A SHORT COURSE ON LEBESGUE MEASURE AND LEBESGUE INTEGRAL

LECTURES BY MANJUNATH KRISHNAPUR
TUTORIALS BY LAKSHMI PRIYA M. E. AND SUBHAJIT GHOSH

1. EXTERIOR MEASURE ON THE LINE

► Exterior measure. For $A \subseteq \mathbb{R}$, define its *external measure*

$$m_*(A) = \inf\left\{\sum_{n=1}^{\infty} |I_n| : \text{ each } I_n \text{ is an interval and } \bigcup_n I_n \supseteq A\right\}.$$

where $|I|$ is the length of the interval I (i.e., $b-a$ if $I=[a,b]$).

► Basic properties of exterior measure.

(1) $0 \le m_*(A) \le \infty$ for all $A \subseteq \mathbb{R}$.

(2) $m_*(A) \le m_*(B)$ whenever $A \subseteq B$. This is called *monotonicity*.

(3) $m_*(\cup_n A_n) \le \sum_n m_*(A_n)$. This is called *countable subadditivity*.

Sketch of proof: Only the last one is non-trivial. To prove it, find a cover for A_n by intervals $I_{n,1}, I_{n,2}, \ldots$ whose lengths sum up to less than $m_*(A_n) + 2^{-n}\epsilon$. The entire collection $\{I_{n,j}\}$ gives a cover for $\cup_n A_n$ by intervals and hence $m_*(\cup_n A_n) \le \sum_n \sum_k |I_{n,k}| \le (\sum m_*(A_n)) + \epsilon$.

► Fact: There exist sets A, B such that $A \cap B = \emptyset$, $A \cup B = [0,1]$ and $m_*(A) = m_*(B) = 1$.

► Further properties of exterior measure.

(1) $m_*(I) = |I|$ for any interval I.

(2) If A, B are disjoint and there is a $\delta > 0$ such that $|x-y| \ge \delta$ for all $x \in A$ and $y \in B$, then $m_*(A \cup B) = m_*(A) + m_*(B)$.

(3) $m_*(A) = \inf\{m_*(G) : G \supseteq A \text{ and } G \text{ is open}\}$.

(4) $m_*(A+x) = m_*(A)$ for $A \subseteq \mathbb{R}$ and any $x \in \mathbb{R}$. $m_*(rA) = rm_*(A)$ for $A \subseteq \mathbb{R}$ and $r > 0$. Here, $A + x = \{a + x : a \in A\}$ and $rA = \{ra : a \in A\}$.

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[0]These are notes for a mini-course on Lebesgue measure and Lebesgue integral, given as part of an NCM Instructional School for Teachers on Analysis and PDE, organized at IISc, Bangalore by T. Gudi and R. Venkatesh during the period May 6-18, 2019. The course consisted of six lectures of 90 minutes duration each, together with six tutorials of one hour each. These notes celebrate the unprecedented event of my covering all the material that I was asked to. References: Stein and Shakarchi's *Real analysis* and Royden's *Real analysis*.

2. Lebesgue measure

▶ Define the *Lebesgue sigma algebra*

$$\mathcal{L} = \{A \subseteq \mathbb{R} : m_*(A \cap E) + m_*(A^c \cap E) = m_*(E) \text{ for all } E \subseteq \mathbb{R}\}.$$

Elements of $\mathcal{L}$ are called measurable sets. For measurable sets, we denote $m_*(A)$ by $m(A)$. Thus m is a mapping from $\mathcal{L}$ into $[0, \infty]$. It is called *Lebesgue measure*.

▶ $\mathcal{L}$ is closed under complements and countable unions. Further, all intervals are in $\mathcal{L}$, and hence also all open sets and closed sets. Any set with zero exterior measure is in $\mathcal{L}$.

