition $G = K \cdot A \cdot N$, carrying out the Laplace transform on such a cone $G/K$ is the same as doing so on $A \cdot N$. This turns out to be a solvable group, and the associated Haar measure is closely related to Lebesgue measure. This provides a ‘natural’ setting for Gindikin’s work and for proving his results – see the 1991 paper of Kostant–Sahi [173] in Adv. Math. for details. As the authors remark, the main ingredient in the above working is the Laplace transform on a self-dual cone; the origins of this can be found in the 1935 paper [264] of Siegel in Ann. of Math.

Gindikin’s work leads us to another recent manifestation of the critical exponent in matrix analysis – specifically, in random matrix theory. We first mention the 1987 paper [179] of integer $n$ to define this is through the Laplace transform of its density function. More precisely, fix an the more standard ‘central’ Wishart distribution, defined by Wishart in 1920 [296]. One way in the language of shape parameters of (non-central) Wishart distributions. We begin with Rossi–Vergne concerning the Wallach set.

Positive cones in formally real Jordan algebras. Lassalle recovered the results of Wallach and

The results of Gindikin, Berezin, Rossi–Vergne, Wallach, and Lassalle can all be interpreted in the language of shape parameters of (non-central) Wishart distributions. We begin with the more standard ‘central’ Wishart distribution, defined by Wishart in 1920 [296]. One way to define this is through the Laplace transform of its density function. More precisely, fix an integer $n \geq 1$; now given the shape parameter $p \in [0, \infty)$ and the scale parameter $\Sigma$, which is a positive definite $n \times n$ real matrix, $\Gamma(p, \Sigma)$ denotes the Wishart distribution, say with density $f$, satisfying:

$$\mathcal{L}\{\Gamma(p, \Sigma)\}(s) := \int_{\mathbb{P}_n(\mathbb{R})} e^{-tr(sA)} f(dA) = det(Id_n + 2s\Sigma)^{-p}, \quad s \in \mathbb{P}_n(\mathbb{R}).$$

It is a well-known fact (see e.g. [55]) that such a distribution exists if and only if $p$ is in the Wallach set $\{0, 1/2, \ldots, (n - 2)/2\} \cup ((n - 1)/2, \infty)$.

We now come to recent work along these lines. The non-central Wishart distribution is similarly defined – now also using a non-centrality parameter $\Omega \in \mathbb{P}_n(\mathbb{R})$ – via its Laplace transform

$$\mathcal{L}\{\Gamma(p, \Sigma, \Omega)\}(s) = det(Id_n + 2s\Sigma)^{-p} e^{-2tr(\Omega\Sigma(Id_n + 2s\Sigma)^{-1})}, \quad s \in \mathbb{P}_n(\mathbb{R}).$$

In 2018, Graczyk–Malecki–Mayerhofer [107] and Letac–Massam [181] showed (akin to above) that such a distribution exists if and only if (a) $p$ belongs to the same Wallach set as above:

$$p \in \{0, 1/2, \ldots, (n - 2)/2\} \cup ((n - 1)/2, \infty),$$

and (b) if $p < (n - 1)/2$ then $\text{rk} \Omega \leq 2p$. The same result was shown by Mayerhofer in Trans. Amer. Math. Soc. in 2019, building on ideas of Faraut [53] (1988) and Peddada–Richards [215] (1991). To do so, Mayerhofer [197] extended prior analysis by the aforementioned authors, on the positivity of generalized binomial coefficients that occur in Euclidean Jordan algebras.
34. Critical exponent for powers preserving $T_N^p$. The Jain–Karlin–Schoenberg kernel. 225

See e.g. [85] for an introduction to this; also see the 2011 work [243] Theorems 1 and 5] by Sahi, for stronger positivity results.

We end with a (superficial) relation between the Wallach set here and the set of powers preserving positivity on $\mathbb{P}_n((0, \infty))$ (and total non-negativity of order $n$ for the powers of the kernel $T_{\Omega}$) studied by FitzGerald–Horn, Karlin, and Jain. These powers were studied in Sections 9 and 15 above, as well as in the present section. It would be interesting to find a deeper, conceptual connection between the two problems.

We begin with the positivity preservers side. Recall from the proof of Theorem 9.3 that if $\alpha \in (0, n - 2) \setminus \mathbb{Z}$, and $x_1, \ldots, x_n \in (0, \infty)$ are pairwise distinct, then the Taylor expansion of the entrywise power $((1 + x_j x_k)^{\alpha})_{j,k=1}^n$ yields

$$((1 + x_j x_k)^{\alpha})_{j,k=1}^n = \sum_{m \geq 0} \binom{\alpha}{m} x^{\text{om}} (x^{\text{om}})^T,$$

where $\binom{\alpha}{m} = \alpha (\alpha - 1) \cdots (\alpha - m + 1)/m!$ and $x^{\text{om}} := (x_1^m, \ldots, x_n^m)^T$. Now the key is that for $m = \lfloor \alpha \rfloor + 2$, the binomial coefficient is negative. Thus one can pre- and post-multiply the above matrix by $u^T, u$ respectively, for some $u \in \mathbb{R}^n$ orthogonal to the smaller entrywise powers of $x$. Using this, one can deduce that $((1 + x_j x_k)^{\alpha})_{j,k=1}^n$ is not positive semidefinite.

The connection to the Wallach set $W$ is via the fact that the analysis for powers that do not lie in $W$ (to show that a (non)central Wishart distribution does not exist) also goes through the negativity of certain generalized binomial coefficients. As a simple example, we look into the argument in the aforementioned work of Peddada–Richards in Ann. Probab. 1991. (A similar computation concludes the proof of [197, Theorem 4.10].) Given integers $k_1 \geq \cdots \geq k_n \geq 0$, define the shifted factorial by:

$$(p)_k := \prod_{j=1}^n (p - \frac{1}{2} (j - 1))_{k_j}, \quad \text{where} \quad (p)_k := p(p + 1) \cdots (p + k - 1)$$

if $k > 0$, and $(p)_0 := 1$. Now it is shown in [215] – via the use of zonal polynomials – that if the (non)central Wishart density with shape parameter $p$ exists, then $(p)_k \geq 0$ for all $n$-tuples $k$ as above. But if $q = 2p \in (0, n - 1) \setminus \mathbb{Z}$, then set

$m := \lfloor q \rfloor + 2, \quad k_1 = \cdots = k_m = 1, \quad k_{m+1} = \cdots = k_n = 0.$

Then

$$\frac{q^2 - 1}{2} \cdots \frac{q - |q| - 1}{2} \cdot (1 \cdot 1 \cdot 1) \geq 0.$$ 

This contradicts the choice of $q$, as desired.
35. Preservers of Pólya frequency functions.

35. **Preservers of Pólya frequency functions.**

35.1. **Preservers of \(TN\) functions.**

35.2. **Preservers of Pólya frequency functions.**

35.3. **Preservers of \(TP\) Pólya frequency functions.**
36. Preservers of Pólya frequency sequences.

Having classified the preservers of $TN$ functions and their subclass of $(TP)$ Pólya frequency functions, we turn to such Toeplitz kernels on distinguished subsets. This section deals with Pólya frequency sequence, i.e. Toeplitz $TN$ kernels on $\mathbb{Z} \times \mathbb{Z}$. However, several of the results will be shown to hold over more general domains $X,Y \subset \mathbb{R}$ with arbitrarily long arithmetic progressions. The full power of these more general domains $SX,Y$ will be revealed in the next section, which concludes this part of the text.

As we saw in the previous three sections, working with kernels on intervals allows one to use powerful tools and results from analysis. These tools are also used in the present section, where we will use PF functions (on $\mathbb{R} \times \mathbb{R}$) to classify the preservers of $(TP)$ PF sequences (on $\mathbb{Z} \times \mathbb{Z}$).

36.1. Preservers of PF sequences.

36.2. Preservers of $(TP)$ PF sequences. Discretization: if $F$ is continuous and preserves PF sequences, then $F$ preserves measurable $TN$ Toeplitz kernels.

36.3. Preservers of one-sided PF sequences.

36.4. Further questions. To conclude this section, here are a few open questions involving Pólya frequency functions and sequences, and their preservers.

Question 36.1. In light of Schoenberg’s theorem characterizing the $TN_2$ functions, classify the preservers of these functions.

For example, the aforementioned theorems imply that all powers $x^\alpha$ preserve the $TN_p$ functions for $p = 2, 3$, if $\alpha \in \mathbb{Z}_{\geq 0} \cup [p - 2, \infty)$. We also saw in that $x^\alpha 1_{x \geq 0}$ is a $TN_p$ function if and only if $\alpha \in \mathbb{Z}_{\geq 0} \cup [p - 2, \infty)$. In light of this, a natural question involves studying the powers preserving total non-negativity of each degree:

Question 36.2. Given an integer $p \geq 2$, classify the powers $x^\alpha$ which preserve the class of $TN_p$ functions. Note by that every such power is in $\mathbb{Z}_{\geq 0} \cup [p - 2, \infty)$.

Coming to Pólya frequency sequences, the above classification results lead to additional, theoretical questions about related sequences, which are mentioned for the interested reader.

Question 36.3. Classify the preservers of one-sided Pólya frequency sequences: (i) that have finitely many diagonals, or (ii) generated by evaluating a polynomial at non-negative integers.

Like a question above, both of these classes of functions have nontrivial power-preservers. Indeed, $x^n$ preserves both of these classes for all integers $n \geq 1$, by Maló’s theorem and a result of Wagner [282], respectively. Akin to the above discussion involving non-integer ‘one-sided powers’ which are $TN_p$, a related question is:

Question 36.4. Classify the power functions $x \mapsto x^\alpha$, such that if $\sum_j a_j x^j$ is a polynomial with positive coefficients and real roots, then so is $\sum_j a_j^\alpha x^j$. In particular, find the ‘critical exponent’ $\alpha_p$ for polynomials of a fixed degree $p$, such that every $\alpha \geq \alpha_p$ satisfies this property on polynomials of degree at most $p$.

Note again by Maló’s theorem [31.8] that every integer $\alpha \geq 1$ satisfies this property. Recently, Wang and Zhang showed in [288] the existence of the threshold $\alpha_p$. Notice by that
this question is connected to entrywise powers preserving PF sequences with up to \( p \) nonzero diagonals, and hence refines the preceding question.

A final, Whitney-type density question is:

**Question 36.5.** Are the totally positive Pólya frequency sequences dense in the set of all Pólya frequency sequences?

Such a result could help obtain the preservers of the \( TP \) PF sequences from the preservers of all PF sequences. (Note that this goal is achieved above via alternate means.)
37. Preservers of TP Hankel kernels.

In this section, we change gears, and work with a setting studied above for matrices: Hankel kernels, now defined on a sub-interval of \( \mathbb{R} \) instead of on the integers \( \{0, 1, \ldots\} \). As should be clear from the previous sections, working with kernels on intervals (e.g. \( \mathbb{R} \times \mathbb{R} \)) allows one to use a host of powerful, classical techniques and results from analysis, which help prove results even on discrete domains (e.g. \( \mathbb{Z} \times \mathbb{Z} \)).

We begin with terminology. Given subsets \( X, Y \subset \mathbb{R} \), define their Minkowski sum \( X + Y := \{x + y : x \in X, y \in Y\} \). Now a kernel \( K : X \times Y \to \mathbb{R} \) is said to be Hankel if there is a function \( f : X + Y \to \mathbb{R} \) such that \( K(x, y) = f(x + y) \) for all \( x \in X, y \in Y \). Note that if \( X = Y \) then any such ‘square’ kernel is symmetric.

A natural class of such kernels in analysis is when \( X \) is an interval, and in this section we focus on this case. In keeping with the above sections, here is a typical example of a Hankel TN kernel: given finitely many positive scalars \( c_1, \ldots, c_n, u_1, \ldots, u_n \), define \( K_{c, u} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( K_{c, u}(x, y) := \sum_{j=1}^{n} c_j u_j^{x+y} \).

We show in Theorem 37.6 that this kernel is TN on \( \mathbb{R} \times \mathbb{R} \) (whence on \( X \times X \) for any subinterval \( X \subset \mathbb{R} \)).

37.1. Preservers of Hankel TN kernels on intervals. In this section, we classify the preservers of Hankel TN and TP kernels, on \( X \times X \) for \( X \) an interval that is always assumed to have positive measure. The first main result addresses Hankel TN kernels: Let \( X \subset \mathbb{R} \) be an interval with positive measure, and \( F : [0, \infty) \to \mathbb{R} \). The following are equivalent:

(1) The composition map \( F \circ - \) preserves total non-negativity on the continuous Hankel TN kernels on \( X \times X \).

(2) The composition map \( F \circ - \) preserves positive semidefiniteness on the continuous Hankel TN kernels on \( X \times X \).

(3) The function \( F \) is a power series with non-negative coefficients: \( F(x) = \sum_{k=0}^{\infty} c_k x^k \) for \( x > 0 \), with all \( c_k \geq 0 \); and \( F(0) \geq 0 \).

The proof of this result uses a discretization technique that will also be useful later:

Lemma 37.2 (Discretization of Hankel kernels). Suppose \( X \subset \mathbb{R} \) is an interval with positive measure, and \( K : X \times X \to \mathbb{R} \). Then each of the following statements implies the next.

(1) \( K \) is TN.

(2) All principal submatrices drawn from \( K \) are TN.

(3) All principal submatrices drawn from \( K \), with arguments lying in an arithmetic progression, are TN.

(4) All principal submatrices drawn from \( K \), with arguments lying in an arithmetic progression, are positive semidefinite.

Conversely, (2) \( \implies \) (1) for all kernels, (3) \( \implies \) (2) for continuous kernels, and (4) \( \implies \) (3) for continuous Hankel kernels.

Proof. The forward implications are immediate from the definitions. Conversely, using the notation in Definition 25.1, if \( x, z \in X^n \) for some \( n \geq 1 \), then the matrix \( K[x; z] \) is a submatrix of \( K[x \cup z; x \cup z] \), where \( x \cup z \) denotes the union of the coordinates of \( x \) and \( z \), together arranged in increasing order. Thus (2) \( \implies \) (1).
Now suppose $K$ is continuous and (3) holds. We will show (2): given $x_1 < \cdots < x_n$ in $X$, let $\epsilon := \min_j (x_{j+1} - x_j)/2$, and approximate each $x_j$ by a rational sequence $z_j^{(k)}$ with:

$$z_1^{(k)} \in [x_1, x_1 + \epsilon), \quad z_n^{(k)} \in (x_n - \epsilon, x_n), \quad z_j^{(k)} \in (x_j - \epsilon, x_j + \epsilon), \quad j \in (1, n).$$

Choose an integer $N_k \geq 1$ such that $z_j^{(k)} \in \frac{1}{N_k} \mathbb{Z}$ for all $k$, and define:

$$z^{(k)} := (z_1^{(k)}, z_2^{(k)} + \frac{1}{N_k}, z_3^{(k)} + \frac{2}{N_k}, \ldots, z_n^{(k)}), \quad z_1^{(k)} := (z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)}).$$

By assumption, the matrix $K[z^{(k)}; z^{(k)}]$ is $TN$, whence so is the submatrix $K[z_1^{(k)}; z_1^{(k)}]$ for all $k$. Let $k \to \infty$; since $K$ is continuous, it follows that $K[x; x]$ is $TN$, where $x = (x_1, \ldots, x_n)$. This shows (2).

Finally, suppose $K$ is continuous and Hankel, and (4) holds. Given an arithmetic progression $x \in X^{n, \uparrow}$, let $A := K[x; x]$ be the corresponding positive semidefinite Hankel matrix. Now define the progression of running averages $y \in X^{n-1, \uparrow}$ by: $y_j := (x_j + x_{j+1})/2$ for $1 \leq j \leq n - 1$; and let $B := K[y; y]$. Since $B$ is positive semidefinite, Hankel, as well as the truncation $A^{(1)}$ of $A$, it follows by Theorem 4.1 that $A$ is $TN$, as desired. $\Box$

Now just as Corollary 4.2 shows that Hankel matrices/kernels form a closed convex cone for $X = \{1, \ldots, n\}$ (for any integer $n \geq 1$), the same immediately follows in the present setting:

**Corollary 37.3.** Suppose $X \subset \mathbb{R}$ is an interval. The continuous Hankel $TN$ kernels on $X \times X$ form a closed convex cone, which is further closed under taking pointwise/Schur products.

**Proof.** This follows from the condition Lemma 37.2 (4) being closed under addition, dilation, taking pointwise limits, and taking pointwise products (by the Schur product theorem). $\Box$

**Remark 37.4.** The last two conditions of Lemma 37.2 can be further refined to ask for the arithmetic progressions in question to be rational (or to belong to a translate of any dense additive subgroup of $\mathbb{R}$). This does not affect either Lemma 37.2 or Corollary 37.3.

The next preliminary result identifies when a continuous Hankel $TN$ (or $TN_p$ for any $p \geq 2$) kernel vanishes at a point. Recall for an interval $X \subset \mathbb{R}$ that $\partial X$ denotes its boundary, i.e. the set of endpoints of $X$.

**Lemma 37.5.** Let $X \subset \mathbb{R}$ be an interval of positive length, and $K : X \times X \to \mathbb{R}$ a Hankel, $TN_2$ kernel. If $K(x, y) = 0$ for some $x, y \in X$, then $K$ vanishes outside `corners, i.e. on $X \times X \setminus \{(x_0, x_0) : x_0 \in \partial X\}$. If moreover $K$ is continuous, then $K \equiv 0$ on $X \times X$.

