Notes on applications of hypercontractivity

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1. *Consider a rooted binary tree of depth n. For each edge assign an independent standard normal as weight. For each leaf define leafsum to be the sum of weights of all edges in the unique path connecting the corresponding leaf and root. Let M_n be the maximum of all the leafsums. Using hypercontractivity we can bound $Var(M_n) \leq C\log(n)$ (optimal bound is $Var(M_n) \leq K$).*

2. *In the first passage percolation model on Z^d with i.i.d. edge weights having absolute value of standard Gaussian, let T_n be the first passage time from origin to $(n, 0, 0, \dots, 0)$. Then $Var(T_n) \leq Cn/\log(n)$*

An useful inequality in proving above results is Talagarand's $L^1 - L^2$ inequality, which is stated as follows.
Let $f : \mathbb{R}^n \to \mathbb{R}$ and γ^n is the standard Gaussian distribution on $\mathbb{R}^n$. Then

$$\mathrm{Var}_{\gamma^n}(f) \leq C \sum_{i=1}^{n} \frac{||\partial_i f||^2_{L^2(\gamma^n)}}{1 + \log \frac{||\partial_i f||_{L^2(\gamma^n)}}{||\partial_i f||_{L^1(\gamma^n)}}}$$

1. is obtained immediately on applying Talagrand's $L^1 - L^2$ inequality. To prove Talagrand's inequality, we use the following framework.

Consider Ornstein-Uhlenbeck process $(X_t)_{t\geq 0}$. The corresponding semi-group $(P_t)_{t\geq 0}$ acting on real valued functions on real line is given by

$$P_t f(x) = \mathbb{E}_x f(X_t) = \mathbb{E} f(e^{-t}x + \sqrt{1 - e^{-2t}}Z)$$

where Z is a standard Gaussian random variable.

Now consider the Gaussian Hilbert space $\mathbb{H} := \mathrm{span}\{\ X\}$ where X has standard Gaussian distribution. For $r < 1, n \in \mathbb{N} \cup \{0\}$ define $A^{:n:} : \mathbb{H}^{:n:} \to \mathbb{H}^{:n:}$ to be $A^{:n:}X = r^n X$. Any $X \in L^2(\Omega, \mathcal{F}(\mathbb{H}), P)$ can be written in the form $X = \sum_{n=0}^{\infty} X_n$ where $X_n \in \mathbb{H}^{:n:}\ \forall n \geq 0$.

Define for $r < 1$, $\Gamma(A) : L^2(\Omega, \mathcal{F}(\mathbb{H}), P) \to L^2(\Omega, \mathcal{F}(\mathbb{H}), P)$ as $\Gamma(A)(X) = \sum_{n=0}^{\infty} A^{:n:} X_n$.

Lemma 1. *$P_t = \Gamma(rI)$ for $r = e^{-t} < 1$.*

Proof. From above

$$\begin{aligned}
P_t f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(e^{-t}x + \sqrt{1-e^{-2t}}z) e^{-z^2/2} dz \\
&= \frac{1}{\sqrt{2\pi(1-r^2)}} \int_{-\infty}^{\infty} f(z) e^{-\frac{(z-rx)^2}{2(1-r^2)}} dz \qquad \text{(by change of variable and } r = e^{-t}) \\
&= \frac{1}{\sqrt{1-r^2}} \int_{-\infty}^{\infty} f(z) e^{-\frac{r^2x^2+r^2y^2-2rxy}{2(1-r^2)}} d\gamma(z) \qquad \text{This can be expanded as} \\
&= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} r^n \frac{h_n(x)h_n(z)}{n!} f(z) d\gamma(z) \qquad (h_n(x) \text{ is } n^{th} \text{ degree monic hermite polynomial}) \\
&= \sum_{n=1}^{\infty} r^n \frac{h_n(x)}{\sqrt{n!}} \int_{-\infty}^{\infty} \frac{h_n(z)}{\sqrt{n!}} d\gamma(z)
\end{aligned}$$

$\square$

An analogous result holds for n-dimensional Ornstein-Uhlenbeck process in which each coordinate is an independent Orstein-Uhlenbeck process.

Proof of Talagrand's $L^1 - L^2$ inequality: For an Ornstein-Uhlenbeck process one can verify the first line of the following.

$$\begin{aligned}
\mathrm{Var}\gamma^n(f) &= \int_0^{\infty} \sum_{i=1}^{n} \mathbb{E}_{\gamma^n}(\partial_i f \partial_i P_t f) dt \\
&= \int_0^{\infty} e^{-t} \sum_{i=1}^{n} \mathbb{E}_{\gamma^n}(\partial_i f P_t \partial_i f) dt \qquad (\nabla P_t = e^{-t} P_t \nabla) \\
&\leq \int_0^{\infty} e^{-t} \sum_{i=1}^{n} ||\partial_i f||_{L^2(\gamma^n)} ||P_t \partial_i f||_{L^2(\gamma^n)} dt \qquad \text{(Cauchy-Schwarz)} \\
&\leq \int_0^{\infty} e^{-t} \sum_{i=1}^{n} ||\partial_i f||_{L^2(\gamma^n)} ||\partial_i f||_{L^p(\gamma^n)} dt \qquad \text{(hypercontractivity, where } p = 1 + e^{-t}) \\
&\leq \int_0^{\infty} e^{-t} \sum_{i=1}^{n} ||\partial_i f||_{L^2(\gamma^n)} ||\partial_i f||_{L^2(\gamma^n)}^{2-\frac{2}{p}} ||\partial_i f||_{L^1(\gamma^n)}^{\frac{2}{p}-1} dt \qquad \text{(Hölder's inequality)}
\end{aligned}$$