▶ Lebesgue measure is countably additive on $\mathcal{L}$. That is, if $A_1, A_2, \ldots$ are in $\mathcal{L}$ and pairwise disjoint, then $m(A_1 \cup A_2 \cup \ldots) = m(A_1) + m(A_2) + \ldots$.

3. Measurable functions

▶ A function $f : \mathbb{R} \mapsto \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is said to be *measurable* if the set $\{x : f(x) < t\}$ (we shall also write $\{f < t\}$ to denote this set) is a measurable set for any $t \in \mathbb{R}$.

▶ More generally, write $f^{-1}(B)$ for $\{f \in A\}$ (which is the same as $\{x \in \mathbb{R} : f(x) \in A\}$ for the inverse image of $B \subseteq \mathbb{R}$. Then $f^{-1}(B^c) = (f^{-1}(B))^c$ and $f^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha f^{-1}(B_\alpha)$ (note that this may even be an uncountable union).

▶ If f is a measurable function, then $\{f \le t\}$, $\{f \ge t\}$, $\{f > t\}$, $f^{-1}(I)$ (where I is an interval) are all measurable sets.

▶ Continuous functions are measurable.

▶ If f, g are measurable, then so are $f + g$, fg, $\max\{f, g\}$, $\min\{f, g\}$. It is easiest to deduce these from the following more general fact: Let $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ be a continuous function. Let $f_1, \ldots, f_d : \mathbb{R} \mapsto \mathbb{R}$ be measurable functions. Then $\varphi(f_1, \ldots, f_d) : \mathbb{R} \mapsto \mathbb{R}$ is a measurable function.

▶ Let f_n, $n = 1, 2, \ldots$, be measurable functions. Then so are $\sup\limits_n f_n$, $\inf\limits_n f_n$, $\limsup\limits_{n \to \infty} f_n$ and $\liminf\limits_{n \to \infty} f_n$.

4. Lebesgue integral

▶ A *simple function* is a function of the form $f = \alpha_1 \chi_{A_1} + \ldots + \alpha_n \chi_{A_n}$ where $\alpha_1, \ldots, \alpha_n$ are real numbers and $A_1, \ldots, A_n$ are measurable sets (and n is any positive integer). Basically these are functions whose range is a finite set. Note that the same simple function can be written in several ways, eg., $\chi_A + \chi_B = \chi_{A \cup B}$ if A and B are disjoint. If one needs a canonical representation, one can insist on distinctness of α_is and pairwise disjointness of A_is.

▶ **Step-1** For a simple function $f = \alpha_1\chi_{A_1} + \ldots + \alpha_n\chi_{A_n}$, we associate a number that we temporarily denote by $L_1(f)$ and defined by

$$L_1(f) = \alpha_1 m(A_1) + \ldots + \alpha_n m(A_n).$$

Justification is needed that this is a proper definition, since there may be multiple ways to write f in the form above.

▶ Check that on the vector space of simple functions, L_1 is linear and positive. This means (1) $L_1(f+g) = L_1(f) + L_1(g)$ and $L_1(\alpha f) = \alpha L_1(f)$ for $\alpha \in \mathbb{R}$, (2) $L_1(f) \geq 0$ if f is a non-negative simple function. As a corollary, if $f \leq g$ are both simple, then $L_1(f) \leq L_1(g)$ (monotonicity). These are fairly easy.

▶ **Step-2** Let f be any non-negative measurable function. Define its integral as

$$L_2(f) = \sup\{L_1(s) : s \text{ is a non-negative simple function and } 0 \leq s \leq f\}.$$

By the monotonicity of L_1 on simple functions, it is easy to check that if f is itself a non-negative simple function, then $L_2(f) = L_1(f)$. Thus, L_2 is an extension of L_1 to a larger class of functions.