**Proof.** Suppose $K$ is as given, and $X$ has interior $(a, b)$ for $-\infty \leq a < b \leq \infty$. Suppose some $K(x, y) = 0$; then so is $K(d_0, d_0)$ for $d_0 = (x + y)/2$, as $K$ is Hankel. Again by this property, it suffices to show $K(d, d) = 0 \forall d \in X \setminus \partial X$. We show this for $d \in (a, d_0)$; the proof is similar for $d \in (d_0, b)$.

Let $d \in (a, d_0)$; the $TN_2$ property of $K[(d, d_0); (d, d_0)]$ gives

$$0 \leq K(d, d_0)^2 \leq K(d, d)K(d_0, d_0) = 0.$$  

This shows $K((d + d_0)/2, (d + d_0)/2) = 0$. Now if $a = -\infty$ then running over all $d$ we are done. Else say $a$ is finite. Then the above argument shows that

$$K(d, d) = 0, \quad \forall d \in ((a + d_0)/2, d_0).$$
Now define the sequence
\[ d_{n+1} := (a + 3d_n)/4 \in ((a + d_n)/2, d_n), \quad n \geq 0. \]
Clearly \( d_n \) decreases from \( d_0 \) to \( a^+ \). Now claim by induction that \( K(d,d) = 0 \) for all \( d \in [d_{n+1}, d_n) \). The base case of \( n = 0 \) was shown above, and the same computations show the induction step as well. Finally, the last assertion is now immediate. \( \square \)

With these preliminaries in hand, we can show the above classification result.

**Proof of Theorem 37.1** Clearly (1) \( \implies \) (2); we next show (3) \( \implies \) (1). Suppose (3) holds and \( K : X \times X \to \mathbb{R} \) is Hankel and \( TN \). There are two cases. First, \( K \) vanishes at a point and hence \( K \equiv 0 \) by Lemma 37.5 in which case \( F(0) \geq 0 \) gives: \( (F \circ K)(0) = F(0)1_{X \times X} \) is \( TN \). Otherwise \( F > 0 \) on \( X \times X \), in which case \( F \circ K \) is again continuous, Hankel, and \( TN \) by Corollary 37.3.

It remains to show (2) \( \implies \) (3). First if \( K \equiv 0 \) then \( F(0)1_{X \times X} = (F \circ K) \) is positive semidefinite, so \( F(0) \geq 0 \). Otherwise \( K > 0 \) by Lemma 37.5. We now appeal to Theorem 19.1 and Remark 19.18. Thus, it suffices to show that \( F[-] \) preserves positivity on the matrices \( (a + bu_j^{i+k})_{j,k=0}^n \) for all \( a,b \geq 0 \) with \( a + b > 0 \) and all \( n \), as well as on all rank-one and all Toeplitz matrices in \( \mathbb{P}_2((0,\infty)) \). For this, it suffices to produce continuous Hankel \( TN \) kernels on \( \mathbb{R} \times \mathbb{R} \) which contain the given test set of matrices at equi-spaced arguments.

First, by the assumptions there exist linear maps \( \varphi_n : \mathbb{R} \to \mathbb{R}, n \geq 1 \) with positive slopes such that \( [0,n] \subset \varphi_n(X) \). Now consider the continuous kernel
\[ K_n(x,y) := a + bu_0^{\varphi_n(x)+\varphi_n(y)}, \quad x,y \in \mathbb{R}. \]
This is a rank-two kernel, and easily verified to be Hankel and \( TN \) on \( \mathbb{R} \times \mathbb{R} \), whence on \( X \times X \). Applying \( F \), it follows that the matrix
\[ (F \circ K_n)(x; x) = (F(a + bu_0^{j+k}))_{j,k=0}^n \]
is positive semidefinite as desired. Here \( x := (\varphi_1^{-1}(0), \ldots, \varphi_n^{-1}(n)) \in X^{n+1,\dagger} \) for all \( n \geq 0 \).

Next, if \( A = \begin{pmatrix} p & q \\ q & q^2/p \end{pmatrix} \) has positive entries and rank one, then consider the continuous Hankel kernel associated to the measure \( p\delta_{q/p} \), i.e., \( K(x,y) := p(q/p)^{x+y} \) for \( x,y \in \mathbb{R} \).

Finally, consider the Toeplitz matrix \( \begin{pmatrix} b & a \\ a & b \end{pmatrix} \), with \( 0 < a < b \). It suffices to produce a continuous Hankel \( TN \) kernel containing this matrix. While one can use Theorem 7.4 we provide a direct proof as well. By rescaling by \( b \), we may assume \( b = 1 \). Now choose any \( \alpha \in (1-a^2,1) \) and consider the continuous Hankel kernel
\[ K(x,y) := \alpha \left( a - \sqrt{\frac{(1-\alpha)(1-a^2)}{\alpha}} \right)^{x+y} + (1-\alpha) \left( a + \sqrt{\frac{\alpha(1-a^2)}{1-\alpha}} \right)^{x+y}, \]
for \( x,y \in \mathbb{R} \). It is easy to check that \( K(0,0) = K(1,1) = 1 \) and \( K(0,1) = a \). Moreover, \( K \) is \( TN \) because we reduce to Theorem 4.1 via Lemma 37.2(4). \( \square \)

### 37.2 Structure of Hankel TP kernels on intervals.

We next turn to Hankel TP kernels. For this, we need to understand both Hankel \( TN \) and \( TP \) kernels in greater detail:

**Theorem 37.6.** Suppose \( X \subset \mathbb{R} \) is an non-empty open interval.

1. The following are equivalent for \( K : X \times X \to \mathbb{R} \) a continuous Hankel kernel:
   a) \( K \) is \( TN \).
234 37. Preservers of TP Hankel kernels.

(b) $K$ is positive semidefinite.
(c) There exists a non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$ such that $K(x, y) = \int_{\mathbb{R}} e^{-(x+y)u} \, d\sigma(u)$ for all $x, y \in X$.

Furthermore, the kernel $K$ is TP if and only if the non-negative measure associated to $\sigma$ has infinite support.

(2) The continuous Hankel TP kernels on $X \times X$ are dense in the continuous Hankel TN kernels on $X \times X$.
(3) The space of continuous Hankel TP kernels on $X \times X$ is a convex cone, which is further closed under taking pointwise/Schur products.

The first part is a representation theorem by Widder (1940) in *Bull. Amer. Math. Soc.*, which solves a moment problem in the spirit of Hamburger, Hausdorff, Stieltjes, and others, but now for ‘exponential moments’ of non-negative measures on the real line. These are termed ‘exponentially convex functions’ by Bernstein. The second part reveals a Whitney-type density result for Hankel kernels on an interval, in the spirit of Whitney’s theorem 6.7 for matrices. In the next few sections, we will see similar variants for other structured kernels.

We now turn to Widder’s proof of Theorem 37.6(1). This uses an intermediate notion of ‘kernels of positive type’, which are now introduced.

**Definition 37.7.** Given real numbers $a \leq b$, a continuous symmetric function $K : [a, b] \times [a, b] \to \mathbb{R}$ is said to be a kernel of positive type on $[a, b]^2$ if for all continuous functions $\xi : [a, b] \to \mathbb{R}$, we have

$$\int_a^b \int_a^b K(s, t)\xi(s)\xi(t) \, ds \, dt \geq 0.$$ 

If now $I \subset \mathbb{R}$ is a sub-interval, and $K : I \times I \to \mathbb{R}$ is continuous, we say $K$ is of positive type if it is so on every closed sub-interval of $I$.

The following result relates positive semidefinite kernels with kernels of positive type.

**Lemma 37.8** (Mercer, 1909). Suppose $a \leq b$ are real numbers and $K : [a, b] \times [a, b] \to \mathbb{R}$ is continuous. Then $K$ is a positive semidefinite kernel if and only if $K$ is of positive type.

Thus, Mercer’s lemma provides an alternate equivalent condition to Theorem 37.6(1) for $X$ compact. It was shown by Mercer in *Phil. Trans. Royal Soc. A* (1909), en route to showing the following famous result (which we do not use, nor pursue in this text):

**Theorem 37.9** (Mercer, 1909). Suppose $K : [a, b]^2 \to \mathbb{R}$ is a kernel of positive type. Then there exists an orthonormal basis $\{e_j : j \geq 1\}$ of $L^2[a, b]$, such that: (a) each $e_j$ is an eigenfunction of the integral operator $T_K \varphi(x) := \int_a^b K(x, s)\varphi(s) \, ds$; (b) the corresponding eigenvalue $\lambda_j$ is non-negative; (c) if $\lambda_j > 0$ then $e_j$ is continuous; and (d) $K$ has the representation

$$K(s, t) = \sum_{j \geq 1} \lambda_j e_j(s)e_j(t).$$

**Proof of Mercer’s lemma 37.8.** If $K$ is a positive semidefinite kernel, and $\xi : [a, b] \to \mathbb{R}$ is continuous, then the double integral

$$\int_a^b \int_a^b K(s, t)\xi(s)\xi(t) \, ds \, dt$$
can be expressed as a limit of Riemann sums:

$$\lim_{n \to \infty} \frac{(b - a)^2}{n^2} \sum_{j,k=0}^{n-1} K(s_j, s_k) \xi(s_j) \xi(s_k) = \lim_{n \to \infty} u_n^T K_n u_n,$$

where

$$s_j := a + j(b - a)/n, \quad K_n := (K(s_j, s_k))_{j,k=0}^{n-1}, \quad u_n = \frac{b - a}{n} (\xi(s_1), \ldots, \xi(s_n))^T.$$

But the right-hand limit is non-negative since $K$ is positive semidefinite.

Conversely, suppose there exist scalars $a \leq s_1 < \cdots < s_n \leq b$ such that the matrix $K_n := (K(s_j, s_k))_{j,k=0}^n$ is not positive semidefinite. Since $K$ is continuous, a perturbation argument allows us to assume $s_0 > a$ and $s_n < b$. Thus there exists an eigenvector $u_n = (\xi_0, \ldots, \xi_n)^T$ such that $u_n^T K_n u_n =: A < 0$.

We now construct a ‘toothsaw’ test function, as follows: we will choose $\epsilon, \eta > 0$, running only over values satisfying:

$$\epsilon + \eta < \min\{s_0 - a, (s_2 - s_1)/2, \ldots, (s_n - s_{n-1})/2, b - s_n\}.$$

Define $\theta_{\epsilon, \eta}(x)$ to be the unique piecewise linear, continuous function on $(a, b)$ such that

$$\theta_{\epsilon, \eta}(x) := \begin{cases} \xi_j, & \text{if } x \in (s_j - \eta, s_j + \eta), \\ 0, & \text{if } x \in (a, s_0 - \epsilon - \eta] \cup [s_n + \epsilon + \eta, b), \\ 0, & \text{if } x \in \bigcup_{j=1}^n [s_j - 1 + \epsilon + \eta, s_j - \epsilon - \eta], \end{cases}$$

and $\theta_{\epsilon, \eta}$ is linear on all remaining sub-intervals $[s_j - \epsilon - \eta, s_j - \eta]$ and $[s_j + \eta, s_j + \epsilon + \eta]$.

Now define $F_n : [-\eta, \eta]^2 \to \mathbb{R}$ via:

$$F_n(x, y) := \sum_{j,k=0}^n K(s_j + x, s_k + y) \xi_j \xi_k.$$

A straightforward computation yields:

$$\int_a^b \int_a^b K(s, t) \theta_{\epsilon, \eta}(s) \theta_{\epsilon, \eta}(t) \, ds \, dt = \sum_{j,k=0}^n \int_{x_j - \epsilon - \eta}^{x_j + \epsilon + \eta} \int_{x_k - \epsilon - \eta}^{x_k + \epsilon + \eta} K(s, t) \theta_{\epsilon, \eta}(s) \theta_{\epsilon, \eta}(t) \, ds \, dt$$

$$= \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} F_n(x, y) \, dx \, dy + J_1,$$

where

$$J_1 = \sum_{j,k=0}^n \int_{Q_{jk}} K(s, t) \theta_{\epsilon, \eta}(s) \theta_{\epsilon, \eta}(t) \, ds \, dt,$$

with $Q_{jk}$ the ‘annular’ region between the squares

$$[x_j - \epsilon - \eta, x_j + \epsilon + \eta] \times [y_k - \epsilon - \eta, y_k + \epsilon + \eta] \quad \text{and} \quad [x_j - \eta, x_j + \eta] \times [y_k - \eta, y_k + \eta].$$

Now for $(s, t) \in Q_{jk}$ we have $|\theta_{\epsilon, \eta}(s) \theta_{\epsilon, \eta}(t)| \leq |\xi_j \xi_k|$, so by an easy computation:

$$|J_1| \leq 4 \epsilon (2\eta + \epsilon) \left( \sum_{j=0}^n |\xi_j| \right)^2 \cdot \|K\|_\infty$$
where \( \|K\|_\infty \) is the supnorm of \( K \) on the compact domain \([a, b]^2\). Note that \( F_n(0, 0) = A < 0 \) (as above), so by continuity there exists a small \( \eta > 0 \) such that \( F_n(x, y) < A/2 \) when \(|x|, |y| < \eta\). Fixing such an \( \eta \),

\[
\int_a^b \int_a^b K(s, t)\theta_{c,\eta}(s)\theta_{c,\eta}(t) \, ds \, dt \leq |J_1| + \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} A/2 \, dx \, dy
\]

for \( \epsilon \) sufficiently small. Choose \( \epsilon > 0 \) small enough such that the right-hand side is negative (since \( A < 0 \)); this yields the desired contradiction. \( \square \)

Another preliminary result required to show Theorem 37.6 is by Bernstein in his 1926 memoir. This is a ‘dimension-free’ analogue of the Boas–Widder theorem 18.10(2) – also recall the related Theorem 19.9 by Boas. In fact, Boas and Widder write that they were motivated to prove their Theorem 18.10(2) ‘in an effort to make more accessible’ the following result of Bernstein, which Widder used in proving Theorem 37.6(1).

**Theorem 37.10** (Bernstein, 1926). Given a sub-interval \((a, b) \subset \mathbb{R}\) and a continuous function \( f : (a, b) \to \mathbb{R} \), if the even-order forward differences (\( \Delta_2^n f(c) := \sum_{j=0}^{2n} \binom{2n}{j} f(c + j\delta) \)) for \( c \in (a, b), \delta \in (0, (b-c)/2n) \) are all non-negative, then \( f \) is analytic in \((a, b)\).

The final preliminary result is by Hamburger (1920) in Math. Z.:

**Proposition 37.11** (Hamburger). If \( f(x) \) is analytic in \((a, b)\) and there exists \( c \in (a, b) \) such that the semi-infinite Hankel matrix \( (f^{j+k}(c))_{j,k \geq 0} \) is positive semi-definite, then

\[
f(x) = \int_{\mathbb{R}} e^{-xu} \, d\sigma(u)
\]

for some non-decreasing function \( \sigma \), with the integral converging on \( x \in (a, b) \).

These two results are used without proofs.

**Proof of Theorem 37.6**

(1) That (a) and (b) are equivalent follows by Lemma 37.2. Next assume (b) holds, let \([a, b] \subset X\), and let the continuous function \( f : X + X \to \mathbb{R} \) be given by: \( f(x+y) = K(x, y) \) for \( x, y \in X \). Suppose \( a \leq c < c + 2n\delta \leq b \) for some integer \( n > 0 \) and scalar \( 0 < \delta < b/c - 2n \). The Hankel matrix \( K_{n,c,\delta} := (f(c+j\delta+k\delta))_{j,k=0}^n \) is positive semidefinite by assumption; we evaluate it against the vector

\[
\xi = (\xi_0, \ldots, \xi_n)^T, \quad \xi_j := \sum_{l=j}^{n} (-1)^{j+1} \binom{j}{l} \eta_l
\]

for some scalars \( \eta_0, \ldots, \eta_n \). In the language of forward differences, this yields:

\[
\xi^T K_{n,c,\delta} \xi = \sum_{j,k=0}^{n} \Delta_\delta^{j+k} f(c)\eta_j\eta_k \geq 0.
\]
Now set \( \eta_0 = \eta_1 = \cdots = \eta_{n-1} = 0, \eta_n = 1 \), we have \( (\Delta_\delta^{2n} f)(c) \geq 0 \). By Theorem 37.10 since \( f \) is continuous, \( f \) is analytic in \((a, b)\). Now replacing \( \eta_j \) in the preceding computation by \( \eta_j / \delta^j \), we obtain as \( \delta \to 0^+ \):

\[
\sum_{j,k=0}^{n} f^{(j+k)}(c) \eta_j \eta_k \geq 0.
\]

As this holds for all integers \( n \geq 0 \) and all \( \eta_0, \ldots, \eta_n \in \mathbb{R} \), Proposition 37.11 implies \( f \) is of the desired form.

Conversely, if (c) holds, then to show that a principal submatrix drawn from \( K \) at arguments \( x_0 < x_1 < \cdots < x_n \) in \( X \) is positive semidefinite, it suffices to consider \( K \) restricted to \([a, b]^2\), where \( a = x_0 \) and \( b = x_n \). Now by Mercer’s lemma 37.8, it suffices to show that \( K|_{[a, b]^2} \) is of positive type. But this is straightforward: given a continuous function \( \xi : [a, b] \to \mathbb{R} \), we have

\[
\int_{a/2}^{b/2} \int_{a/2}^{b/2} f(s+t)\xi(s)\xi(t) \, ds \, dt = \int_{a/2}^{b/2} \int_{a/2}^{b/2} \xi(s)\xi(t) \int_{\mathbb{R}} e^{-(s+t)u} \, d\sigma(u) \, ds \, dt
\]

\[
= \int_{\mathbb{R}} \left( \int_{a/2}^{b/2} e^{-su} \xi(s) \, ds \right)^2 \, d\sigma(u) \geq 0.
\]

The final step involves a change of order of integration, which is justified because the integral representation of \( f \) converges uniformly in \([a, b]\).

