MA341 – Matrix Analysis and Positivity
2019 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Friday, September 6* in TA's office hours, or previously in class)

Question 1. Suppose $u \in \mathbb{R}^n$ is a nonzero vector. Classify all matrices $A \in \mathbb{P}_n(\mathbb{R})$ is such that $uu^T - A$ is also positive semidefinite.

Question 2 (R. Horn)**.** Suppose G is a (finite simple undirected) graph, and A_G is its adjacency matrix (with zeros on the diagonal, and all entries $0, 1$). Prove that $A + \mathrm{Id}$ is positive semidefinite if and only if every connected component of G is complete.

Question 3. Compute the Moore–Penrose inverse of the following matrices:

(1) uv^T, where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$.

(2) $\sum_{j=1}^k c_j u_j u_j^T$, where $u_1, \dots, u_k \in \mathbb{R}^n$ are orthonormal, and $c_j \in \mathbb{R}$ are nonzero scalars.

Question 4. If the columns of A are linearly independent, show that $A^\dagger = (A^* A)^{-1} A^*$.

Question 5. Show that if $m, n > 0$ and $0 < r \leq \min(m, n)$ are integers, and $D_{m \times m}, D'_{n \times n}$ are positive definite diagonal matrices, then $DA_{m \times n}D'$ is TP_r if and only if A is TP_r.

Question 6. Show that the (real symmetric) Toeplitz TN matrices fail to form a convex cone. Specifically, for each $n \geq 3$, produce two $n \times n$ real symmetric Toeplitz TN matrices, whose sum is not TN.

Question 7 ((De)compression trick, cont.)**.** Suppose A is positive semidefinite (in fact, real symmetric will suffice). Prove that the eigenvalues of the 'decompressed' matrix $(a_{jk} \mathbf{1}_{m_j \times m_k})_{j,k=1}^n$ are 0 with multiplicity $\sum_j m_j - n$, and the n eigenvalues of $\sqrt{D} A \sqrt{D}$, where D is the diagonal matrix with diagonal entries $m_1, \dots, m_n$.