▶ Check that if f, g are non-negative measurable functions, then (1) $L_2(f+g) = L_2(f) + L_2(g)$, (2) $L_2(\alpha f) = \alpha L_2(f)$ for $\alpha > 0$, (3) $L_2(f) \geq 0$. As before, it follows that if $0 \leq f \leq g$ are both measurable, then $L_2(f) \leq L_2(g)$ (monotonicity). Observe that non-negative measurable functions don't form a vector space, hence the restriction to $\alpha > 0$ in the second claim.

The second and third statements are trivial. The first one is *the most non-trivial point* in the entire construction and we explain it a little.

▶ Given a non-negative measurable function f, there exist simple functions s_n such that $s_n(x)$ increases to $f(x)$ for all $x \in \mathbb{R}$.

Proof: Divide $[0, \infty)$ into intervals $[0, 2^{-n})$, $[2^{-n}, 2.2^{-n})$, ...,$[n - 2^{-n}, n)$, and $[n, \infty)$. If $f(x)$ belongs to $[k2^{-n}, (k+1)2^{-n})$, define $f_n(x)$ to be $k2^{-n}$. If $f(x) \geq n$, define $f_n(x) = n$. Then $f_n(x)$ is an increasing sequence (for this it is important that we chose the small intervals to have length 2^{-n}. it would not work if we chose them to have length $1/n$, for example) and converges to $f(x)$. This gives the existence of s_ns.

▶ If $0 \leq s_1 \leq s_2 \leq \ldots$ are simple functions and $s_n(x) \to f(x)$ for all $x \in \mathbb{R}$, then $L_1(s_n) \uparrow L_2(f)$.

Proof: By monotonicity, $L_1(s_n)$ increases to some $\ell \leq +\infty$. Also, $L_1(s_n) \leq L(f)$ since $s_n \leq f$, hence $\ell \leq L_2(f)$. To show the other way inequality, let $0 \leq t \leq f$ be any simple function and let $0 < c < 1$ be any number. Write $t = \alpha_1\chi_{A_1} + \ldots + \alpha_k\chi_{A_k}$ with disjoint A_ks. Let $B_{n,j} := A_j \cap \{s_n \geq c\alpha_j\} \uparrow A_j$ for each $j \leq k$. Then $s_n \geq c\alpha_1\chi_{B_{n,1}} + \ldots + c\alpha_k\chi_{B_{n,k}}$ which shows that $L_1(s_n) \geq c\alpha_1 m(B_{n,1}) + \ldots + c_k\alpha_k m(B_{n,k})$. As $s_n \uparrow f$, it is easy to see that $B_{n,j} \uparrow A_j$ and hence $m(B_{n,j}) \uparrow m(A_j)$. This shows that $\lim L_1(s_n) \geq cL_2(t)$. Take $c \to 1$ and then supremum over all t to get $\ell \geq L_2(f)$.

▶ **Step-3** Let f be any measurable function. Let $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Then f_+, f_- are non-negative measurable functions and $f = f_+ - f_-$. If it so happens that $L_2(f_+)$ and $L_2(f_-)$ (defined in step-2) are finite, then say that f is integrable and set $L_3(f) = L_2(f_+) - L_2(f_-)$.

If only one of $L_2(f_+)$ and $L_2(f_-)$ is equal to $+\infty$ and the other is finite, then we do not say that f is integrable, although it is convenient to define $L_3(f)$ as $\pm\infty$ (depending on which of them is infinite).

▶ Prove that integrable functions form a vector space and that L_3 is linear and positive on this vector space. That is (1) $L_3(f+g) = L_3(f) + L_3(g)$, (2) $L_3(\alpha f) = \alpha L_3(f)$ for $\alpha \in \mathbb{R}$, (3) $L_3(f) \geq 0$.

Proofs: Again the second and third claims are obvious. To show the first one, observe that $f+g$ can be written as $(f+g)_+ - (f+g)_-$ and also as $(f_+ - f_-) + (g_+ - g_-)$. Equating the two and rearranging, we see that $(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+$. All these are non-negative measurable functions. Using additivity of L_2, it follows that $L_2((f+g)_+) + L_2(f_-) + L_2(g_-) = L_2((f+g)_-) + L_2(f_+) + L_2(g_+)$. Rearranging again, we see that $L_3(f+g) = L_3(f) + L_3(g)$.