This ends the proof of the equivalence. Now suppose the measure associated to \( \sigma \) has infinite support. Then the kernel \( K(x, y) = f(x+y) \) is \( TP \), by Proposition 6.2. If needed, we can use the fact that a kernel \( K(x, y) \) on \( X \times Y \) (for \( X, Y \subset \mathbb{R} \)) is \( TN/TP \) if and only if the kernel \( K(-x, -y) \) is so, because drawing square submatrices from one or the other kernel are equivalent, modulo applying the order reversing permutation to the rows as well as the columns.

Next, suppose the measure for \( \sigma \) has finite support, say with mass \( c_k \) at \( u_k \in \mathbb{R} \) for \( k = 1, \ldots, r \). Given two \( n \)-tuples of points \( x, y \in X^{n, r} \), we see that

\[
K[x; y] = \sum_{k=1}^{r} c_k \left( e^{-[(x_i+y_j)u_k]} \right)_{i,j=1}^{n} = \sum_{k=1}^{r} c_k x_0^{ou_k} y_0^{ou_k}, \tag{37.12}
\]

where \( x_0 = (e^{-x_i})_{i=1}^{n}, y_0 = (e^{-y_j})_{j=1}^{n} \). Thus \( K[x; y] \) has rank \( \leq r \), hence is singular if \( n > r \). It follows that \( K \) is not \( TP \).

(2) Given a continuous Hankel \( TN \) kernel \( K(x, y) = f(x+y) \) on \( X \times X \), with the function \( f \) as in the previous part, \( K \) is the limit as \( \epsilon \to 0^+ \), of the kernels

\[
K_\epsilon(x, y) := K(x, y) + \epsilon \int_{0}^{1} e^{-(x+y)u} \, du.
\]

Since each underlying measure in \( K_\epsilon \) has infinite support, \( K_\epsilon \) is \( TP \) by the previous part.

(3) Note that \( TP \) kernels are closed under dilations. If now \( K, K' \) are \( TP \), with underlying representative functions \( \sigma, \sigma' \), then the measures corresponding to these have infinite supports, whence the same holds for \( \sigma + \sigma' \). Hence \( K + K' \) is also \( TP \). Finally, \( K \cdot K' \) is \( TN \) by Corollary 37.3. If it is not \( TP \), then the representative function \( \tau \) has underlying measure of finite support, say of size \( r \). Now choose an arithmetic progression \( x \in X^{r+1, r} \); then the principal submatrices \( K[x; x] \) and \( K'[x; x] \) are Hankel
by the choice of $x$ and $TP$ by assumption. Hence so is their Schur product, which is precisely $(K \cdot K')[x; x]$, by Theorem 4.1. In particular, this is a principal submatrix of $K \cdot K'$ of rank $r + 1$, which contradicts the choice of $r$, say by (37.12) for $K \cdot K'$.

□

37.3. Preservers of Hankel $TP$ kernels. To conclude, we classify the preservers of Hankel $TP$ kernels, parallel to the $TN$ version in Theorem 37.1.

Theorem 37.13. Let $X \subseteq \mathbb{R}$ be an open sub-interval with positive measure, and a function $F : (0, \infty) \to \mathbb{R}$. The following are equivalent:

1. The composition map $F \circ -$ preserves total positivity on the continuous Hankel $TP$ kernels on $X \times X$.

2. The composition map $F \circ -$ preserves positive definiteness on the continuous Hankel $TP$ kernels on $X \times X$.

3. The function $F$ is a power series with non-negative coefficients: $F(x) = \sum_{k=0}^{\infty} c_k x^k$ for $x > 0$, with all $c_k \geq 0$; and $F$ is non-constant.

Proof. Clearly (1) $\Rightarrow$ (2). Next, suppose (3) holds, with $c_{n_0} > 0$ for some $n_0 > 0$. Let $K : X \times X \to \mathbb{R}$ be a continuous $TP$ Hankel kernel, then so is $K^{n_0}$ by Theorem 37.6. Moreover, Corollary 37.3 shows $G \circ K$ is a continuous $TN$ Hankel kernel, where $G(x) := F(x) - c_{n_0} x^{n_0}$. Now $G \circ K$ and $K^{n_0}$ have integral representations as above, say with corresponding non-negative measures $\nu$ and $\mu$ respectively. Since $\mu$ has infinite support from above, so does $\nu + c_{n_0} \mu \geq 0$. Hence $F \circ K = G \circ K + c_{n_0} K^{n_0}$ is TP.

It remains to show (2) $\Rightarrow$ (3). First note by Theorem 7.4 that every $2 \times 2$ TP matrix occurs as a ‘principal submatrix’ of a continuous Hankel $TP$ kernel on $\mathbb{R} \times \mathbb{R}$, drawn from a function evaluated at equi-spaced arguments. Hence by Lemma 12.14, $f$ is continuous, positive, and strictly increasing on $(0, \infty)$. It follows that $f$ preserves continuous Hankel $TN$ kernels, hence is of the desired form by Theorem 37.6(2). Clearly $f$ is non-constant, and the proof is complete. □
38. Total positivity preservers: All kernels.

We now bring together many of the techniques and results discussed above – not only in this part, but in the previous parts of this text – to solve the motivating problem in this part: \textit{Classify the functions preserving totally non-negative/positive kernels on } \( X \times Y \), \textit{where } \( X, Y \) \textit{are (arbitrary) non-empty totally ordered sets.}

Recall the characterizations in Theorems \[12.11\] and \[12.13\] which resolved this question for \( X, Y \) finite. Similarly, Theorem \[12.15\] and Corollary \[12.17\] respectively classified the preservers of \( \text{TN} \) and \( \text{TP} \) symmetric kernels, for \( X = Y \) finite.

38.1. Finite-continuum kernels. In this section, we answer the above question when at least one of \( X, Y \) is infinite. The first step is to resolve this when exactly one of \( X, Y \) is finite and the other is infinite; in this case we do not consider symmetric kernels.

The key result which is required to solve the classification question is a recent extension of Whitney’s density theorem, which uses discretized Gaussian convolution.

\textbf{Theorem 38.1.} \textit{Given an integer } \( p \geq 2 \), \textit{and a bounded } \( \text{TN}_p \) \textit{kernel } \( K \) \textit{on } \( \mathbb{R} \times \mathbb{R} \), \textit{let } \( C \subset \mathbb{R}^2 \) \textit{denote the points of continuity of } \( K \). \textit{Then there exists a sequence of } \( \text{TP}_p \) \textit{kernels } \( (K_l)_{l \geq 1} \) \textit{that converge to } \( K \) \textit{locally uniformly on } \( C \). \textit{If moreover } \( K \) \textit{is ‘symmetric’, i.e. } \( K(x, y) = K(y, x) \) \textit{for all } \( x, y \in \mathbb{R} \), \textit{then the sequence } \( K_l \) \textit{may also be taken to be symmetric for all } \( l \geq 1 \).

Theorem 38.1 is due to Belton, Guillot, Khare, and Putinar (2020), for arbitrary subsets \( X, Y \subset \mathbb{R} \) – which is not more general because one can always extend such a kernel to one on \( \mathbb{R} \times \mathbb{R} \), by padding by zeros. Notice also that the assumption that \( X = Y = \mathbb{R} \) is itself not unnecessarily restrictive, given Lemma 7.2 (For \( p = 1 \), a \( \text{TP}_1 \) kernel is merely a positive-valued function, and so \( K + \frac{1}{4}1 \) approximates any \( \text{TN}_1 \) kernel \( K \).)
Most of the material in this part is taken from Belton–Guillot–Khare–Putinar [23], and we discuss the remaining references.

Theorem 37.6(1) is a representation theorem for totally non-negative continuous Hankel kernels on an open interval, shown by Widder [294]. Mercer’s lemma 37.8 and theorem 37.9 are from [201]. Bernstein’s theorem 37.10 guaranteeing analyticity from the positivity of even-order forward differences is from [31]. Hamburger’s theorem 37.11 is from [121]. (See also [32] and [6, Theorem 5.5.4].)
Part 6:
Entrywise polynomials preserving positivity in fixed dimension
Part 6: Entrywise polynomials preserving positivity in fixed dimension

Classification of sign patterns.

In Part 3 of this text, we classified the entrywise functions preserving positivity in all dimensions; these are precisely the power series with non-negative coefficients. Earlier in Part 2, we had classified the entrywise powers preserving positivity (as well as total positivity and total non-negativity) in fixed dimension. In this final part of this text, we study polynomials that entrywise preserve positive semidefiniteness in fixed dimension.

Recall from the Schur product theorem [3.10] and its converse, the Schoenberg–Rudin theorem [16.2] that the only polynomials that entrywise preserve positivity in all dimensions are the ones with all non-negative coefficients. Thus, if one fixes the dimension $N \geq 3$ of the test set of positive matrices, then it is reasonable to expect that there should exist more polynomial preservers – in other words, polynomial preservers with negative coefficients. However, this problem remained completely open until very recently (∼2016): not a single polynomial preserver was known with a negative coefficient, nor was a non-existence result proved!

In this final part, we answer this existence question, as well as stronger variants of it. Namely, not only do we produce such polynomial preservers, we also fully resolve the more challenging question: which coefficients of polynomial preservers on $N \times N$ matrices can be negative? Looking ahead in this part:

- We classify the sign patterns of entrywise polynomial preservers on $\mathbb{P}_N$, for fixed $N$.
- We extend this to all power series; but also countable sums of real powers, such as $\sum_{\alpha \in \mathbb{Q}, \alpha \geq N-2} c_\alpha x^\alpha$. This case is more subtle than that of polynomial preservers.
- We will also completely classify the sign patterns of polynomials that entrywise preserve Hankel totally non-negative matrices of a fixed dimension. Recall from the discussions around Theorems [12.19] and [19.1] that this is expected to be very similar to (if not exactly the same as) the classification for positivity preservers.

In what follows, we work with $\mathbb{P}_N((0,\rho))$ for $N > 0$ fixed and $0 < \rho < \infty$. Since we work with polynomials and power series, this is equivalent to working over $\mathbb{P}_N((0,\rho))$, by density and continuity. If $\rho = +\infty$, one can prove results that are similar to the ones shown below; but for a ‘first look’ at the proofs and techniques used, we restrict ourselves to $\mathbb{P}_N((0,\rho))$. For full details of the $\rho = +\infty$ case, as well as ramifications and applications of the results below, we refer the reader to the 2017 preprint “On the sign patterns of entrywise positivity preservers in fixed dimension” by Khare and Tao.

39.1. Horn-type necessary condition; matrices with negative entries. In this section and beyond, we work with polynomials or power series

$$f(x) = c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots, \quad \text{with } n_0 < n_1 < \cdots,$$

and $c_{n_j} \in \mathbb{R}$ typically nonzero. Recall the (stronger) Horn–Loewner theorem [17.1] which shows that if $f \in C^{(N-1)}(I)$ for $I = (0,\infty)$, and $f[-]$ preserves positivity on (rank two Hankel $TN$ matrices in) $\mathbb{P}_N(I)$, then $f, f', \ldots, f^{(N-1)} \geq 0$ on $I$. In the special case that $f$ is a polynomial or a power series, one can say more, and under weaker assumptions:

**Lemma 39.2** (Horn-type necessary condition). Fix an integer $N > 0$. Let $\rho > 0$ and $f : (0,\rho) \to \mathbb{R}$ be of the form (39.1), and assume that $f[-]$ preserves positivity on rank-one Hankel $TN$ matrices in $\mathbb{P}_N((0,\rho))$. If $c_m < 0$ for some $m$, then $c_n > 0$ for at least $N$ values $n < m$. 
Remark 39.3. In both (39.1) as well as Lemma 39.2 (and its proof below), we have deliberately not insisted on the exponents $n_j$ being non-negative integers. In fact, one can choose \{n_j : j \geq 0\} to be an arbitrary increasing sequence of real numbers.

Proof of Lemma 39.2. Suppose the result is false. Then $f(x) = \sum_{j=0}^{m} c_j x^{n_j}$, with $m = n_k$ for some $0 \leq k < N$ and $c_0, \ldots, c_{n_k - 1} > 0$. Choose any $u_0 \in (0, 1)$ and define $u := (1, u_0, \ldots, u_0^{N-1})^T \in \mathbb{R}^N$. Then $u^{n_0}, \ldots, u^{n_k}$ are linearly independent, forming (some of) the columns of a generalized Vandermonde matrix. Hence there exists $v \in \mathbb{R}^N$ such that

\[ v \perp u^{n_0}, \ldots, u^{n_k-1} \quad \text{and} \quad v^T u^{n_0} = v^T u^{n_k} = 1. \]

For $0 < \epsilon < \rho$, we let $A_{\epsilon} := c u u^T$, which is a rank-one Hankel moment matrix in $\mathbb{P}_N((0, \rho))$ (and hence $TN$). Now compute using the hypotheses:

\[ 0 \leq v^T f[A_{\epsilon}] v = v^T \left( \sum_{j=0}^{m} c_j \epsilon^{n_j} u^{n_j} (u^{n_j})^T \right) v = c_{n_k} \epsilon^{n_k} + o(\epsilon^{n_k}). \]

Thus $0 \leq \lim_{\epsilon \to 0^+} \frac{v^T f[A_{\epsilon}] v}{\epsilon^{n_k}} < 0$, which is a contradiction. Hence $k \geq N$, proving the claim. □

Thus, by Lemma 39.2, any entrywise polynomial preserver of positivity on $\mathbb{P}_N((0, \rho))$ must have its $N$ nonzero Maclaurin coefficients of ‘lowest degree’ to be positive. The obvious question is if any of the other terms can be negative, e.g. the immediate next coefficient.

We tackle this question in the remainder of this text, and show that in fact every other coefficient can indeed be negative. For now, we point out that working with positive matrices with other entries can not provide such a structured answer (in the flavor of Lemma 39.2).

As a simple example, consider the family of polynomials

\[ p_{k,t}(x) := t(1 + x^2 + \cdots + x^{2k}) - x^{2k+1}, \quad t > 0, \]

where $k \geq 0$ is an integer. Now claim that $p_{k,t}[\cdot]$ can never preserve positivity on $\mathbb{P}_N((-\rho, \rho))$ for $N \geq 2$. Indeed, if $u := (1, -1, 0, \ldots, 0)^T$ and $A := (\rho/2) uu^T \in \mathbb{P}_N((-\rho, \rho))$, then

\[ u^T p_{k,t}[A] u = -4(\rho/2)^{2k+1} < 0. \]

Therefore $p_{k,t}[A]$ is not positive semidefinite for any $k \geq 0$. If one allows complex entries, similar examples with higher-order roots of unity can be constructed, in which such negative results (compared to Lemma 39.2) can be obtained.

39.2. Classification of sign patterns for polynomials. In light of the above discussion, henceforth we restrict ourselves to working with matrices in $\mathbb{P}_N((0, \rho))$ for $0 < \rho < \infty$. By Lemma 39.2, every polynomial preserver on $\mathbb{P}_N((0, \rho))$ must have its $N$ lowest-degree Maclaurin coefficients (which are nonzero) to be positive.

We are interested in understanding if any (or every) other coefficient can be negative. If say the next lowest-degree coefficient could be negative, this would achieve two goals:

- It would provide (the first example of) a polynomial preserver in fixed dimension, which has a negative Maclaurin coefficient.
- It would provide (the first example of) a polynomial that preserves positivity on $\mathbb{P}_N((0, \rho))$, but necessarily not on $\mathbb{P}_{N+1}((0, \rho))$. In particular, this would show that the Horn-type necessary condition in Lemma 39.2 is “best possible”. (See Remark 17.3 in the parallel setting of entrywise power preservers, for the original Horn condition.)

We show in this part of the text that these goals are indeed achieved:
Theorem 39.4 (Classification of sign patterns, fixed dimension). Fix integers \( N > 0 \) and \( 0 \leq n_0 < n_1 < \cdots < n_{N-1} \), as well as a sign \( \varepsilon_M \in \{-1,0,1\} \) for each integer \( M > n_{N-1} \). Given reals \( \rho, c_{n_0}, c_{n_1}, \ldots, c_{n_{N-1}} > 0 \), there exists a power series
\[
f(x) = c_{n_0}x^{n_0} + \cdots + c_{n_{N-1}}x^{n_{N-1}} + \sum_{M > n_{N-1}} c_M x^M,
\]
satisfying the following properties:

1. \( f \) is convergent on \((0, \rho)\).
2. \( f[-] : \mathbb{P}_N((0, \rho)) \to \mathbb{P}_N \).
3. \( \text{sgn}(c_M) = \varepsilon_M \) for each \( M > n_{N-1} \).

This is slightly stronger than classifying the sign patterns, in that the ‘initial coefficients’ are also specified. In fact, this result can be strengthened in two different ways; see (i) Theorem 39.7, in which the set of powers allowed is vastly more general; and (ii) Theorem 42.12 and the discussion preceding it, in which the coefficients for \( M > n_{N-1} \) are also ‘specified’.

