Probability Space $(\Omega, \mathcal{F}, P)$.

Def 1:- Gaussian linear Space is a real linear space of r.v, defined on $(\Omega, \mathcal{F}, P)$ such that each variable in Space is centered Gaussian. (it is a Subspace of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, P)$). we use norm and inner Product of L^2 on it.

Def 2:- A Gaussian Hilbert Space is Gaussian linear space which is complete i.e closed subspace of $L^2(\Omega, \mathcal{F}, P)$ Consisting of Centered Gaussian r.v

Result 1:- $G \subset L^2_R(\Omega, \mathcal{F}, P)$ is a Gau. Lin. Space, then $\bar{G}$ in L^2 is Gau. Hil. space. explanation:- $\xi_n \in G$, $\xi_n \to \xi$ in $L^2 \Rightarrow \xi \stackrel{d}{=} N(0, \sigma^2)$ $\therefore$ ξ in $\bar{G}$ has normal dist.

Result 2:- Any set of rv in GLS (Gaussian Linear Space) G has a joint normal distribution.

explanation:-

$\sum_{i=1}^{n} t_i \xi_i \in G$ and has normal distribution. for arbitary t_i's.

Therefore $\{\xi_i\}_{i=1}^{n}$ has a joint normal distribution.

L^P norms are all proportional on a GLS G.

Closure of G in any L^P $0 < p < \infty$ equals GHS $\bar{G}$.

GHS $\bar{G}$ is closed Subspace of $L^P_R(\Omega, \mathcal{F}, P)$.

Examples:-

(i) Let ~~ξ be a rv.~~ ξ be any non-degenerate, normal variable with mean zero. Then $\{t\xi : t \in \mathbb{R}\}$ is a one-dimensional Gaussian Hilbert Space.

(ii) Let $\xi_1, \xi_2 \cdots \xi_n$ have a joint normal distribution with mean zero. Then the linear span $\{\sum_1^n t_i \xi_i : t_i \in \mathbb{R}\}$ is a finite dimensional Gaussian Hilbert space.

(iii) More generally, if $\{\xi_\alpha\}$ is any set of centered jointly normal variables, then linear span of $\{\xi_\alpha\}$ is a Gaussian linear space. closed linear span of $\{\xi_\alpha\}$ is GHS.

$$\left\{ \sum_\alpha a_\alpha \xi_\alpha : \sum_\alpha a_\alpha^2 < \infty \right\} \text{ is GHS.}$$

(iv) B_t, $0 \le t < \infty$ SBM Brownian Motion.

Closed linear span of $\{B_t\}_{t \geq 0}$ is a GHS.

denoted by $H(B)$. This space has a simple representation in terms of stochastic integrals

$H(B) = \left\{ \int_0^\infty f(t)\, dB_t \right\}$ where f ranges over set of deterministic functions $L^2_{\mathbb{R}}([0,\infty), dt)$.

H be GHS on $(\Omega, \mathcal{F}, P)$.

variables in $H \in L^p$ for every finite p (Holder's inequality).

Def: $n \geq 0$. $\overline{P}_n(H)$ be closure in $L^2(\Omega, \mathcal{F}, P)$ of the linear space. $P_n(H) = \{ P(\xi_1, \xi_2, \dots \xi_m) : P$ is a polynomial of degree $\le n$; $\xi_1, \xi_2 \dots \xi_m \in H$; $m < \infty \}$.

let $H^{:n:} = \overline{P}_n(H) \ominus \overline{P}_{n-1}(H) = \overline{P}_n(H) \cap \overline{P}_{n-1}(H)^\perp$

for $n = 0$, we let $H^{:0:} = \overline{P}_0(H)$, space of constants.

If H is finite dim, then $\overline{P}_n(H) = P_n(H)$.

If H has infinite dimension.

$\{\xi_i\}_{i=1}^\infty$ ONB in H then $\sum_{i=1}^\infty \frac{1}{2^i} \xi_i^2 \in \overline{P}_2(H)$

but it is not in $P_2(H)$.

[requires proof].

By def., $\{\bar{P}_n(H)\}_{n=0}^{\infty}$ is an increasing seq of closed subspaces of L^2, while spaces $H^{:n:}$ are orthogonal.

$$\bar{P}_n(H) = \bigoplus_{0}^{n} H^{:k:}$$

and thus.

$$\bigoplus_{0}^{\infty} H^{:k:} = \overline{\bigcup_{0}^{\infty} \bar{P}_n(H)}$$

The latter space, infact, consists of all square integrable functions that are measurable with respect to σ-field generated by H.