▶ Henceforth, write $\int_{\mathbb{R}} f(x) dm(x)$ or $\int f dm$ in short for $L_3(f)$. To integrate on subsets of $\mathbb{R}$, we write $\int_A f dm$ to mean $\int_{\mathbb{R}} f \chi_A dm$.

▶ If f is an integrable function and g is another function such that $m\{f \neq g\} = 0$. Then g is also measurable and integrable and $\int f dm = \int g dm$. That is, Lebesgue integral is insensitive to changes of values on sets of zero measure. In particular, if $f \geq 0$ and $\int f dm = 0$, then it is not necessary that $f = 0$. We can only say that $f = 0$ $a.e.$ (read as f is equal to zero almost everywhere and meaning that $m\{f \neq 0\} = 0$).

▶ Since $\pm\infty$ are allowed values of measurable functions, adding functions can be problematic if there exists points for which we get $\infty - \infty$ which is undefined. However, observe that if f is an integrable function, then $-\infty < f < +\infty$ $a.e.$, hence by changing f on a set of zero measure, we can make it finite-valued. When adding two integrable functions, the ambiguity of $\infty - \infty$ can occur at most on a set of zero measure, and if we ignore that set (or set the sum to be 0, say), then the sum is well-defined. More about this $a.e.$ business later.

▶ We summarize the main properties of the Lebesgue integral. It is **linear** and **positive**. Further, for any integrable function $|\int f \, dm| \leq \int |f| \, dm$ (since the left side is the difference of two non-negative numbers $\int f_+ \, dm$ and $\int f_- \, dm$ while the right side is the sum of the same numbers).

5. RIEMANN AND LEBESGUE INTEGRALS

▶ A function can fail to be integrable either because it is too big or because the the local structure of the function is bad. For example, if we go back to Riemann integral on $[0,1]$, the function $1/x$ fails to be integrable for the first reason, while the function $\chi_{\mathbb{Q}}$ (one on rationals, zero on irrationals) fails to be integrable for the second reason (upper Riemann sums are always 1 and lower Riemann sums are always 0). The first difficulty is not surprising, hence let us assume in this section that all our functions are bounded function on $[0,1]$.

▶ (Vitali-Lebesgue): A bounded function on $[0,1]$ is Riemann integrable if and only if the set of discontinuity points of f has zero Lebesgue measure.

▶ From the construction of Lebesgue integral, it is clear that all bounded *measurable* functions are integrable in the Lebesgue sense.

▶ Check that if f is continuous $a.e.$, then it is measurable. Hence, Riemann integrable functions form a subset of Lebesgue integrable functions. Further, where both are defined, they agree. Therefore, Lebesgue integral is strictly better.

▶ In some books one finds examples of functions that are not Lebesgue integrable but do have an improper Riemann integral (eg., $\frac{\sin x}{x}$). But these are not absolutely integrable.

6. CONVERGENCE THEOREMS FOR LEBESGUE INTEGRAL

▶ To appreciate this section, go back and look at the clumsy theorems about exchanging limits and integrals in Riemann integration. But never use them because now you have better theorems!

▶ **Monotone convergence theorem** Let $0 \le f_1 \le f_2 \le \ldots$ be a increasing sequence of measurable functions and let $f_n \uparrow f$ (then f is measurable of course). Then $\int f_n dm \uparrow \int f dm$.