Proof. Suppose we can prove the theorem in the special case when exactly one \( \varepsilon_M \) is negative. Then for each \( M > n_{N-1} \), there exists \( 0 < \delta_M < \frac{1}{M} \) such that
\[
f_M(x) := \sum_{j=0}^{N_1} c_{n_j} x^{n_j} + c_M x^M
\]
preserves positivity on \( \mathbb{P}_N((0, \rho)) \) whenever \( |c_M| \leq \delta_M \). Set \( c_M := \varepsilon_M \delta_M \) for each \( M > n_{N-1} \) and define \( f(x) := \sum_{M > n_{N-1}} 2^{n_{N-1} - M} f_M(x) \). If \( x \in (0, \rho) \), then we have
\[
|f(x)| \leq \sum_{M > n_{N-1}} 2^{n_{N-1} - M} |f_M(x)| \leq \sum_{M > n_{N-1}} 2^{n_{N-1} - M} \left( \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + \delta_M x^M \right)
\]
\[
\leq \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + e^x < \infty.
\]
Hence \( f \) converges on \((0, \rho)\). As each \( f_M[-] \) preserves positivity and \( \mathbb{P}_N \) is a closed convex cone, \( f[-] \) also preserves positivity. It therefore remains to show that the result holds when one coefficient is negative. But this follows from Theorem 39.5 below. \( \square \)

Thus, it remains to show the following ‘qualitative’ result:

Theorem 39.5. Let \( N > 0 \) and \( 0 \leq n_0 < n_1 < \cdots < n_{N-1} < M \) be integers, and \( \rho, c_{n_0}, c_{n_1}, \ldots, c_{n_{N-1}} > 0 \) be real. Then the function \( f(x) = \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c_M x^M \) entrywise preserves positivity on \( \mathbb{P}_N((0, \rho)) \), for some \( c_M < 0 \).

We will show this result in the next two sections.

39.3. Classification of sign patterns for sums of real powers. Below, after proving Theorem 39.5, we further strengthen it by proving a quantitative version – see Theorem 42.1 – which gives a sharp lower bound on \( c_M \). For now, we list a special case of that result (without proof, as we show the more general Theorem 42.1 below):

Theorem 39.6. Theorem 39.5 holds even when the exponents \( n_0, n_1, \ldots, n_{N-1}, M \) are real and lie in the set \( \mathbb{Z}^\geq 0 \cup [N-2, \infty) \).
With Theorem 39.6 in hand, it is possible to classify the sign patterns of a more general family of preservers, of the form $f(x) = \sum_{j=0}^{\infty} c_{n_j} x^{n_j}$, where $n_j \in \mathbb{Z}_{\geq 0} \cup [N-2, \infty)$ are an arbitrary countable collection of pairwise distinct non-negative (real) exponents.

**Theorem 39.7** (Classification of sign patterns of power series preservers, fixed dimension). Let $N \geq 2$ and let $n_0, n_1, \ldots$ be a sequence of pairwise distinct real numbers in $\mathbb{Z}_{\geq 0} \cup [N-2, \infty)$. For each $j \geq 0$, let $\varepsilon_j \in \{-1, 0, 1\}$ be a sign such that whenever $\varepsilon_{j_0} = -1$, one has $\varepsilon_j = +1$ for at least $N$ choices of $j$ satisfying $n_j < n_{j_0}$. Then for every $\rho > 0$, there exists a series with real exponents and real coefficients

$$f(x) = \sum_{j=0}^{\infty} c_{n_j} x^{n_j}$$

which is convergent on $(0, \rho)$, which entrywise preserves positivity on $\mathbb{P}_N((0, \rho))$, and in which $\text{sgn}(c_{n_j}) = \varepsilon_j$ for all $j \geq 0$.

That the sign patterns must satisfy the given hypotheses follows from Lemma 39.2. In particular, Theorem 39.7 shows that the Horn-type necessary condition in Lemma 39.2 remains the best possible in this generality as well.

**Remark 39.8.** A key difference between the classifications in Theorems 39.4 and 39.7 is that the latter is more flexible, since the sequence $n_0, n_1, \ldots$ can now contain an infinite decreasing subsequence of exponents. This is more general than even Hahn or Puiseux series, not just power series. For instance, the sum may be over all rational powers in $\mathbb{Z}_{\geq 0} \cup [N-2, \infty)$.

**Proof of Theorem 39.7**. Given any set $\{n_j : j \geq 0\}$ of (pairwise distinct) non-negative powers,

$$\sum_{j \geq 0} \frac{x^{n_j}}{j! |n_j|!} < \infty, \quad \forall x > 0. \tag{39.9}$$

Indeed, if we partition $\mathbb{Z}_{\geq 0}$ into the disjoint union of $J_k := \{j \geq 0 : n_j \in (k-1, k]\}$, $k \geq 0$, then using Tonelli’s theorem, we can estimate:

$$\sum_{j \geq 0} \frac{x^{n_j}}{j! |n_j|!} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j \in J_k} \frac{x^{n_j}}{j!} \leq e + \sum_{k \geq 1} \frac{1}{k!} \sum_{j \in J_k} \frac{x^k + x^{k-1}}{j!} < e + e^{x^k + x^{k-1}} < \infty.$$

We now turn to the proof. Let $J := \{j : \varepsilon_j = -1\} \subset \mathbb{Z}_{\geq 0}$. For each $l \in J$, by Lemma 39.2 there exist $j_1(l), \ldots, j_N(l)$ such that $\varepsilon_{j_k(l)} = 1$ and $n_{j_k(l)} < n_l$, for $k = 1, \ldots, N$. Define

$$f_l(x) := \sum_{k=1}^{N} \frac{x^{n_{j_k(l)}}}{[n_{j_k(l)}]!} \delta_l \frac{x^{n_l}}{[n_l]!}$$

where $\delta_l \in (0, 1)$ is chosen such that $f_l[-]$ preserves positivity on $\mathbb{P}_N((0, \rho))$ by Theorem 39.6. Let $J' \subset \mathbb{Z}_{\geq 0}$ consist of all $j \geq 0$ such that $\varepsilon_j = +1$ but $j \neq j_l(l)$ for any $l \in J$, $k \in [1, N]$. Finally, define

$$f(x) := \sum_{l \in J} f_l(x) \frac{1}{l!} + \sum_{j \in J'} \frac{x^{n_j}}{j! |n_j|!}, \quad x > 0.$$

Repeating the calculation in (39.9), one can verify that $f$ converges absolutely on $(0, \infty)$ and hence on $(0, \rho)$. By the above hypotheses and the Schur product theorem, it follows that $f[-]$ preserves positivity on $\mathbb{P}_N((0, \rho))$. \qed
The goal in this section and the next is to prove the ‘qualitative’ Theorem 39.5 in the previous section. Thus, we work with polynomials of the form
\[ f(x) = \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c_M x^M, \]
where \( N > 0 \) and \( 0 \leq n_0 < n_1 < \ldots < n_{N-1} < M \) are integers, and \( \rho, c_{n_0}, c_{n_1}, \ldots, c_{n_{N-1}} > 0 \) are real.

40.1. Basic properties of Schur polynomials. We begin by defining the key tool required in this section: Schur polynomials. These functions are then used – via the Cauchy–Binet identity – to understand when polynomials of the above form entrywise preserve positivity on a ‘generic’ rank-one matrix in \( \mathbb{P}_N((0, \rho)) \).

Definition 40.1. Fix integers \( m, N > 0 \), and define \( n_{\text{min}} := (0, 1, \ldots, N-1) \). Now suppose \( 0 \leq n'_0 \leq n'_1 \leq \cdots \leq n'_{N-1} \) are also integers.

(1) A column-strict Young tableau, with shape \( n' := (n'_0, n'_1, \ldots, n'_{N-1}) \) and cell entries \( 1, 2, \ldots, m \), is a left-aligned two-dimensional rectangular array \( T \) of cells, with \( n'_0 \) cells in the bottom row, \( n'_1 \) cells in the second lowest row, and so on, such that:
   - Each cell in \( T \) has integer entry \( j \) with \( 1 \leq j \leq m \).
   - Entries weakly decrease in each row, from left to right.
   - Entries strictly decrease in each column, from top to bottom.

(2) Given variables \( u_1, u_2, \ldots, u_m \) and a column-strict Young tableau \( T \) as above, define its weight to be
   \[ \text{wt}(T) := \prod_{j=1}^{m} u_j^{f_j}, \]
   where \( f_j \) equals the number of cells in \( T \) with entry \( j \).

(3) Given an increasing sequence of integers \( 0 \leq n_0 < \cdots < n_{N-1} \), define the tuple \( n := (n_0, n_1, \ldots, n_{N-1}) \), and the corresponding Schur polynomial over \( u := (u_1, u_2, \ldots, u_m)^T \) to be
   \[ s_n(u) := \sum_T \text{wt}(T), \quad (40.2) \]
   where \( T \) runs over all column-strict Young tableaux of shape \( n' := n - n_{\text{min}} \) with cell entries \( 1, 2, \ldots, m \). (We will also abuse notation slightly and write \( s_n(u_1, \ldots, u_m) \) on occasion.)

Example 40.3. Suppose \( N = m = 3 \), and \( n = (0, 2, 4) \). The column-strict Young Tableaux with shape \( n - n_{\text{min}} = (0, 1, 2) \) and cell entries \( 1, 2, 3 \) are:

```
  3 3
  2 1

  3 3
  2 1

  3 2
  2 1

  3 2
  2 1

  3 1
  2 1

  3 1
  2 1

  2 2
  2 1

  2 1
```

As a consequence,
\[
\begin{align*}
  s_{(0,2,4)}(u_1, u_2, u_3) &= u_1^2 u_2 + u_2^2 u_3 + u_3 u_2^2 + 2 u_3 u_2 u_1 + u_3 u_1^2 + u_2^2 u_1 + u_2 u_1^2 \\
  &= (u_1 + u_2)(u_2 + u_3)(u_3 + u_1).
\end{align*}
\]
Remark 40.4. A visible notational distinction with the literature is that column-strict tableaux traditionally have entries that are \emph{increasing} down columns, and weakly increasing as one moves across rows. Since we only work with sets of tableaux through the sums of their weights occurring in Schur polynomials, this distinction is unimportant in the text, for the following reason: define an involutive bijection \( \iota : j \mapsto m + 1 - j \), where \( \{1, \ldots, m\} \) is the alphabet of possible cell entries. Then the column-strict Young tableaux in our notation bijectively correspond under \( \iota \) – applied to each cell entry – to the ‘usual’ column-strict Young tableaux (in the literature); and as Schur polynomials are symmetric under permuting the variables by \( \iota \) (see Proposition 40.6), the sums of weights of the two sets of tableaux coincide.

Remark 40.5. Schur polynomials are fundamental objects in type \( A \) representation theory (of the general linear group, or the special linear Lie algebra), and are characters of irreducible finite-dimensional polynomial representations (over fields of characteristic zero). The above example 40.3 is a special case, corresponding to the adjoint representation for the Lie algebra of \( 3 \times 3 \) traceless matrices. This interpretation will not be used below.

Schur polynomials are always homogeneous – and also symmetric, because they can be written as a quotient of two generalized Vandermonde determinants. This is Cauchy’s definition; the definition \( 40.2 \) using Young tableaux is by Littlewood. One can show that these two definitions are equivalent, among other basic properties:

Proposition 40.6. Fix integers \( m = N > 0 \) and \( 0 \leq n_0 < n_1 < \cdots < n_{N-1} \).

1. (Cauchy’s definition.) If \( F \) is a field and \( u = (u_1, \ldots, u_N)^T \in F^N \), then
   \[
   \det(u^{n_0} | u^{n_1} | \cdots | u^{n_{N-1}})_{N \times N} = V(u)s_n(u),
   \]
   where \( V(u) := \prod_{1 \leq j < k \leq N}(u_k - u_j) \) is the Vandermonde determinant as in (17.5). In particular, \( s_n(u) \) is symmetric and homogeneous of degree \( \sum_{j=0}^{N-1}(n_j - j) \).

2. (Principal Specialization Formula.) For any \( q \in F \) that is not a root of unity or else has order \( \geq N \), we have
   \[
   s_n(1, q, q^2, \ldots, q^{N-1}) = \prod_{0 \leq j < k \leq N-1} \frac{q^{n_k} - q^{n_j}}{q^k - q^j}.
   \]

3. (Weyl Dimension Formula.) Specialized to \( q = 1 \), we have:
   \[
   s_n(1, 1, \ldots, 1) = \frac{V(n)}{V(n_{\min})} \in \mathbb{N}.
   \]
   In particular, there are \( V(n)/V(n_{\min}) \) column-strict tableaux of shape \( n - n_{\min} \) and cell entries \( 1, \ldots, N \). Here and below, we will mildly abuse notation and write \( V(n) \) for a tuple/row vector \( n \) to denote \( V(n^T) \).

Proof. The first part is proved in Theorem 43.1 below. Using this, we show the second part. Set \( u := (1, q, q^2, \ldots, q^{N-1})^T \) with \( q \) as given. Then it is easy to verify that
   \[
   s_n(u) = \frac{\det(u^{n_0} | u^{n_1} | \cdots | u^{n_{N-1}})}{V(u)} = \frac{V((q^{n_0}, \ldots, q^{n_{N-1}}))}{V((q^0, \ldots, q^{N-1}))} = \prod_{0 \leq j < k \leq N-1} \frac{q^{n_k} - q^{n_j}}{q^k - q^j},
   \]
as desired.

Finally, to prove the Weyl Dimension Formula, notice that by the first part, the Schur polynomial has integer coefficients and hence makes sense over \( \mathbb{Z} \), and then specializes
to \( s_n(u) \) over any ground field. Now work over the ground field \( \mathbb{Q} \), and let \( f_n(T) := s_n(1, T, \ldots, T^{N-1}) \in \mathbb{Z}[T] \) be the corresponding ‘principally specialized’ polynomial. Then,

\[ V((q^0, \ldots, q^{N-1}))f_n(q) = V((q^{n_0}, \ldots, q^{n_{N-1}})), \quad \forall q \in \mathbb{Q}. \]

In particular, for every \( q \neq 1 \), dividing both sides by \((q - 1)(q^2)\), we obtain:

\[
\prod_{0 \leq j < k \leq N-1} (q^{n_j} + q^{n_j+1} + \cdots + q^{n_k-1}) - f_n(q) \prod_{0 \leq j < k \leq N-1} (q^j + q^{j+1} + \cdots + q^{k-1}) = 0,
\]

for all \( q \in \mathbb{Q} \setminus \{1\} \). This means that the left-hand side is (the specialization of) a polynomial with infinitely many roots, whence the polynomial vanishes identically on \( \mathbb{Q} \). Specializing this polynomial to \( q = 1 \) now yields the Weyl Dimension Formula:

\[
\frac{V(n)}{V(n_{\text{min}})} = \prod_{0 \leq j < k \leq N-1} \frac{n_k - n_j}{k - j} = f_n(1) = s_n(1, \ldots, 1).
\]

The final assertion now follows from Littlewood’s definition \([40.2]\) of \( s_n(u) \). \( \Box \)

### 40.2. Polynomials preserving positivity on individual rank-one positive matrices.

We return to proving Theorem \([39.5]\) and hence Theorem \([39.4]\) on sign patterns. As we have shown, it suffices to prove the theorem for one higher degree (leading) term with a negative coefficient. Before proving the result in full, we tackle the following (simpler) versions. Thus, we are given a real polynomial as above:

\[
f(x) = \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c_M x^M, \text{ where } c_{n_j} > 0 \forall j.
\]

1. Does there exist \( c_M < 0 \) such that \( f[-]: \mathbb{P}_N((0, \rho)) \to \mathbb{P}_N? \)

   Here is a reformulation: dividing the expression for \( f(x) \) throughout by \( |c_M| = 1/t > 0 \), define

   \[
p_t(x) := t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^m, \quad \text{where } c_{n_j} > 0 \forall j.
   \]

   Then it is enough to ask for which \( t > 0 \) (if any) does \( p_t[-]: \mathbb{P}_N((0, \rho)) \to \mathbb{P}_N? \)

2. Here are two simplifications: Can we produce such a constant \( t > 0 \) for only the subset of rank-one matrices in \( \mathbb{P}_N((0, \rho)) \)? How about for a single rank-one matrix \( uu^T \)?

3. An even more specific (and special) case: let \( u \) be ‘generic’, in that \( u \in (0, \infty)^N \) has distinct coordinates, and \( p_t \) is as above. Can one now compute all \( t > 0 \) such that \( p_t[uu^T] \in \mathbb{P}_N \)? How about all \( t > 0 \) such that \( \det p_t[uu^T] \geq 0 \)?

We begin by answering the last of these questions – the answer crucially uses Schur polynomials. The following result shows that in fact, \( \det p_t[uu^T] \geq 0 \) implies \( p_t[uu^T] \) is positive semidefinite!

**Proposition 40.8.** With notation as in \([40.7]\), define the vectors

\[
n := (n_0, \ldots, n_{N-1}), \quad n_j := (n_0, \ldots, n_{j-1}, \rho_j, n_{j+1}, \ldots, n_{N-1}, M), \quad 0 \leq j < N,
\]

where \( 0 \leq n_0 < \cdots < n_{N-1} < M \). Now if the \( n_j \) and \( M \) are integers, and a vector \( u \in (0, \infty)^N \) has pairwise distinct coordinates, then the following are equivalent:

1. \( p_t[uu^T] \) is positive semidefinite.
2. \( \det p_t[uu^T] \geq 0 \).
3. \( t \geq \sum_{j=0}^{N-1} s_{n_j}(u)^2 \).


In particular, at least for ‘most’ rank-one matrices, it is possible to find polynomial preservers of positivity (on that one matrix), with a negative coefficient.