Here Hölder's inequality is applied by writing $|\partial_i f|^p = |\partial_i f|^{2p-2} |\partial_i f|^{2-p}$, and observe that $\frac{1}{2p-2}$ and $\frac{1}{2-p}$ are conjugates.

$$
\begin{aligned}
\mathrm{Var}\gamma^n(f) &\leq \sum_{i=1}^{n} ||\partial_i f||^2_{L^2(\gamma^n)} \int_0^\infty e^{-t} \left(\frac{||\partial_i f||_{L^2(\gamma^n)}}{||\partial_i f||_{L^1(\gamma^n)}} \right)^{\tanh(t)} dt \\
&\leq \sum_{i=1}^{n} ||\partial_i f||^2_{L^2(\gamma^n)} \int_0^\infty e^{-t} \left(\frac{||\partial_i f||_{L^2(\gamma^n)}}{||\partial_i f||_{L^1(\gamma^n)}} \right)^{1-e^{-t}} dt \qquad (\because \tanh t \geq 1 - e^{-t}) \\
&\leq \sum_{i=1}^{n} ||\partial_i f||^2_{L^2(\gamma^n)} \int_0^1 \left(\frac{||\partial_i f||_{L^2(\gamma^n)}}{||\partial_i f||_{L^1(\gamma^n)}} \right)^{u} du \\
&\leq \sum_{i=1}^{n} ||\partial_i f||^2_{L^2(\gamma^n)} \frac{C}{1 + \log\left(\frac{||\partial_i f||_{L^2(\gamma^n)}}{||\partial_i f||_{L^1(\gamma^n)}} \right)}
\end{aligned}
$$

□

Remark: If we use the trivial bound (i.e $||P_t \partial_i f||_{L^2(\gamma^n)} \leq ||\partial_i f||_{L^2(\gamma^n)}$) instead of hypercontractivity, we obtain Poincaré inequality which is $\mathrm{Var}_{\gamma^n}(f) \leq \sum_{i=1}^{n} ||\partial_i f||^2_{L^2(\gamma^n)}$

Example: Let $X_1, X_2, \ldots, X_n$ are i.i.d. standard Gaussian random variables. Let $M_n := \max_{1 \leq i \leq n} X_i$. Define $f : \mathbb{R}^n \to \mathbb{R}$ as $f(x_1, x_2, \ldots, x_n) = \max_{1 \leq i \leq n} x_i$. It can be easily seen that $\partial_i f = 1_{\{ X_i \geq X_j \text{for all } j\}}$ so,

$$
\mathrm{Var}(M_n) \leq \begin{cases} 1 & \text{(Using Poincaré inequality)} \\ \frac{C}{\log(n)} & \text{(Using Talagrand's } L^1 - L^2 \text{ inequality)} \end{cases}
$$

We can see an improvement when we use hypercontractivity inequality instead of Cauchy-Schwarz.

*Computation for **1.**:* Enumerate edges of the rooted(root vertex r) binary tree such that $e_{2^k+1}, e_{2^k+2}, \ldots, e_{2^{k+1}}$ are the edges of the tree at depth k Let $X_{e_1}, X_{e_2}, \ldots, X_{e_{2^{n+1}-2}}$ be the weights of the edges of the tree (which are i.i.d. standard Gaussians). For every leaf vertex l define

$$
W_l = \sum_{\substack{e\,:\,e \text{ is in the unique} \\ \text{path joining } l \text{ and } r}} X_e
$$

$$
M_n = \max_{l:\, l \text{ is a leaf of the tree}} W_l
$$

We can define the corresponding f on $\mathbb{R}^{2^{n+1}-2}$.

For $2^{k+1}-1 \leq i \leq 2^{k+2}-2$

$$\begin{aligned}
|\partial_i f| &= \mathbb{1}_{\{\ e_i \text{ is in the optimal path}\}} \\
||\partial_i f||_{L^2(\gamma^n)} &= \frac{1}{2^{k/2}} \\
||\partial_i f||_{L^1(\gamma^n)} &= \frac{1}{2^k}
\end{aligned}$$

From Talagrand's $L^1 - L^2$ inequality we obtain

$$\begin{aligned}
\mathrm{Var}(M_n) = \mathrm{Var}_{\gamma^n}(f) &\leq C \sum_{k=1}^{n} \sum_{i=2^k-1}^{2^{k+1}-2} \frac{1/2^k}{1+\log(2^{k/2})} \\
&\leq C' \log(n)
\end{aligned}$$

This is an improvement over Poincaré inequality which gives $\mathrm{Var}(M_n) \leq n$. $\square$

References

[1] Janson, Svante. Gaussian Hilbert spaces. Vol. 129. Cambridge university press, 1997.

[2] Chatterjee, S. Superconcentration and Related Topics. Springer International Publishing, 2014.