Proof: That $\int f_n dm$ is increasing in n and bounded above by $\int f dm$ is clear. To show the other way inequality, it suffices to produce non-negative simple functions t_n that increase to f and such that $t_n \le f_n$ (because then $\int f_n dm \ge \int t_n dm$ and the latter increases to $\int f dm$ by the preliminary version of MCT that we proved earlier). To produce such a sequence of t_ns, pick non-negative simple functions $s_{n,k}$ that increase to f_n as $k \to \infty$ and define $t_k = \max\{s_{1,k}, \ldots, s_{k,k}\}$ It is clear that t_ks increase. Further, for any n we see that $t_k \ge s_{n,k}$ for $n \ge k$. Hence $\lim t_k \ge f_n$. As this holds for all n, we see that $t_k \uparrow f$.

▶ **Fatou's lemma** Let f_n be non-negative measurable functions. Then $\int (\liminf_{n\to\infty} f_n) dm \le \liminf(\int f_n dm)$.

Proof: Let $g_n = \inf\limits_{k:k\ge n} f_k$ and $g = \liminf f_n$. Then $g_k \uparrow g$ and since all are non-negative, by MCT it follows that $\int g dm = \lim \int g_n dm$. But $g_n \le f_n$ and hence $\int g_n dm \le \int f_n dm$ which implies that $\int g dm \le \liminf \int f_n dm$.

▶ **Dominated convergence theorem** Let f_n be measurable functions such that $f_n \to f$ pointwise. If there is a integrable function g that dominates f_ns (i.e., $|f_n| \le g$ for all n), then $\int f_n dm \to \int f dm$. In fact $\int |f_n - f| dm \to 0$.

Proof: As $-g \le f_n \le g$, the functions $g - f_n$ and $g + f_n$ are non-negative. They converge to $g - f$ and $g + f$ respectively. By Fatou's lemma, it follows that $\int (g \pm f) dm \le \liminf \int (g - f_n) dm$. As g is integrable and dominates f_ns (and hence dominates f), we see that f_n, f are also integrable. Hence the integrals can be separated and we get

$$\int g dm + \int f dm \le \int g dm + \liminf \int f_n dm, \qquad \int g dm - \int f dm \le \int g dm - \limsup \int f_n dm.$$

Cancelling the $\int g dm$ terms, we get $\limsup \int f_n dm \le \int f dm \le \liminf \int f_n dm$, from which the first result follows.

Now apply the same result to $|f_n - f|$ in place of f_n and 0 in place of f. As these functions are dominated by $2g$, it follows that $\int |f_n - f| dm \to 0$.

7. ALMOST EVERYWHERE

► We saw that if $f \geq 0$ is a measurable function, then $\int f dm = 0$ if and only if $f = 0$ $a.e.$ (meaning $m\{f \neq 0\} = 0$). This is in contrast to integrals of continuous functions where the conclusion would have been that $f = 0$ identically. As a consequence, if f is an integrable function and we get g by changing f on a set of zero measure (i.e., $m\{g \neq f\} = 0$), then $\int g dm$ is the same as $\int f dm$. In fact, $\int |f - g| dm = 0$.

► The moral is that in measure theory the value of a function at a point is not so meaningful. Functions have only some sort of a global meaning (in contrast to continuous functions, where neighbouring values fix the value at a point). This is made more precise by defining a relationship $f \sim g$ if $f = g$ $a.e.$ (i.e., $m\{f \neq g\} = 0$). Check that this is indeed an equivalence relation. All functions in one equivalence class are thought of as essentially the same function.

For later purposes, let us be more precise. Let $\mathcal{M}$ denote the set of all measurable functions on $\mathbb{R}$ that take finite values except on a set of zero measure. That is,

$$\mathcal{M} = \{f : \mathbb{R} \mapsto \overline{\mathbb{R}} : f \text{ is measurable and } -\infty < f < +\infty \ a.e.\}.$$

Say that $f, g \in \mathcal{M}$ are equivalent if $f = g$ $a.e.$ Check that this is an equivalence relation. Let $\mathcal{M}_0$ denote the collection of all equivalence classes. Check that $\mathcal{M}_0$ has a vector space structure under addition and scalar multiplication. For example, to define the sum of two equivalence classes, we take a representative function from each, sum the functions, and take the equivalence class containing the sum, etc. In doing this, the problem of $\infty - \infty$ occurs at most on a set of zero measure, and we may choose to define it any way, for example, let $\infty - \infty = 0$ (note that the equivalence class of the resulting function does not change).