The proof of Proposition 40.8 uses the following even more widely applicable equivalence between the non-negativity of the determinant and of the entire spectrum for ‘special’ linear pencils of matrices:

**Lemma 40.10.** Fix $\mathbf{w} \in \mathbb{R}^N$ and a positive definite matrix $H$. Define the linear pencil $P_t := tH - \mathbf{ww}^T$, for $t > 0$. Then the following are equivalent:

1. $P_t$ is positive semidefinite.
2. $\det P_t \geq 0$.
3. $t \geq \mathbf{w}^TH^{-1}\mathbf{w} = 1 - \frac{\det(H - \mathbf{ww}^T)}{\det H}$.

This lemma is naturally connected to the theory of (generalized) Rayleigh quotients, although we do not pursue this further.

**Proof.** We show a cyclic chain of implications. That (1) $\implies$ (2) is immediate.

(2) $\implies$ (3) : Using the identity (2.32) from the section on Schur complements, we obtain by taking determinants:

$$
\det \begin{pmatrix} A & B \\ B' & D \end{pmatrix} = \det D \cdot \det(A - BD^{-1}B')
$$

whenever $A, D$ are square matrices, with $D$ invertible. Using this, we compute:

$$
0 \leq \det(tH - \mathbf{ww}^T) = \det \begin{pmatrix} tH & \mathbf{w} \\ \mathbf{w}^T & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & \mathbf{w}^T \\ \mathbf{w} & tH \end{pmatrix} = \det(tH) \det(1 - \mathbf{w}^T(\mathbf{H})^{-1}\mathbf{w}).
$$

Since the last quantity is a scalar, and $\det(tH) > 0$ by assumption, it follows that

$$
1 \geq t^{-1}(\mathbf{w}^TH^{-1}\mathbf{w}) \implies t \geq \mathbf{w}^TH^{-1}\mathbf{w}.
$$

Now substitute $t = 1$ in the above computation, to obtain:

$$
\frac{\det(H - \mathbf{ww}^T)}{\det H} = \frac{1 - \mathbf{w}^TH^{-1}\mathbf{w}}{\det H} \geq 1 - t,
$$

which implies (3).

(3) $\implies$ (1) : It suffices to show that $\mathbf{x}^TP_t \mathbf{x} \geq 0$ for all nonzero vectors $\mathbf{x} \in \mathbb{R}^N$. Using a change of variables $\mathbf{y} = \sqrt{H} \mathbf{x} \neq \mathbf{0}$, we compute:

$$
\mathbf{x}^TP_t \mathbf{x} = t\mathbf{y}^T \mathbf{y} - (\mathbf{y}^T \sqrt{H}^{-1} \mathbf{w})^2 \\
= \|\mathbf{y}\|^2(t - ((\mathbf{y}')^T \sqrt{H}^{-1} \mathbf{w})^2), \quad \text{where } \mathbf{y}' := \frac{\mathbf{y}}{\|\mathbf{y}\|}
$$

$$
\geq \|\mathbf{y}\|^2(t - \|\mathbf{y}'\|^2 \left\|\sqrt{H^{-1}} \mathbf{w}\right\|_2^2) \quad \text{(using Cauchy–Schwarz)}
$$

$$
\geq \|\mathbf{y}\|^2(\mathbf{w}^T H^{-1} \mathbf{w} - \mathbf{w}^T H^{-1} \mathbf{w}) = 0 \quad \text{(by assumption)}.
$$

We can now answer the last of the above questions on positivity preservers, for generic rank-one matrices.
We are interested in the following matrix and its determinant:

\[ p_t[uu^T] = t \sum_{j=0}^{N-1} c_{n_j} (u^{on_j}) (u^{on_j})^T - (u^{oM}) (u^{oM})^T. \]

We first work more generally: over any field \( F \), and with matrices \( uv^T \), where \( u, v \in F^N \). Thus, we study

\[ p_t[uv^T] = t \sum_{j=0}^{N-1} c_{n_j} u^{on_j} (v^{on_j})^T - u^{oM} (v^{oM})^T, \]

where \( t, c_{n_j} \in F \), and \( c_{n_j} \neq 0 \ \forall j \). Setting \( D = \text{diag}(tc_{n_0}, \ldots, tc_{n_{N-1}}, -1) \), we have the decomposition

\[ p_t[uv^T] = U(u)DU(v)^T, \quad \text{where } U(u)_{N \times (N+1)} := (u^{on_0} | \ldots | u^{on_{N-1}} | u^{oM}). \]

Applying the Cauchy–Binet identity to \( A = U(u), B = DU(v)^T \), as well as Cauchy’s definition in Proposition \( 40.6(1) \), we obtain the following general determinantal identity, valid over any field:

\[ \det p_t[uv^T] = V(u)V(v)t^{N-1} \prod_{j=0}^{N-1} c_{n_j} \cdot \left( s_n(u)s_n(v)t - \sum_{j=0}^{N-1} \frac{s_{n_j}(u)s_{n_j}(v)}{c_{n_j}} \right), \quad (40.11) \]

Now specialize this identity to \( F = \mathbb{R} \), with \( t, c_{n_j} > 0 \) and \( u = v \in \mathbb{R}^N \) having distinct coordinates. From this we deduce the following consequences. First, set

\[ H := \sum_{j=0}^{N-1} c_{n_j} (uu^T)^{on_j} = U'(u)D'U'(u)^T, \quad w := u^{oM}, \]

where \( U'(u) := (u^{on_0} | \ldots | u^{on_{N-1}}) \) is a non-singular generalized Vandermonde matrix and \( D' := \text{diag}(c_{n_0}, \ldots, c_{n_{N-1}}) \) is positive definite. From this it follows that \( H \) is positive definite, whence Lemma \( 40.10 \) applies. Moreover, \( H - w^Tw = p_t[uu^T] \), so using the above calculation \( (40.11) \) and the Cauchy–Binet identity respectively, we have:

\[
\det(H - w^Tw) = V(u)^2 \prod_{j=0}^{N-1} c_{n_j} \cdot s_n(u)^2 \left( 1 - \sum_{j=0}^{N-1} \frac{s_{n_j}(u)^2}{c_{n_j} s_n(u)^2} \right),
\]

\[ \det H = V(u)^2 \prod_{j=0}^{N-1} c_{n_j} \cdot s_n(u)^2. \]

In particular, from Lemma \( 40.10(3) \) we obtain:

\[ w^TH^{-1}w = \sum_{j=0}^{N-1} \frac{s_{n_j}(u)^2}{c_{n_j} s_n(u)^2}. \]

Now the proposition follows directly from Lemma \( 40.10 \) since \( P_t = p_t[uu^T] \) for all \( t > 0 \). \( \square \)
In the previous section, we computed the exact threshold for the leading term of a polynomial
\[
p_t(x) := t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M,
\]
where \(c_{n_j} > 0 \forall j\) (and where \(0 \leq n_0 < \cdots < n_{N-1} < M\) are integers), such that \(p_t[uu^T] \in \mathbb{P}_N\) for a single vector \(u \in (0, \infty)^N\) with pairwise distinct coordinates. Recall that our (partial) goal is to find a threshold that works for all rank-one matrices \(uu^T \in \mathbb{P}_N((0, \rho))\) — i.e., for \(u \in (0, \sqrt{\rho})^N\).

Thus, we need to show that the supremum of the threshold over all such \(u\) is bounded:

\[
\sup_{u \in (0, \sqrt{\rho})^N} \frac{\sum_{j=0}^{N-1} s_{n_j}(u)^2}{c_{n_j} u_n(u)^2} < \infty.
\]

Since we only consider vectors \(u\) with positive coordinates, it suffices to bound \(s_{n_j}(u)/s_n(u)\) from above, for each \(j\). In turn, for this it suffices to find lower and upper bounds for every Schur polynomial evaluated at \(u \in (0, \infty)^N\). This is achieved by the following result:

**Theorem 41.1** (First-order approximation / Leading term of Schur polynomials). Say \(N > 0\) and \(0 \leq n_0 < \cdots < n_{N-1}\) are integers. Then for all real numbers \(0 < u_1 \leq u_2 \leq \cdots \leq u_N\), we have the bounds

\[
1 \times u^{n-n_{\text{min}}} \leq s_n(u) \leq \frac{V(n)}{V(n_{\text{min}})} \times u^{n-n_{\text{min}}},
\]

where \(u^{n-n_{\text{min}}} = u_1^{n_0-0} u_2^{n_1-1} \cdots u_N^{n_{N-1}-(N-1)}\), and \(V(n)\) is as in (17.5). Moreover the constants 1 and \(\frac{V(n)}{V(n_{\text{min}})}\) cannot be improved.

**Proof.** Recall that \(s_n(u)\) is obtained by summing the weights of all column-strict Young tableaux of shape \(n-n_{\text{min}}\) with cell entries \(1, \ldots, N\). Moreover, by the Weyl Dimension Formula in Proposition 40.6(3), there are precisely \(V(n)/V(n_{\text{min}})\) such tableaux. Now each such tableau can have weight at most \(u^{n-n_{\text{min}}}\), as follows: the cells in the top row each have entries at most \(N\); the cells in the next row at most \(N-1\), and so on. The tableau \(T_{\text{max}}\) obtained in this fashion has weight precisely \(u^{n-n_{\text{min}}}\). Hence by definition, we have:

\[
u^{n-n_{\text{min}}} = \text{wt}(T_{\text{max}}) \leq \sum_T \text{wt}(T) = s_n(u) \leq \sum_T \text{wt}(T_{\text{max}}) = \frac{V(n)}{V(n_{\text{min}})} u^{n-n_{\text{min}}}.
\]

This proves the bounds; we claim that both bounds are sharp. If \(n = n_{\text{min}}\) then all terms in the claimed inequalities are 1, and we are done. Thus, assume henceforth that \(n \neq n_{\text{min}}\).

Let \(A > 1\) and define \(u(A) := (A, A^2, \ldots, A^N)\). Then \(\text{wt}(T_{\text{max}}) = A^M\) for some \(M > 0\). Hence for every column-strict tableau \(T \neq T_{\text{max}}\) as above, \(\text{wt}(T)\) is at most \(\text{wt}(T_{\text{max}})/A\) and at least \(1 = \text{wt}(T_{\text{max}})/A^M\). Now summing over all such tableaux \(T\) yields:

\[
s_n(u(A)) \leq u(A)^{n-n_{\text{min}}} \left(1 + \left(\frac{V(n)}{V(n_{\text{min}})} - 1\right) \frac{1}{A}\right),
\]

\[
s_n(u(A)) \geq u(A)^{n-n_{\text{min}}} \left(1 + \left(\frac{V(n)}{V(n_{\text{min}})} - 1\right) \frac{1}{A^M}\right).
\]

Divide throughout by \(u(A)^{n-n_{\text{min}}}\); now taking the limit as \(A \to \infty\) yields the sharp lower bound 1, while taking the limit as \(A \to 1^+\) yields the sharp upper bound \(V(n)/V(n_{\text{min}})\). \(\square\)
41. First-order approximation / leading term of Schur polynomials.

From rank-one matrices to all matrices.

We now use Theorem 41.1 and Proposition 40.8 in the previous section, to find a threshold for \( t > 0 \) beyond which \( p_t(\cdot) \) preserves positivity on all rank-one matrices in \( \mathbb{P}_N((0, \rho)) \) — and in fact, on all matrices in \( \mathbb{P}_N((0, \rho)) \).

**Theorem 41.2.** Fix integers \( N > 0 \) and \( 0 \leq n_0 < n_1 < \cdots < n_{N-1} < M \), and scalars \( \rho, t, c_{n_0}, \ldots, c_{n_{N-1}} > 0 \). The polynomial \( p_t(x) := t \sum_{j=0}^{N-1} c_j x^{n_j} - x^M \) entrywise preserves positivity on \( \mathbb{P}_N((0, \rho)) \), if \( t \geq t_0 := \sum_{j=0}^{N-1} \frac{V(n_j)^2}{c_{n_j} V(n_{\min})^2} \rho^{M-n_j} \).

The notation in Definition 25.1 is useful here and in the sequel. Specifically, \( X^{N,\neq} \) for a set \( X \) and an integer \( N \geq 1 \) denotes the set of \( N \)-tuples from \( X \) with pairwise distinct entries.

**Proof of Theorem 41.2.** Given \( u \in (0, \sqrt{\rho})^{N,\neq} \), from Proposition 40.8 it follows that \( p_t(uu^T) \in \mathbb{P}_N \) if and only if \( t \geq \frac{2}{\sum_{j=0}^{N-1} s_{n_j}(u)^2} \sum_{j=0}^{N-1} \frac{u^{2(n_j-n_{\min})}V(n_j)^2}{c_{n_j} u^{2(n_{\min})}} = \sum_{j=0}^{N-1} \frac{V(n_j)^2}{c_{n_j} V(n_{\min})^2} u^{2(n_j-n)} \), and this is bounded above by \( t_0 \), since if \( v := \sqrt{\rho}(1, \ldots, 1)^T \) then \( u^{2(n_j-n)} \leq v^{2(n_j-n)} = \rho^{M-n_j} \) for all \( j \). Thus, we conclude that

\[
 t \geq t_0 \implies p_t(uu^T) \in \mathbb{P}_N \forall u \in (0, \sqrt{\rho})^{N,\neq} \implies p_t(uu^T) \in \mathbb{P}_N \forall u \in (0, \sqrt{\rho})^N, \]

where the first implication was proved above, the second follows by (the symmetric nature of Schur polynomials and by) relabelling the rows and columns of \( uu^T \) to rearrange the entries of \( u \) in increasing order, and the third implication follows from the continuity of \( p_t \) and the density of \( (0, \sqrt{\rho})^{N,\neq} \) in \( (0, \sqrt{\rho})^N \).

This validates the claimed threshold \( t_0 \) for all rank-one matrices. To prove the result on all of \( \mathbb{P}_N((0, \rho)) \), we use induction on \( N \geq 1 \), with the base case of \( N = 1 \) already done since \( 1 \times 1 \) matrices have rank one.

For the induction step, recall the Extension Theorem 9.12 which said that: Suppose \( I = (0, \rho) \) or \((-\rho, \rho)\) or its closure, for some \( 0 < \rho \leq \infty \). If \( h \in C^1(I) \) is such that \( h[-] \) preserves positivity on rank-one matrices in \( \mathbb{P}_N(I) \) and \( h'[-] : \mathbb{P}_{N-1}(I) \rightarrow \mathbb{P}_{N-1} \), then \( h'[-] : \mathbb{P}_N(I) \rightarrow \mathbb{P}_N \).

We will apply this result to \( h(x) = p_{t_0}(x) \), with \( t_0 \) as above. By the extension theorem, we need to show that \( h'[-] : \mathbb{P}_{N-1}((0, \rho)) \rightarrow \mathbb{P}_{N-1} \). Note that

\[
h'(x) = t_0 \sum_{j=0}^{N-1} n_j c_{n_j} x^{n_j-1} = M x^{M-1} = Mg(x) + t_0 n_0 c_{n_0} x^{n_0-1},
\]

where we define

\[
g(x) := t_0 \frac{1}{M} \sum_{j=1}^{N-1} n_j c_{n_j} x^{n_j-1} - x^{M-1}.
\]

We claim that the entrywise polynomial map \( g[-] : \mathbb{P}_{N-1}((0, \rho)) \rightarrow \mathbb{P}_{N-1} \). If this holds, then by the Schur product theorem, the same property is satisfied by \( Mg(x) + t_0 n_0 c_{n_0} x^{n_0-1} \) (regardless of whether \( n_0 = 0 \) or \( n_0 > 0 \)). But this function is precisely \( g' \), and the theorem would follow.
It thus remains to prove the claim, and we do so via a series of reductions and simplifications – i.e., “working backward”. By the induction hypothesis, Theorem 41.2 holds in dimension $N - 1 \geq 1$, for the polynomials

$$q_t(x) := t \sum_{j=1}^{N-1} n_j c_{n_j} x^{n_j - 1} - x^{M-1}.$$ 

For this family, the threshold is now given by

$$\sum_{j=1}^{N-1} \frac{V(n_j')^2}{n_j c_{n_j} V(n_{\min}')^2} \rho^{M-1-(n_j-1)},$$

where $n_{\min}' := (0, \ldots, N-2), \ n_j' := (n_1, \ldots, n_{j-1}, \tilde{n}_j, n_{j+1}, \ldots, n_{N-1}, M) \ \forall j > 0.$

Thus, the proof is complete if we show that

$$\sum_{j=1}^{N-1} \frac{V(n_j')^2}{n_j c_{n_j} V(n_{\min}')^2} \rho^{M-1-(n_j-1)} \leq \frac{t_0}{M} = \sum_{j=0}^{N-1} \frac{V(n_j)^2}{M c_{n_j} V(n_{\min})^2} \rho^{M-n_j}.$$ 

In turn, comparing just the $j$th summand for each $j > 0$, it suffices to show that

$$\frac{V(n_j')}{\sqrt{n_j' V(n_{\min}'')}} \leq \frac{V(n_j)}{\sqrt{MV(n_{\min})}}, \quad \forall j > 0.$$ 

Dividing the right-hand side by the left-hand side, and cancelling common factors, we obtain the expression

$$\prod_{k=1}^{N-1} \frac{n_k - n_0}{k} \cdot \frac{\sqrt{n_j}}{\sqrt{M}} \cdot \frac{M - n_0}{n_j - n_0}.$$ 

Since every factor in the product term is at least 1, it remains to show that

$$\frac{M - n_0}{n_j - n_0} \geq \frac{\sqrt{M}}{\sqrt{n_j}}, \quad \forall j > 0.$$ 

But this follows from a straightforward calculation:

$$(M - n_0)^2 n_j - (n_j - n_0)^2 M = (M - n_j)(Mn_j - n_0^2) > 0,$$

and the proof is complete. \qed

Finally, we recall our original goal of classifying the sign patterns of positivity preservers in fixed dimension – see Theorem 39.4. We showed this result holds if one can prove its special case, Theorem 39.5. Now this latter result follows from Theorem 41.2, by setting

$$c_M := -t_0^{-1}, \text{ where } t_0 = \sum_{j=0}^{N-1} \frac{V(n_j)^2}{c_{n_j} V(n_{\min})^2} \rho^{M-n_j} \text{ as in Theorem 41.2}. \quad \square$$
42. Exact quantitative bound: monotonicity of Schur ratios. Real powers and power series.