Thm :- The spaces $H^{:n:}$, $n \geq 0$ are mutually orthogonal closed subspaces of $L^2 = L^2(\Omega, \mathcal{F}, P)$ and

$$\bigoplus_{0}^{\infty} H^{:n:} = L^2(\Omega, \mathcal{F}(H), P). \text{ where}$$

$\mathcal{F}(H)$ is σ-field generated by r.V in H.

This decomposition of $L^2(\Omega, \mathcal{F}(H), P)$ is called wiener chaos decomposition. $X = \sum_{0}^{\infty} X_n$, $X_n \in H^{:n:}$

$X \in L^2(\Omega, \mathcal{F}(H), P)$.

ex:- $H = \{t\xi;\ t \in \mathbb{R}\}$ ξ is $N(0,1)$ distributed r.v.

$H^{:n:}$ is spanned by ~~t_n then~~ h_n, where $\{h_n\}_{n=0}^{\infty}$ is sequence of orthogonal polynomials with respect to stand Gaussian measure $dp = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

$h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$.

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$P(H) = \bigcup_0^\infty P_n(H)$ space of polynomials in elements of H.

$\overline{P}_*(H) = \bigcup_0^\infty \overline{P}_n(H) = \sum_0^\infty H^{:n:}$

i.e. space of all elements in $L^2(\Omega, \mathcal{F}(H), P)$ having finite chaos decomposition.

Again in finite dim H, $\overline{P}_*(H)$ equals $P(H)$.

but if H has infinite dim, then $\overline{P}_*(H)$ is strictly larger.

Thm :- The set of polynomial variables $P(H) = \bigcup_n P_n(H)$ is a dense subspace of $L^p(\Omega, \mathcal{F}(H), P)$.

Def :- for $n \geq 0$, Π_n denotes the orthogonal projection of L^2 onto $H^{:n:}$.

If $\xi_1, \xi_2 \ldots \xi_n$ is a finite sequence of elements of GHS H, their wick product $:\xi_1 \xi_2 \ldots \xi_n: \in H^{:n:}$ is given by $:\xi_1 \xi_2 \ldots \xi_n: = \Pi_n(\xi_1 \xi_2 \ldots \xi_n)$.

for $n = 0$, $:: = 1 \in H^{:0:}$

Define the general wick product by

$X \odot Y = \Pi_{m+n}(XY)$ if $X \in H^{:m:}$ and $Y \in H^{:n:}$ for some $m, n \geq 0$ and extend $\odot$ by bilinearity to $\overline{P}_*(H)$.

$:\xi: = \xi$, $:\xi_1 \xi_2: = \xi_1 \xi_2 - E(\xi_1 \xi_2)$.

$:\xi^n: = \sigma^n h_n(\xi/\sigma)$ $\quad \xi \sim N(0, \sigma^2)$.

$\{h_n\}$ is sequence of hermite polynomials

Thm:- Let H be a Gaussian Hilbert space and let $\{\xi_i\}_{i\in I}$ be an orthonormal basis in H (finite or infinite, possibly even uncountable). If $\alpha = (\alpha_i)_{i\in I}$ is a multi index, ie sequence of non-negative integers with only finitely many elements different from 0, then

$$: \prod_i \xi_i^{\alpha_i} : = \prod_i h_{\alpha_i}(\xi_i).$$

for each $n \geq 0$, the set $\{ (\prod_i \alpha_i!)^{-1/2} : \prod_i \xi_i^{\alpha_i} : \}$, where (α_i) ranges over all multi-indices is an orthonormal basis in $L^2(\Omega, \mathcal{F}(H), P)$. The subset of all such variables with $|\alpha| = \sum \alpha_i = n$ is an orthonormal basis in $H^{:n:}$ ~~and~~

Properties, definitions:-

Define:- Given a zero mean Gaussian r.v X its wick exponential is defined as follows

$$: \exp\{X\} : = \exp\{ X - E(X^2)/2 \}.$$

Given jointly Gaussian random variables $\underset{\sim}{X} = (X_1, X_2 \ldots X_k)$ and integers $\underset{\sim}{n} = (n_1, n_2 \ldots n_k)$.