► Measure zero sets need not be small in other senses. For example, the Cantor set is uncountable but has zero measure.

8. p-NORMS

► Consider $\mathbb{R}^2$, a two dimensional real vector space. For $p > 0$, define $\|x\|_p := (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Then $\|\alpha x\|_p = |\alpha| \|x\|_p$ (homogeneity) and $\|x\|_p \geq 0$ with equality if and only if $x = 0$ (positivity). What remains to make it a norm is the triangle inequality $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in \mathbb{R}^2$. This is true for $1 \leq p < \infty$ but fails for $0 < p < 1$. Thus, we have the norms $\|\cdot\|_p$ for $1 \leq p < \infty$.

► It is an elementary exercise to show that $\|x\|_p \to \max\{|x_1|, |x_2|\}$ as $p \to \infty$, for any $x \in \mathbb{R}^2$. This suggests defining $\|x\|_\infty := \max\{|x_1|, |x_2|\}$. Check that $\|cdot\|_\infty$ is also a norm.

► Of all the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$ on R^2, the most special one is $\|\cdot\|_2$. This is because this norm comes from an inner product. More precisely, if we consider define $\langle x, y \rangle = x_1 y_1 + x_2 y_2$, then this is an inner product and it induces the norm $\langle x, x\rangle = \|x\|_2$. It is a good exercise to show that $\|\cdot\|_p$ does not come from an inner product, for any $p \neq 2$.

▶ The dual space of $\mathbb{R}^2$ is isomorphic to $\mathbb{R}^2$ itself, by identifying the linear functional $x \mapsto ax_1 + bx_2$ with the vector (a, b). Given a norm $\|\cdot\|$ on $\mathbb{R}^2$, we define its dual norm as $\|(a,b)\|_* := \max\{|ax_1 + bx_2| : \|x\| = 1\}$. It is not hard to show that the dual norm of $\|\cdot\|_p$ (for $1 \le p \le \infty$) is $\|\cdot\|_q$ where q is related to p by $\frac{1}{p} + \frac{1}{q} = 1$.

▶ Given a vector space with a norm, it becomes a metric space with the distance between x and y defined as $\|x - y\|$.

▶ All this is to motivate the definition of L^p spaces in terms of Lebesgue integral. The definition can also be made with Riemann integral, for example, consider the space of all Riemann integrable functions on $\mathbb{R}$. The problem with this space is that it is not complete. When we go to Lebesgue integrable functions, we get a complete space. This may be taken as one of the purposes of developing Lebesgue integration theory.

9. LEBESGUE SPACES

▶ Consider the space of all integrable functions on $\mathbb{R}$. We have seen that it forms a vector space. If we try to define a norm on this space by setting $\|f\| := \int_{\mathbb{R}} |f| dm$, then it does satisfy homogeneity ($\|\alpha f\| = |\alpha| \|f\|$) and triangle inequality ($\|f + g\| \le \|f\| + \|g\|$). It is also true that $\|f\| \ge 0$, but it is not true that $\|f\| = 0$ implies that $f = 0$. Hence it is not a norm.

▶ To overcome this problem, consider $\mathcal{M}_0$, the collection of equivalence classes of measurable functions defined above. Observe that if one function in an equivalence class is integrable, so are all other functions, and their integrals are the same. Thus, we can define the space

$$L^1 = \{[f] : f \text{ is integrable }\}$$

and a norm $\|[f]\|_1 = \int_{\mathbb{R}} |f| dm$. All this is well-defined (i.e., independent of the representative chosen) and the norm is a genuine norm.