In the last few sections, we proved the existence of a negative threshold \( c_M \) for polynomials
\[
f(x) = \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c_M x^M
\]
to entrywise preserve positivity on \( P_N((0, \rho)) \). (Here, \( N > 0 \) and \( 0 \leq n_0 < \cdots < n_{N-1} < M \) are integers.) We now compute the exact value of this threshold, more generally for real powers; this has multiple consequences which are described below. Thus, our goal is to prove the following quantitative result, for real powers – including negative powers:

**Theorem 42.1.** Fix an integer \( N > 0 \) and real powers \( n_0 < \cdots < n_{N-1} < M \). Also fix real scalars \( \rho > 0 \) and \( c_{n_0}, \ldots, c_{n_{N-1}}, c_M \), and define
\[
f(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c_M x^M. \tag{42.2}
\]

Then the following are equivalent:

1. The entrywise map \( f[-] \) preserves positivity on all rank-one matrices in \( P_N((0, \rho)) \).
2. The map \( f[-] \) preserves positivity on rank-one Hankel \( TN \) matrices in \( P_N((0, \rho)) \).
3. Either all \( c_{n_j}, c_M \geq 0 \); or \( c_{n_j} > 0 \ \forall \ j \) and \( c_M \geq -C^{-1} \), where
\[
C = \sum_{j=0}^{N-1} \frac{V(n_j)^2}{c_{n_j} V(n)^2 \rho^{M-n_j}}. \tag{42.3}
\]

Here \( V(u), n, n_j \) are defined as in (17.5) and (40.9).

If moreover we assume that \( n_j \in \mathbb{Z}_{\geq 0} \cup [N-2, \infty) \) for all \( j \), then the above conditions are further equivalent to the ‘full-rank’ version:

4. The entrywise map \( f[-] \) preserves positivity on \( P_N([0, \rho]) \).

**Theorem 42.1** is a powerful result. It has multiple applications; we now list some of them.

1. Suppose \( M = N \) and \( n_j = j \) for \( 0 \leq j \leq N - 1 \). Then the result provides a complete characterization of which polynomials of degree \( \leq N \) entrywise preserve positivity on \( P_N((0, \rho)) \) – or more generally, on any intermediate set between \( P_N((0, \rho)) \) and the rank-one Hankel \( TN \) matrices inside it.
2. In fact a similar result to the previous characterization is implied, whenever one considers linear combinations of at most \( N + 1 \) monomial powers.
3. The result provides information on positivity preservers beyond polynomials, since \( n_j, M \) are now allowed to be real, even negative if one works with rank-one matrices.
4. In particular, the result implies Theorem 39.6 and hence Theorem 39.7 (see its proof). This latter theorem provides a full classification of the sign patterns of possible “countable sums of real powers” which entrywise preserve positivity on \( P_N((0, \rho)) \).
5. The result also provides information on preservers of total non-negativity on Hankel matrices in fixed dimension; see Corollary 42.9 below.
6. There are further applications, two of which are (i) to the matrix cube problem and to sharp linear matrix inequalities/spectrahedra involving entrywise powers; and (ii) to computing the simultaneous kernels of entrywise powers and a related “Schubert cell-type” stratification of the cone \( P_N(\mathbb{C}) \). These are explained in the 2016 paper of
Belton, Guillot, Khare, and Putinar in *Adv. in Math.*; see also the 2017 preprint by Khare and Tao (mentioned a few lines above (39.1)).

7. Theorem 42.1 is proved using a monotonicity phenomenon for ratios of Schur polynomials; see Theorem 42.6 below. This latter result is also useful in extending a 2011 conjecture by Cuttler–Greene–Skandera (and its proof). In fact, this line of attack ends up characterizing weak majorization using Schur polynomials. See the 2017 preprint by Khare and Tao for more details.

8. One further application is Theorem 42.12 which finds a threshold for bounding by $\sum_{j=0}^{N-1} c_{nj} A^n_{nj}$, any power series – and more general ‘Laplace transforms’ – applied entrywise to a positive matrix $A$. This extends Theorem 42.1 where the ‘power series’ is simply $x^M$, because Theorem 42.1 says in particular that $(x^M)[A] = A^\circ M$ is dominated by a multiple of $\sum_{j=0}^{N-1} c_{nj} A^n_{nj}$.

9. As mentioned in the remarks prior to Theorem 39.4, Theorem 42.1 also provides examples of ‘power series’ preservers on $\mathbb{P}_N((0, \rho))$ with negative coefficients; and of such functions which preserve positivity on $\mathbb{P}_N((0, \rho))$ but not on $\mathbb{P}_{N+1}((0, \rho))$.

42.1. Monotonicity of ratios of Schur polynomials. The proof of Theorem 42.1 uses the same ingredients as developed in previous sections. A summary of what follows is now provided. In the rank-one case, we use a variant of Proposition 40.8 for an individual matrix; the result does not apply as is, since the powers may now be real. Next, in order to find the sharp threshold for all rank-one matrices, even for real powers we crucially appeal to the result does not apply as is, since the powers may now be real. Next, in order to find the sharp threshold for all rank-one matrices, even for real powers we crucially appeal to the integer power case. Namely, we will first understand the behavior and supremum of the function $s_n(u)/s_n(u)$ over $u \in (0, \sqrt{\rho})^N$ (and for each $0 \leq j \leq N - 1$). One may hope that these suprema behave well enough that the sharp threshold can be computed for rank-one matrices; the further hope would be that this threshold bound is tight enough to behave well with respect to the extension principle in Theorem 9.12 and hence to work for all matrices in $\mathbb{P}_N((0, \rho))$. Remarkably, these two hopes are indeed justified, proving the theorem.

We begin with the key result required to be able to take suprema over ratios of Schur polynomials $s_m(u)/s_n(u)$. To motivate the result, here is a special case.

**Example 42.4.** Suppose $N = 3$, $u = (0, 2, 3)$, $m = (0, 2, 4)$. As above, we have $u = (u_1, u_2, u_3)^T$ and $n_{\text{min}} = (0, 1, 2)$. Now let $f(u) := \frac{s_m(u)}{s_n(u)} : (0, \infty)^N \to (0, \infty)$. This is a rational function, whose numerator sums weights over tableaux of shape $(2, 1)$, and hence by Example 40.3 above, equals $(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)$. The denominator sums weights over tableaux of shape $(1, 1)$; there are only three such:

\[
\begin{array}{ccc}
3 & & \\
2 & 3 & \\
& 1 & 2 \\
& & 1
\end{array}
\]

and hence,

\[f(u) := \frac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1u_2 + u_2u_3 + u_3u_1}, \quad u_1, u_2, u_3 > 0.\]

Notice that the numerator and denominator are both Schur polynomials, hence positive combinations of monomials (this is called ‘monomial positivity’). In particular, they are both non-decreasing in each coordinate. One can verify that their ratio $f(u)$ is not a polynomial; moreover, it is not a priori clear if $f(u)$ shares the same coordinatewise monotonicity property. However, we claim that this does hold, i.e., $f(u)$ is non-decreasing in each coordinate on $u \in (0, \infty)^N$. 
To see why: by symmetry, it suffices to show that $f$ is non-decreasing in $u_3$. Using the quotient rule of differentiation, we claim that the expression

$$s_n(u)\partial_{u_3}s_m(u) - s_m(u)\partial_{u_3}s_n(u)$$

is non-negative on $(0, \infty)^3$. Indeed, computing this expression yields:

$$(u_1 + u_2)(u_1u_3 + 2u_1u_2 + u_2u_3)$$

and this is clearly non-negative as desired. More strongly, the expression (42.5) turns out to be monomial positive, which implies non-negativity.

Here is the punchline: an even stronger phenomenon holds. Namely, when we write the expression (42.5) in the form

$$\sum_{j \geq 0} p_j(u_1, u_2)u_j$$

each polynomial $p_j$ is Schur positive! This means that it is a non-negative integer-linear combination of Schur polynomials:

$p_0(u_1, u_2) = 0,$

$$p_1(u_1, u_2) = 2u_1u_2^2 + 2u_1^2u_2 = 2 \begin{array}{c} 2 \\ 1 \end{array} + 2 \begin{array}{c} 2 \\ 1 \end{array} = 2s_{(1,3)}(u_1, u_2),$$

$$p_2(u_1, u_2) = (u_1 + u_2)^2 = \begin{array}{c} 2 \\ 2 \end{array} + \begin{array}{c} 2 \\ 1 \end{array} + \begin{array}{c} 1 \\ 1 \end{array} + \begin{array}{c} 2 \\ 1 \end{array} = s_{(0,3)}(u_1, u_2) + s_{(1,2)}(u_1, u_2),$$

modulo a mild abuse of notation. This yields the sought-for non-negativity, as each $s_n(u)$ is monomial positive by definition. (See the discussion following (29.13) for the ‘original’ occurrence of monomial positivity and its ‘upgrade’ to (skew) Schur positivity.)

The remarkable fact is that the phenomena described in the above example also occur for every pair of Schur polynomials $s_m(u), s_n(u)$ for which $m \geq n$ coordinatewise:

**Theorem 42.6** (Monotonicity of Schur polynomial ratios). Suppose $0 \leq n_0 < \cdots < n_{N-1}$ and $0 \leq m_0 < \cdots < m_{N-1}$ are integers satisfying: $n_j \leq m_j \forall j$. Then the symmetric function

$$f : (0, \infty)^N \to \mathbb{R}, \quad f(u) := \frac{s_m(u)}{s_n(u)}$$

is non-decreasing in each coordinate.

More strongly, viewing the expression

$$s_n(u) \cdot \partial_{u_N}s_m(u) - s_m(u) \cdot \partial_{u_N}s_n(u)$$

as a polynomial in $u_N$, the coefficient of each monomial $u_N^j$ is a Schur positive polynomial in $(u_1, u_2, \ldots, u_{N-1})^T$.

Theorem 42.6 is an application of a deep result in representation theory/symmetric function theory, by Lam, Postnikov, and Pylyavsky in *Amer. J. Math.* (2007). The proof of this latter result is beyond the scope of this text, and hence is not pursued further; but its usage means that in the spirit of the previous sections, the proof of Theorem 42.6 once again combines analysis with symmetric function theory. Moreover, this 2007 result in [178] arose out of prior work of Skandera [267] in 2004, on determinant inequalities for minors of totally non-negative matrices.
42. Exact quantitative bound: monotonicity of Schur ratios.

42.2. Proof of the quantitative bound. Using Theorem 42.1, we can now prove the main result in this section.

Proof of Theorem 42.1. We first work only with rank-one matrices. Clearly (1) \(\implies\) (2), and we show that (2) \(\implies\) (3) \(\implies\) (1).

If all coefficients \(c_n, c_M \geq 0\) then \(f[-]\) preserves positivity on rank-one matrices. Otherwise, by the Horn-type necessary conditions in Lemma 39.2 (now for real powers, possibly negative!), it follows that \(c_n, \ldots, c_{n-1} > 0 > c_M\). In this case, the discussion that opens Section 40.2 allows us to reformulate the problem using

\[
p_t(x) := \sum_{j=0}^{N-1} c_n x^j - x^M, \quad t > 0,
\]

and the goal is to find a sharp positive lower bound for \(t\), above which \(p_t[-]\) preserves positivity on rank-one Hankel \(TN\) matrices \(uu^T \in \mathbb{P}_N((0,\rho))\).

But now one can play the same game as in Section 40.2. In other words, Lemma 40.10 shows that the ‘real powers analogue’ of Proposition 40.8 holds: \(p_t[uu^T] \geq 0\) if and only if

\[
t \geq \sum_{j=0}^{N-1} \frac{\det(u^{\circ n_j})^2}{c_n \det(u^{\circ n})^2},
\]

for all generic rank-one matrices \(uu^T\), with \(u \in (0,\sqrt{\rho})^{N,\#}\). By the same reasoning as in the proof of Theorem 41.2 (see the previous section), \(p_t[-]\) preserves positivity on a given test set of rank-one matrices \(\{uu^T : u \in S \subset (0,\sqrt{\rho})^N\}\), if and only if (by density and continuity,) \(t\) exceeds the following supremum:

\[
t \geq \sup_{u \in S \cap (0,\sqrt{\rho})^{N,\#}} \sum_{j=0}^{N-1} \frac{\det(u^{\circ n_j})^2}{c_n \det(u^{\circ n})^2}. \quad (42.7)
\]

This is, of course, subject to \(S \cap (0,\sqrt{\rho})^{N,\#}\) being dense in the set \(S\), which is indeed the case if \(\{uu^T : u \in S \cap (0,\sqrt{\rho})^N\}\) equals the set of rank-one Hankel \(TN\) matrices as in assertion (2).

Thus, to prove (2) \(\implies\) (3) \(\implies\) (1) in the theorem, it suffices to prove: (i) the supremum (42.7) is bounded above by the value \(\sum_{j=0}^{N-1} \frac{\det(u^{\circ n_j})^2}{c_n \det(u^{\circ n})^2} V(n_j)^2 \rho^{M-n_j};\) and (ii) this value is attained on (a countable set of) rank-one Hankel \(TN\) matrices, whence it equals the supremum.

We now prove both of these assertions. By Theorem 42.1, each ratio \(\det(u^{\circ n})/\det(u^{\circ n})\) is coordinatewise non-decreasing, hence its supremum on \((0,\sqrt{\rho})^{N,\#}\) is bounded above by (and in fact equals) its limit as \(u \to \sqrt{\rho}(1^-,\ldots,1^-)\). To see why this limit exists, note that every vector \(u \in (0,\rho)^N\) is bounded above – coordinatewise – by a vector of the form

\[
u(\epsilon) := \sqrt{\rho}(\epsilon^2, \ldots, \epsilon^N)^T \in (0,\sqrt{\rho})^{N,\#}, \quad \epsilon \in (0,1).
\]

In particular, by Theorem 42.1 the limit as \(u \to \sqrt{\rho}(1^-,\ldots,1^-)\) exists and equals the limit by using the rank-one Hankel \(TN\) family \(u(\epsilon)u(\epsilon)^T\), for any sequence of \(\epsilon \to 1^-\) – provided this latter limit exists. We show this presently; thus, we work with a countable sequence of \(\epsilon \to 0^+\) in place of Lemma 39.2 above, and another countable sequence of \(\epsilon \to 1^-\) in what follows. First observe:
Lemma 42.8 (Principal Specialization Formula for real powers). Suppose $q > 0$ and $n_0 < n_1 < \cdots < n_{N-1}$ are real. If $n := (n_0, \ldots, n_{N-1})$ and $u := (1, q, \ldots, q^{N-1})^T$, then
\[
\det(u^{\circ n}) = \prod_{0 \leq j < k \leq N-1} (q^{n_k} - q^{n_j}) = V(q^{\circ n}).
\]
The proof is exactly the same as of Proposition 40.6(2), since the transpose of $u^{\circ n}$ is a usual Vandermonde matrix.

We can now complete the proof of Theorem 42.1. The above lemma immediately implies:
\[
\frac{\det(u(\epsilon)^{\circ n})}{\det(u(\epsilon)^{\circ 1})} = \sqrt{\rho^{M-n_j} V(e^{\circ n_j}) / V(e^{\circ 1})}, \quad \forall 0 \leq j \leq N - 1.
\]
Dividing the numerator and denominator by $(1 - \epsilon)^{\binom{N}{2}}$ and taking the limit as $\epsilon \to 1^-$ using L'Hôpital's rule, we obtain the expression $\sqrt{\rho^{M-n_j} V(n_j) / V(n)}$. Since all of these suprema/limits occur as $\epsilon \to 1^-$, we finally have:
\[
\lim_{\epsilon \to 1^-} \sup_{u \in (0, \sqrt{\rho})^N} \sum_{j=0}^{N-1} \frac{\det(u^{\circ n_j})^2}{c_{n_j} \det(u^{\circ 1})^2} = \lim_{\epsilon \to 1^-} \sum_{j=0}^{N-1} \frac{\det(u(\epsilon)^{\circ n_j})^2}{c_{n_j} \det(u(\epsilon)^{\circ 1})^2} = \sum_{j=0}^{N-1} \frac{V(n_j)^2 \rho^{M-n_j}}{V(n)^2 c_{n_j}}.
\]
This proves the equivalence of assertions (1)–(3) in the theorem, for rank-one matrices.

Finally, suppose all $n_j \in \mathbb{Z}_{\geq 0} \cup [N - 2, \infty)$. In this case $4 \implies 1$ is immediate. Conversely, given that (1) holds, we prove (4) using once again the integration trick of FitzGerald–Horn, as isolated in Theorem 9.12. The proof and calculation are similar to that of Theorem 41.2 above, and are left to the interested reader as an exercise.