wick monomial $: X_1^{n_1} X_2^{n_2} \ldots X_k^{n_k} :$ is defined as,

and

$$: X_1^{n_1} X_2^{n_2} \ldots X_k^{n_k} : = \left(\frac{\partial^{n_1+n_2 \ldots n_k}}{\partial t_1^{n_1} \partial t_2^{n_2} \ldots \partial t_k^{n_k}} : \exp\{t_1 X_1 \ldots + t_k X_k\} : \right)_{t_1 = t_2 \ldots t_k = 0}$$

wick product of two wick monomials is defined as

$$:(:x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}:)(:x_1^{m_1} \dots x_k^{m_k}:): = :x_1^{n_1+m_1} \dots x_k^{n_k+m_k}:$$

wick polynomials are linear combinations of wick monomials. wick product extends by linearity from wick monomials to wick polynomials.

Proposition :- The wick product (defined for wick polynomials) is commutative, associative and distributive with respect to linear combinations. That is, given the wick polynomials P, Q, R and real numbers α, β we have

$$:PQ: = :QP: \quad , \quad :(:PQ:)R: = :P(:QR:): \quad ,$$

$$:P(\alpha Q + \beta R): = \alpha :PQ: + \beta :PR:$$

Proposition :- Let $X = (X_1, X_2 \dots X_m, Y_1, Y_2, \dots Y_n)$ be jointly Gaussian. Then

$$E(:X_1 X_2 \dots X_m: :Y_1 Y_2 \dots Y_n:) = \delta_{m,n} \sum_{\sigma \in perm(n)} \prod_{i=1}^{n} E(X_i Y_{\sigma(i)}).$$

Thm :- If $A : H_1 \longrightarrow H_2$ is a bounded linear map between two GHS, defined on prob. spaces (Ω_i, f_i, P_i), $i = 1, 2$, then

$$:\xi_1 \xi_2 \ldots \xi_n: \longrightarrow :A\xi_1 A\xi_2 \ldots A\xi_n:$$

defines a bounded linear operator $A^{:n:} : H_1^{:n:} \longrightarrow H_2^{:n:}$ with $\|A^{:n:}\| = \|A\|^n$. These operators combine to an algebra homomorphism $\bar{P}_*(H_1) \longrightarrow \bar{P}_\alpha(H_2)$, and, provided moreover $\|A\| \leq 1$, a linear operator

$$\Gamma(A) : L^2(\Omega_1, f(H_1), P_1) \longrightarrow L^2(\Omega_2, f(H_2), P_2)$$

with $\|\Gamma(A)\| = 1$.

Remark :- we also have the functorial property

$\Gamma(AB) = \Gamma(A)\Gamma(B)$, if $B : H_0 \longrightarrow H_1$ is another contraction.

Exa :- H be GHS and γ be real. $|\gamma| \leq 1$. Then letting I denote the ~~ideten~~ identity operator on various spaces, $:(\gamma I)^n: = \gamma^n I$ on $H^{:n:}$. Hence $\Gamma(\gamma I)$ is linear operator on $L^2(\Omega, f(H), P)$ i.e given by

$$\Gamma(\gamma I)\left(\sum_0^\infty X_n\right) = \sum_0^\infty \gamma^n X_n \quad X_n \in H^{:n:}$$

i.e every $H^{:n:}$ is an eigenspace with eigenvalue γ^n.

Thm :-

Let $A : H_1 \longrightarrow H_2$ be a linear operator between two Gaussian Hilbert spaces, defined on probability spaces $(\Omega_i, \mathcal{F}_i, P_i)$ such that $\|A\| \leq 1$. Then $\Gamma(A)$ can be (uniquely) extended to a continuous linear operator $L^1(\Omega_1, \mathcal{F}(H_1), P_1) \longrightarrow L^1(\Omega_2, \mathcal{F}(H_2), P_2)$, ~~w~~ which we also denote by $\Gamma(A)$. Furthermore,

(i) $\|\Gamma(A)X\|_p \leq \|X\|_p$ for any $X \in L^p$, $1 \leq p \leq \infty$

(ii) if $X \geq 0$ a.s, then $\Gamma(A)X \geq 0$ a.s.

(iii) $|\Gamma(A)X| \leq \Gamma(A)|X|$ a.s for any $X \in L^1$.