▶ More generally, for any $p > 0$, we define the space

$$L^p = \{[f] : |f|^p \text{ is integrable }\}$$

and write $\|[f]\|_p = \left(\int_{\mathbb{R}} |f|^p \, dm\right)^{1/p}$. We also define

$$L^\infty = \{[f] : |f| \text{ is bounded }\}$$

with $\|[f]\|_\infty$ defined as the essential supremum of $|f|$, defined as the infimum of all $t > 0$ such that $m\{|f| > t\} = 0$. A key point about these norms is that if $f \sim g$, then f and g have the same norm (in fact $\|f - g\| = 0$). Hence, when we move to equivalence classes (which is the same as quotienting by the subspace of functions that are zero $a.e.$), they define norms on the reduced space.

▶ **Theorem** For $1 \le p \le \infty$, the space L^p is a vector space and $\|\cdot\|_p$ is a norm on it. Further, L^p is complete in the metric induced by the norm.

▶ This theorem is a fundamental theorem in analysis. Note that we could have consider the space of all Riemann integrable functions. That would be a vector space, and $f \mapsto \int |f|$ would have been a norm on it (some equivalence would have to be defined first). However, it would not have been complete. The above theorem assures us that with Lebesgue integral "all the holes are filled", and there is no need for any further integral.

▶ The proof of the above theorem needs some work. Even the fact that $\|\cdot\|_p$ is a norm is not obvious. It is easy to see in the three most important cases of $p = 1, 2, \infty$.

▶ **Proof of completeness of L^1:** Given a Cauchy sequence in L^1, take representatives f_n in the equivalence classes. The Cauchy property implies that $\|f_n - f_m\|_1 \to 0$ as $m, n \to \infty$. Hence we may find $n_1 < n_2 < n_3 < \ldots$ such that $\|f_n - f_m\|_1 \le 2^k$ if $n, m \ge n_k$. We shall show that f_{n_j} converges $a.e.$ to a function f and then argue that this convergence is also in L^1 metric (i.e., $\|f_{n_k} - f\|_1 \to 0$ as $k \to \infty$). Then it follows that f_n converges to f in L^1 metric (the reason is that if a Cauchy sequence in a metric space has a convergent subsequence, then the whole sequence converges). Then $[f_n] \to [f]$ in L^1, showing the completeness.

To show that the sequence $h_j = f_{n_j}$ converges, write $h_k = h_1 + (h_2 - h_1) + \ldots + (h_k - h_{k-1})$. The limit, if it exists, should be the corresponding infinite series, but we should first make sure that the series makes sense. Hence, we define

$$g = |h_1| + \sum_{k=1}^{\infty} |h_{k+1} - h_k| = |h_1| + \lim_{m\to\infty} \sum_{k=1}^{m} |h_{k+1} - h_k|.$$

This is well-defined, although the value could be $+\infty$ at some x. However, by MCT, $\int g\, dm = \int |h_1| dm + \sum_{k=1}^{\infty} \int |h_{k+1} - h_k| dm$, which is finite since the kth terms is at most 2^{-k} (by choice of n_k). Thus, g is integrable, and hence $g < \infty$ $a.e.$ Therefore, we may define

$$f(x) = \begin{cases} h_1(x) + \sum_{k=1}^{\infty}(h_{k+1}(x) - h_k(x)) & \text{if } g(x) < \infty \\ 0 & \text{if } g(x) = \infty. \end{cases}.$$

When $g(x) < \infty$, the series converges absolutely and hence $f(x)$ has a well-defined finite value. But of course, this means that $h_k \to f$ $a.e.$ (since the partial sums are just h_k). We claim that this convergence is also in L^1. Indeed,

$$\int |f - h_m| dm = \int \Big| \sum_{k=m}^{\infty} (h_{k+1} - h_k) \Big|\, dm \le \sum_{k=m}^{\infty} 2^k = 2^{-m+1}.$$

Thus $\|f - h_m\|_1 \to 0$, completing the proof.