42.3. Applications to Hankel $TN$ preservers in fixed dimension and to power series preservers. We conclude by discussing some applications of Theorem 42.1. First, the result implies in particular that $A^{\circ M}$ is bounded above by a multiple of $\sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j}$. In particular, the proof of Theorem 39.7 above goes through; thus, we have classified the sign patterns of all entrywise ‘power series’ preserving positivity on $\mathbb{P}_N((0, \rho))$.

Second, the equivalent conditions in Theorem 42.1 classifying the (entrywise) polynomial positivity preservers on $\mathbb{P}_N((0, \rho))$ – or on rank-one matrices – also end up classifying the polynomial preservers of total non-negativity on the corresponding test sets:

Corollary 42.9. With notation as in Theorem 42.1, if we restrict to all real powers and only rank-one matrices, then assertions (1)–(3) in Theorem 42.1 are further equivalent to:

(1$'$) $f[-]$ preserves total non-negativity on all rank-one matrices in $HTN_N$ with entries in $(0, \rho)$.

If moreover all $n_j$ lie in $\mathbb{Z}_{\geq 0} \cup [N - 2, \infty)$, then these conditions are further equivalent to:

(4$'$) $f[-]$ preserves total non-negativity on all matrices in $HTN_N$ with entries in $[0, \rho]$.

Recall here that by Definition 12.18, $HTN_N$ denotes the set of $N \times N$ Hankel totally nonnegative matrices.

Proof. Clearly, (1$'$) implies assertion (2) in Theorem 42.1. Conversely, we claim that assertion (1) in Theorem 42.1 implies (1$'$) via Theorem 4.1. Indeed, if $A \in HTN_N$ has rank one and entries in $(0, \rho)$, then $f[A] \in \mathbb{P}_N$ by Theorem 42.1(1). Similarly, $A^{(1)} \oplus (0)_{1 \times 1} \in \mathbb{P}_N((0, \rho))$ and has rank one, whence $f[A^{(1)}]$ is also positive semidefinite, and hence Theorem 4.1 applies as desired. The same proof works to show the equivalence between (4$'$) and Theorem 42.1(1).
The third and final application is to bounding \( g[A] \), where \( g(x) \) is a power series – or more generally, a sum of real powers – by a threshold times \( \sum_{j=0}^{N-1} c_{n_j} A^{n_j} \). This extends Theorem \[42.1\] in which \( g(x) = x^M \). The idea is that if we fix \( 0 \leq n_0 < \cdots < n_{N-1} \) and \( c_{n_j} \) for \( j = 0, \ldots, N - 1 \), then

\[
A^M \leq t_M \sum_{j=0}^{N-1} c_{n_j} A^{n_j}, \quad \text{where } t_M := \sum_{j=0}^{N-1} \frac{V(n_j)^2}{c_{n_j} V(n)^2} \rho^{M-n_j}, \quad (42.10)
\]

and this linear matrix inequality holds for all \( A \in \mathbb{P}_N((0, \rho)) \) – possibly of rank one if the \( n_j \) are allowed to be arbitrary non-negative real numbers, else of all ranks if all \( n_j \in \mathbb{Z}^{\geq 0} \cup [N-2, \infty) \). Here the \( \leq \) stands for the positive semidefinite ordering, or Loewner ordering – see e.g. Definition \[14.7\]. Moreover, the constant \( t_M \) depends on \( M \) through \( n_j \) and \( \rho^{M-n_j} \).

If now we consider a power series \( g(x) := \sum_{M \geq n_{N-1}+1} c_M x^M \), then by adding several linear matrix inequalities of the form \( (42.10) \), it follows that

\[
g[A] \leq t_g \sum_{j=0}^{N-1} c_{n_j} A^{n_j}, \quad \text{where } t_g := \sum_{M \geq n_{N-1}+1} \max(c_M,0)t_M,
\]

and this is a valid linear matrix inequality as long as the sum \( t_g \) is convergent. Thus, we now explore when this sum converges.

Even more generally: notice that a power series is the sum/integral of the power function, over a measure on the powers which is supported on the integers. Thus, given any real measure \( \mu \) supported in \( [n_{N-1} + \varepsilon, \infty) \), one can consider its corresponding ‘Laplace transform’

\[
g_\mu(x) := \int_{n_{N-1}+\varepsilon}^{\infty} x^t \, d\mu(t). \quad (42.11)
\]

The final application of Theorem \[42.1\] explores in this generality, when a threshold exists to bound \( g_\mu[A] \) by a sum of \( N \) lower powers.

**Theorem 42.12.** Fix \( N \geq 2 \) and real exponents \( 0 \leq n_0 < \cdots < n_{N-1} \) in the set \( \mathbb{Z}^{\geq 0} \cup [N-2, \infty) \). Also fix scalars \( \rho, c_{n_j} > 0 \) for all \( j \).

Now suppose \( \varepsilon, \varepsilon' > 0 \) and \( \mu \) is a real measure supported on \( [n_{N-1} + \varepsilon, \infty) \) such that \( g_\mu(x) \) – defined as in \( (42.11) \) – is absolutely convergent at \( \rho(1+\varepsilon') \). Then there exists a finite constant \( t_\mu \in (0, \infty) \), such that the map

\[
t_\mu \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - g_\mu(x)
\]

entrywise preserves positivity on \( \mathbb{P}_N((0, \rho)) \). Equivalently, \( g_\mu[A] \leq t_\mu \sum_{j=0}^{N-1} c_{n_j} A^{n_j} \), for all \( A \in \mathbb{P}_N((0, \rho)) \).

**Proof.** If \( \mu = \mu_+ - \mu_- \) denotes the decomposition of \( \mu \) into its positive and negative parts, then notice (e.g. by the FitzGerald–Horn Theorem \[9.3\]) that

\[
\int_{\mathbb{R}} A^M \, d\mu_-(M) \in \mathbb{P}_N, \quad \forall A \in \mathbb{P}_N((0, \rho)).
\]

Hence, it suffices to show that

\[
t_\mu := \int_{n_{N-1}+\varepsilon}^{\infty} t_M \, d\mu_+(M) = \int_{n_{N-1}+\varepsilon}^{\infty} \sum_{j=0}^{N-1} \frac{V(n_j)^2}{c_{n_j} V(n)^2} \rho^{M-n_j} \, d\mu_+(M) < \infty, \quad (42.13)
\]
since this would imply:

\[
t^{\mu} \sum_{j=0}^{N-1} c_n A^{\circ n_j} - g_{\mu}[A] = \int_{n_{N-1}+\varepsilon}^{\infty} \left( t_M \sum_{j=0}^{N-1} c_n A^{\circ n_j} - A^{\circ M} \right) d\mu_+(M) + \int_{n_{N-1}+\varepsilon}^{\infty} A^{\circ M} d\mu_-(M),
\]

and both integrands and integrals are positive semidefinite.

In turn, isolating the terms in (42.13) that depend on \(M\), it suffices to show for each \(j\) that

\[
\int_{n_{N-1}+\varepsilon}^{\infty} \prod_{k=0, k \neq j}^{N-1} (M - n_k)^2 \rho^M d\mu_+(M) < \infty.
\]

By linearity, it suffices to examine the finiteness of the integrals

\[
\int_{n_{N-1}+\varepsilon}^{\infty} M^k \rho^M d\mu_+(M), \quad k \geq 0.
\]

But by assumption, \(\int_{n_{N-1}+\varepsilon}^{\infty} \rho^M (1 + \varepsilon')^M d\mu_+(M)\) is finite; and moreover, for any fixed \(k \geq 0\) there is a threshold \(M_k\) beyond which \((1 + \varepsilon')^M \geq M^k\). (Indeed, this happens when \(\frac{\log M}{M} \leq \frac{\log(1 + \varepsilon')}{k}\).) Therefore,

\[
\int_{n_{N-1}+\varepsilon'}^{\infty} M^k \rho^M d\mu_+(M) \leq \int_{n_{N-1}+\varepsilon'}^{M_k} M^k \rho^M d\mu_+(M) + \int_{M_k}^{\infty} \rho^M (1 + \varepsilon')^M d\mu_+(M) < \infty,
\]

which concludes the proof. \(\square\)
43. Appendix E: Cauchy’s and Littlewood’s definitions of Schur polynomials.  

For completeness, in this section we show the equivalence of four definitions of Schur polynomials, two of which are named identities. To proceed, first recall two other families of symmetric polynomials: the elementary symmetric polynomials are simply

\[ e_1(u_1, u_2, \ldots) := u_1 + u_2 + \cdots, \quad e_2(u_1, u_2, \ldots) := u_1u_2 + u_1u_3 + u_2u_3 + \cdots, \]

and in general,

\[ e_k(u_1, u_2, \ldots) := \sum_{1 \leq j_1 < j_2 < \cdots} u_{j_1}u_{j_2} \cdots \]

These symmetric functions crucially feature while decomposing polynomials into linear factors.

We also recall the complete homogeneous symmetric polynomials

\[ h_k(u_1, u_2, \ldots) := \sum_{1 \leq j_1 \leq j_2 \leq \cdots} u_{j_1}u_{j_2} \cdots \]

By convention, we set \( e_0 = h_0 = 1 \), and \( e_k = h_k = 0 \) for \( k < 0 \). Now we have:

**Theorem 43.1.** Fix an integer \( N \geq 1 \) and any unital commutative ground ring. Given a partition of \( N \) – i.e., an \( N \)-tuple of non-increasing non-negative integers \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \) with \( \sum_j \lambda_j = N \) – the following four definitions give the same expression \( s_{\lambda+\delta}(u_1, u_2, \ldots, u_N) \).

1. (Littlewood’s definition.) The sum of weights over all column-strict Young tableaux of shape \( \lambda \) with cell entries \( u_1, \ldots, u_N \).

2. (Cauchy’s definition, aka the type A Weyl Character Formula.) The ratio of the (generalized) Vandermonde determinants \( a_{\lambda+\delta}/a_\delta \), where \( \delta := (N-1, N-2, \ldots, 0) \), and \( a_\lambda := \det(u_{\lambda_j+N-k}^{j}) \).

3. (The Jacobi–Trudi identity.) The determinant \( \det(h_{\lambda_j-j+k}^{j})_{j,k=1}^{N} \).

4. (The dual Jacobi–Trudi identity, or von Nägelsbach–Kostka identity.) The determinant \( \det(e_{\lambda'_j-j+k}^{j}) \), where \( \lambda' \) is the dual partition, meaning \( \lambda'_k := \#\{j : \lambda_j \geq k\} \).

From this result, we deduce the equivalence of these definitions of the Schur polynomial for fewer numbers of variables \( u_1, \ldots, u_n \), where \( n \leq N \).

**Corollary 43.2.** Suppose \( 1 \leq r < N \) and \( \lambda_{r+1} = \cdots = \lambda_N = 0 \). Then the four definitions in Theorem 43.1 agree for the smaller set of variables \( u_1, \ldots, u_r \).

**Proof.** Using fewer numbers of variables in the definitions (3), (4) amounts to specializing the remaining variables \( u_{r+1}, \ldots, u_N \) to zero. The same holds for definition (1) since weights involving the ‘extra’ variables \( u_{r+1}, \ldots, u_N \) now get set to zero. It follows that definitions (1), (3), and (4) agree for fewer numbers of variables.

We will show that Cauchy’s definition (2) in Theorem 43.1 has the same property. In this case the definitions are different: Given \( u_1, \ldots, u_r \) for \( 1 \leq r \leq N \), the corresponding ratio of alternating polynomials would only involve \( \lambda_1 \geq \cdots \geq \lambda_r \), and would equal \( \det(u_{\lambda_j+N-r}^{j})_{j,k=1}^{r} / \det(u_{\lambda_j-r}^{j})_{j,k=1}^{r} \). Now claim that this equals the ratio in (2), by downward induction on \( r \leq N \). Note that it suffices to show the claim for \( r = N - 1 \). But here, if we set \( u_N := 0 \) then both generalized Vandermonde matrices have last column \((0, \ldots, 0, 1)^T\). In particular, we may expand along their last columns. Now cancelling the common factors of
$u_1 \cdots u_{N-1}$ from each of the previous columns reduces to the case of $r = N - 1$, and the proof is completed by similarly continuing inductively. \hfill \Box

The remainder of this section is devoted to proving Theorem 43.1. We will show that (4) $\iff$ (1) $\iff$ (3) $\iff$ (2), and over the ground ring $\mathbb{Z}$, which then carries over to arbitrary ground rings. To do so, we use an idea due to Karlin–Macgregor (1959), Lindström (1973), and Gessel–Viennot (1985), which interprets determinants in terms of tuples of weighted lattice paths. The approach below is due to Bressoud–Wei (1993).

**Proposition 43.3.** The definitions (1) and (3) are equivalent.

**Proof.** The proof is divided into steps, for ease of exposition.

**Step 1:** In this step we define the formalism of lattice paths and their weights. Define points in the plane

$$P_k := (N - k + 1, N), \quad Q_k := (N - k + 1 + \lambda_k, 1), \quad k = 1, 2, \ldots, N,$$

and consider (ordered) $N$-tuples $p$ of (directed) lattice paths satisfying the following properties:

1. The $k$th path starts at some $P_j$ and ends at $Q_k$, for each $k$.
2. No two paths start at the same point $P_j$.
3. From $P_j$, and at each point $(a, b)$, a path can go either East or South. Weight each East step at height $(a, b)$ by $u_{N+1-b}$.

Notice that one can assign a unique permutation $\sigma = \sigma_p \in S_N$ to each tuple of paths $p$, so that paths go from $P_{\sigma(k)}$ to $Q_k$ for each $k$.

We now assign a weight to each tuple $p$, defined to be $(-1)^{\sigma_p}$ times the product of the weights at all East steps in $p$. For instance, if $\lambda = (3, 1, 1, 0, 0)$ partitions $N = 5$, then here is a typical tuple of paths:

- For $k = 4, 5$, $P_k, Q_k$ are each connected by vertical straight lines (i.e., four South steps each).
- $P_2$ and $Q_3$ are connected by a vertical straight line (i.e., four South steps).
- The steps from $P_3$ to $Q_2$ are "SESESS".
- The steps from $P_1$ to $Q_1$ are "SEESSES".

This tuple $p$ corresponds to the permutation $\sigma_p = (13245)$, and has weight $-u_3^2u_3u_4$.

**Step 2:** The next goal is to examine the generating function of the tuples, i.e., $\sum_p \text{wt}(p)$. Note that given $\sigma$, among all tuples $p$ with $\sigma_p = \sigma$, the $k$th path contributes a monomial of total degree $\lambda_k - k + \sigma(k)$, which can be any monomial in $u_1, \ldots, u_N$ of this total degree. It follows that the generating function equals

$$\sum_p \text{wt}(p) = \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{k=1}^{N} h_{\lambda_k - k + \sigma(k)} = \det(h_{\lambda_k - k + j})_{j,k=1}^{N}.$$

**Step 3:** We next rewrite the above generating function to obtain $\sum_T \text{wt}(T)$ (the sum of weights over all column-strict Young tableaux of shape $\lambda$ with cell entries $u_1, \ldots, u_N$), which is precisely $s_{\lambda^t}(u_1, \ldots, u_N)$ by definition. To do so, we will pair off the tuples $p$ of intersecting paths into pairs, whose weights cancel one another.

Suppose $p$ consists of intersecting paths. Define the final intersection point of $p$ to be the lattice point with maximum $x$-coordinate where at least two paths intersect, and if there are more than one such points, then the one with minimal $y$ coordinate. Now claim that exactly
two paths in \( p \) intersect at this point. Indeed, if three paths intersect at any point, then all of them have to go either East or South at the next step. By the pigeonhole principle, there are at least two paths that proceed in the same direction. It follows that a point common to three paths in \( p \) cannot be the final intersection point, as desired.

Define the \textit{tail} of \( p \) to be the two paths to the East and South of the final intersection point in \( p \). Given an intersecting tuple of paths \( p \), there exists a unique other tuple \( p' \) with the same final intersection point between the same two paths, but with the tails swapped. It is easy to see that the paths \( p, p' \) satisfy have opposite signs (for their permutations \( \sigma_p, \sigma_{p'} \)), but the same monomials in their weights. Therefore \( \text{wt}(p) = -\text{wt}(p') \), and the intersecting paths pair off as desired.

\textbf{Step 4:} From Step 3, the generating function \( \sum_p \text{wt}(p) \) equals the sum over only tuples of non-intersecting paths. Each of these tuples necessarily has \( \sigma_p = \text{id} \), so all signs are positive. In such a tuple, the monomial weight for the \( k \)th path naturally corresponds to a weakly increasing sequence of \( \lambda_k \) integers in \([1, N]\). That the paths do not intersect corresponds to the entries in the \( k \)th sequence being strictly smaller than the corresponding entries in the \((k + 1)\)st sequence. This yields a natural weight-preserving bijection from the tuples of non-intersecting paths to the ‘column-strict’ Young tableaux of shape \( \lambda \) with cell entries \( 1, \ldots, N \). (Notice that these tableaux are in direct bijection to the column-strict Young tableaux studied earlier in this part, by switching the cell entries \( j \leftrightarrow N + 1 - j \).) This concludes the proof.

\textbf{Proposition 43.4.} The definitions (1) and (4) are equivalent.

\textit{Proof.} The proof is a variant of that of Proposition 43.3. Now we consider all tuples of paths such that the \( k \)th path goes from \( P_{\sigma(k)} \), to the point

\[ Q'_k := (N - k + 1 + \lambda'_k, 1), \]

and moreover, each of these paths has at most one East step at each fixed height – i.e., no two East steps are consecutive.

Once again, in summing to obtain the generating function, given a permutation \( \sigma = \sigma_p \), the \( k \)th path in \( p \) contributes a monomial of total degree \( \lambda'_k - k + \sigma(k) \), but now runs over all monomials with individual variables of degree at most 1 – i.e., all monomials in \( e_{\lambda'_k - k + \sigma(k)} \).

It follows that

\[ \sum_p \text{wt}(p) = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{k=1}^N e_{\lambda'_k - k + \sigma(k)} = \det(e_{\lambda'_k - k + j})_{j,k=1}^N. \]

On the other side, we once again pair off tuples – this time, leaving the ones that do not overlap. In other words, paths in tuples may intersect at a point, but do not share an East/South line segment. As in the previous proof, notice that at most two paths can overlap at a given point. Now given a tuple containing two overlapping paths, define the \textit{final overlap point} similarly as in Proposition 43.3. Then for every tuple of paths \( p \) that overlaps, there exists a unique other tuple \( p' \) with the same final overlap point between the same two paths, but with the (new version of) tails swapped. It is easy to see that \( p, p' \) have the same monomials as weights, but with opposite signs, so they pair off and cancel weights.

This leaves us with tuples of non-overlapping paths, all of which again corresponding to \( \sigma_p = \text{id} \). In such a tuple, from the \( k \)th path we obtain a strictly increasing sequence of \( \lambda'_k \) integers in \([1, N]\). That the paths do not overlap corresponds to the entries in the \( k \)th sequence being at most as large as the corresponding entries in the \((k + 1)\)st sequence. This
gives a bijection to the *conjugates* of column-strict Young tableaux of shape \( \lambda \), and hence we once again have \( \sum_p \text{wt}(p) = \sum_T \text{wt}(T) \) in this setting.

\[ \text{Corollary 43.5. Schur polynomials are symmetric and homogeneous.} \]

\[ \text{Proof. This follows because Definition (4) is symmetric and homogeneous in the variables } u_j. \]

Finally, we show:

\[ \text{Proposition 43.6. The definitions (2) and (3) are equivalent.} \]

\[ \text{Proof. Once again, this proof is split into steps, for ease of exposition. In the proof below, we use the above results and assume that the definitions (1), (3), and (4) are all equivalent. Thus, our goal is to show that} \]

\[ \tau \text{-permuted paths is independent of } N. \]

Thus, our goal is to show that \( \text{det}(u_j^{N-k})_{j,k=1}^N \cdot \text{det}(h_{\lambda_j-j+k})_{j,k=1}^N = \text{det}(u_j^{\lambda_k+N-k})_{j,k=1}^N. \)

**Step 1:** We explain the formalism, which is a refinement of the one in the proof of Proposition 43.3 Thus, we return to the setting of paths between \( P_k = (N - k + 1, N) \) and \( Q_k = (N - k + 1 + \lambda_k, 1) \) for \( k = 1, \ldots, N \), but now equipped also with a permutation \( \tau \in S_N \). The weight of an East step now depends on its height; at height \( N + 1 - b \), an East step has weight \( u_{\tau(b)} \) instead of \( u_b \). Now consider tuples of paths over all \( \tau \); let us write their weights as \( \text{wt}_\tau(p) \) for notational clarity. In what follows, we also use \( p \) or \( (p, \tau) \) depending on the need to specify and work with \( \tau \in S_N \).

For each fixed \( \tau \in S_N \), notice first that the generating function \( \sum_p \text{wt}_\tau(p) \) of the \( \tau \)-permuted paths is independent of \( \tau \), by Corollary 43.5.

Now we define a new weight for these \( \tau \)-permuted paths \( p \). Namely, given \( p = (p, \tau) \), recall there exists a unique permutation \( \sigma_p \in S_N \); now define

\[ \text{wt}'(p) := (-1)^\tau \mu(\tau) \cdot \text{wt}(p), \quad \text{where } \mu(\tau) := u_{\tau(1)}^{N-1} u_{\tau(2)}^{N-2} \cdots u_{\tau(N-1)}. \]

The new generating function is

\[ \sum_{\tau \in S_N} \sum_p \text{wt}'(p) = \sum_{\tau \in S_N} (-1)^\tau \mu(\tau) \sum_p \text{wt}(p) = \text{det}(h_{\lambda_k-k+j})_{j,k=1}^N \cdot \text{det}(u_j^{N-k})_{j,k=1}^N, \]

where the final equality follows from the above propositions, given that the inner sum is independent of \( \tau \) from above.

**Step 2:** Say that a tuple \( p = (P_{\sigma_p(k)} \rightarrow Q_k)_k \) is *high enough* if for every \( 1 \leq k \leq N \), the \( k \)th path has no East steps below height \( N + 1 - k \). Now claim that (summing over all \( \tau \in S_N \)) the \( \tau \)-tuples that are not high enough once again pair up, with cancelling weights.

Modulo the claim, we prove the theorem. The first reduction is that for a fixed \( \tau \), we may further restrict to the \( \tau \)-tuples that are high enough and are non-intersecting (as in the proof of Proposition 43.3). Indeed, defining the final intersection point and the tail of \( p \) as in that proof, it follows that switching tails in tuples \( p \) of intersecting paths changes neither the monomial part of the weight, nor the high-enough property; and it induces the opposite sign to that of \( p \).

Thus, the generating function of all \( \tau \)-tuples (over all \( \tau \)) equals that of all non-intersecting, high-enough \( \tau \)-tuples (also summed over all \( \tau \in S_N \)). But each such tuple corresponds to \( \sigma_p = \text{id} \), and in it, all East steps in the first path must occur in the topmost row/height/\( Y \)-coordinate of \( N \). Hence all East steps in the second path must occur in the next highest row, and so on. It follows that the non-intersecting, high-enough \( \tau \)-tuples \( p = (p, \tau) \) are in...
bijection with \( \tau \in S_N \); moreover, each such tuple has weight \((-1)^\tau \mu(\tau) u_{\tau(1)}^{\lambda_1} u_{\tau(2)}^{\lambda_2} \cdots u_{\tau(N)}^{\lambda_N}\). Thus, the above generating function is shown to equal

\[
\det(u_j^{\lambda_k+N-k})_{j,k=1}^N,
\]

and the proof is complete.

**Step 3:** It thus remains to show the claim in Step 2 above. Given parameters

\[
\sigma \in S_N, \quad k \in [1,N], \quad j \in [1,N-k],
\]

let \(NH_{\sigma,k,j}\) denote the \(\tau\)-tuples of paths \(p = (p,\tau)\) (with \(\tau\) running over \(S_N\)), which satisfy the following properties:

1. \(p\) is not high \((NH)\) enough.
2. In \(p\), the \(k\)th path has an East step at most by height \(N-k\), but the paths labelled \(1,\ldots,k-1\) are all high enough.
3. Moreover, \(j\) is the height of the lowest East step in the \(k\)th path; thus \(j \in [1,N-k]\).
4. The permutation associated to the start and end points of the paths in the tuple is \(\sigma_p = \sigma \in S_N\).

Note that the set \(NH\) of tuples of paths that are not high enough can be partitioned as:

\[
NH = \bigcup_{\sigma \in S_N, \ k \in [1,N], \ j \in [1,N-k]} NH_{\sigma,k,j}.
\]

We now construct an involution of sets \(\iota : NH \to NH\) which permutes each subset \(NH_{\sigma,k,j}\), and such that \(p\) and \(\iota(p)\) have the same monomial attached to them but different \(\tau, \tau'\), leading to cancelling signs \((-1)^\tau \neq (-1)^{\tau'}\).

Thus, suppose \(p\) is a \(\tau\)-tuple in \(NH_{\sigma,k,j}\). Now define \(\tau' := \tau \circ (N-j, N+1-j)\); in other words,

\[
\tau'(i) := \begin{cases} 
\tau(i+1), & \text{if } i = N-j; \\
\tau(i-1), & \text{if } i = N-j+1; \\
\tau(i), & \text{otherwise}.
\end{cases}
\]

In particular,

\[
(-1)^{\tau'} = -(-1)^{\tau} \quad \text{and} \quad \mu(\tau') = \mu(\tau) u_{\tau(N+1-j)} u_{\tau(N-j)}^{-1}.
\]

With \(\tau'\) in hand, we can define the tuple \(\iota(p) = (\iota(p), \tau') \in NH_{\sigma,k,j}\). First, change the weight of each East step at height \(N+1-b\), from \(u_{\tau(b)}\) to \(u_{\tau'(b)}\). Next, we keep unchanged the paths labelled \(1,\ldots,k-1\), and in the remaining paths we do not change the source and target nodes either (since \(\sigma\) is fixed). Notice that weights change at only two heights \(j, j+1\); hence the first \(k-1\) paths do not see any weights change.

The changes in the (other) paths are now described. In the \(k\)th path, change only the numbers \(n_l\) of East steps at height \(l = j, j+1\), via: \((n_j, n_{j+1}) \mapsto (n_{j+1}+1, n_j-1)\). Note, the product of weights of all East steps in this path changes by a multiplicative factor of \(u_{\tau(N+1-j)} u_{\tau(N-j)}^{-1}\) which cancels the above change from \(\mu(\tau)\) to \(\mu(\tau')\). Finally, in the \(m\)th path for each \(m > k\), if \(n_l\) again denotes the number of East steps at height \(l\), then we swap \(n_j \leftrightarrow n_{j+1}\) steps in the \(m\)th path. This leaves unchanged the weight of those paths, and hence of the tuple \(p\) overall.

It is now straightforward to verify that the map \(\iota\) is an involution that preserves each of the sets \(NH_{\sigma,k,j}\). Since \(\text{wt}(\iota(p)) = -\text{wt}(p)\) for all \(p \in NH\), the claim in Step 2 is true, and the proof of the theorem is complete. \(\square\)
BIBLIOGRAPHIC NOTES AND REFERENCES

Most of the material in this part is taken from Belton–Guillot–Khare–Putinar [17] (see also [18] and [19] for summaries) and Khare–Tao [163] (see also its summary [164]). We list the remaining references. For preliminaries on Schur polynomials, the standard reference is Macdonald’s monograph [191]. Theorem 42.6 is proved using a deep result of Lam–Postnikov–Pylyavskyy [178], following previous work by Skandera [267]. There are other ways to show this result, e.g. using Chebyshev blossoming as shown by Ait-Haddou in joint works [4, 5], or by Dodgson condensation (see [163]). Theorem ?? is partly from [163] (the coordinatewise non-decreasing property on $(0,\infty)^N$), but the stronger assertions are shown in [24]. Appendix C follows Bressoud–Wei [52], relying on the works of Karlin–McGregor [155, 156], Lindström [184], and Gessel–Viennot [104].
REFERENCES


272 Bibliography


INDEX

A
absolutely monotonic function, 115
Andréief’s Identity, 35

B
Basic Composition Formula, 34
Bernstein’s theorem, 115
Boas–Widder theorem, 114, 133
de Bruijn’s theorem: zeros in sectors, 202
Budan–Fourier theorem, 56

C
Cauchy functional equation, 61
Cauchy mean-value theorem for divided differences, 114
Cauchy–Binet formula, 32
Chebyshev polynomials, 104, 105, 189
chordal graph, 83
column-strict Young tableau, 247
complete homogeneous symmetric polynomials $h_k(u)$, 265
compound matrix $C_r(A)$, 44
conditionally positive semidefinite matrix, 100, 101
contiguous minor, 27
continuity of roots, 46, 89
convolution, 194
correlation matrix, 16, 73, 104
covariance matrix, 16, 73, 104
critical exponent for:
  a graph $G$, 79
  Loewner convexity, 90
  Loewner monotonicity, 80
  Loewner positivity, 51
  Loewner super-additivity, 81
  one-sided $TN_p$ powers, 223
  positive definiteness, 54
  real-rootedness, 229

dual Jacobi–Trudi identity, 265

D
Descartes’ rule of signs, 31, 55, 56, 85, 176
Dirac function, 170
measure $\delta_x$, 18
discretization of kernels, 231
divided differences, 114
dominated convergence theorem (Lebesgue), 112
dual Jacobi–Trudi identity, 265

E
elementary symmetric polynomials $e_k(u)$, 185
entrywise
  map/function/transform $f[A]$, 51
  polynomial positivity preserver, 244
  powers preserving positivity $H_G$, 79

F
Fekete–Schoenberg lemma, 29
forward difference $(\Delta^k f(x))$, 113, 133
Further questions, 84, 92, 223, 229

G
gambler’s ruin, symmetric, 165
Gaussian kernel $T_G$, 37, 101, 169, 182, 186
Gegenbauer polynomials, 104, 105
Gram matrix, 16, 103, 104

H
Hadamard composition of polynomials/power series, 22, 202
Hamburger’s theorem, 118
Hankel
  $TN_n$ matrices $H_{TN_n}$, 71
  kernel, 231
  moment matrix $H_p$, 18
Heaviside function $H_d$, 196
Hermite–Poulain theorem, 200
Hilbert sphere $S_\infty$, 102
Historical notes, 18, 27, 50, 97, 99, 101, 111
  117, 123, 151, 176, 188, 199, 213, 217, 223
holomorphic function, 131
homotopy argument, 32, 86
Horn–Loewner theorem, stronger form, 107
Hurwitz matrix (generalized), 188

I
Identity theorem, 128
integration trick, 32, 54, 80, 82, 91, 261

J
Jacobi
  complementary minor formula, 183
  formula, 15
  matrix, 189
  Jacobi–Trudi identity (dual), 186, 265
  Jensen multiplier sequence, 202

283
Kronecker product \( A \otimes B \), 23

Kronecker’s theorem, 45

Laguerre–Pólya theorem, 206, 213
Laguerre–Poulain–Rolle trick, 31, 56, 200, 204
Laplace transform, bilateral, 193
Lee–Yang (circle) theorem, 211
Loewner
- convex function, 90, 153, 158
- monotone function, 80, 153, 158
- monotone function theorem, 98
- order, 158
- positive function, 51, 149
- super-additive function, 80

Málo’s theorem, 202
Markov chain, 165
matrix completion problems, 39
Mercer’s lemma, 234
Mercer’s theorem, 234
metric
- geometry, 99
- space, 99
mid-convex
- additive function, 63
- multiplicative function, 65
- Ostrowski’s theorem, 63
- Sierpiński’s theorem, 64
mollifier, 111
moment problem:
- Hamburger, 18
- Stieltjes, 27
moment-sequence \( s(\mu) \), 18, 118
monomial positive polynomial, 185, 258
Montel’s theorem, 131
Moore–Penrose inverse \( A^\dagger \), 20
Morera’s theorem, 131
Motzkin, Theodore, 123
multiplier sequence, 201

Perron’s theorem, 43
Pólya frequency (PF) function, 37, 169, 217
sequence, 181
sequence, finite, 188
sequence, generating function \( \Psi_n(x) \), 182
sequence, one-sided, 184
Pólya–Benz theorem, 201
Pólya–Schur theorem, 208, 209, 213
positive definite
- function, 101, 103, 104
- kernel, 113, 149
- matrix, 13
positive semidefinite (psd)
- kernel, 149
- matrices \( \mathbb{P}_n(I) \), 51
- matrices with zero pattern/sparsity \( \mathbb{P}_G \), 19
- matrix, 13
- positivity certificate, 123
- positivity-certificate trick, 125
Principal Specialization Formula, 248, 261
Rayleigh–Ritz theorem, 144
real analytic function \( f \in C^\omega(I) \), 127
Riemann hypothesis, 210
Routh–Hurwitz stability, 188
s.o.s. polynomial, 123
Schoenberg’s theorem:
- original form, 97
- stronger form, 117, 118
Schur
- complement, 19
- composition, 201
- composition theorem, 202
- polynomial \( s_n(u) \), 247
- polynomial, leading term of, 253
- positive polynomial, 186, 259
- product of matrices \( A \otimes B \), 22
- product theorem, 23
Schur–Szegő composition theorem, 205
sector in \( \mathbb{C} \) (open), 202
sign changes \( S^-() \), 26, 55, 56, 58, 85, 170
sign non-reversal property:
- \( TN \) matrices, 179
- \( TP \) matrices, 177
- spectral theorem, 13
- spherical distance \( \angle \), 102
Sylvester’s criterion, 14
Toeplitz cosine matrix, 17, 20
Toeplitz kernel \( T_f \) associated to a function \( f \), 164
totally non-negative \( (TN, TN_p) \)
- function, 164
- function of order \( p \), 164
kernel, 34
kernel of order $p$, 34
matrix, 25
matrix of order $p$, 25
totally positive ($TP, TP_p$)
kernel, 34
kernel of order $p$, 34
matrix, 25
matrix of order $p$, 25
TP completion problems, 39, 41
truncation $A^{(1)}$ of a matrix $A$, 27, 70, 232

V
Vandermonde determinant, 108, 248
Vandermonde matrix, generalized, 31, 40
variation diminishing property:
functions, 173
history, 176
polynomials, 174
sequences, 176
sign regular matrices, 26
$TN$ matrices, 172
$TP$ matrices, 170
Vasudeva’s theorem:
original form, 97
stronger form, 115

W
Wallach set (Berezin, Gindikin), 223
weak majorization, 258
Weyl Dimension Formula, 248
Weyl’s inequality, 144
Whitney-type density, 36, 234, 239
Wiener norm, 123
Wishart distribution, (non)central, 224

Z
zeros of polynomials in sectors, 186, 188, 189, 202