PROBLEM SET 7
(MEASURE THEORY)

TO BE DISCUSSED ON 28ST MARCH IN TUTORIALS. PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. State whether true or false and justify. Whenever we write L^p, it is assumed that $1 \le p \le \infty$.

(1) $L^\infty(\mathbb{R}, \mathcal{B}, \lambda_1)$ is not separable (has no countable dense subset).

(2) If $p_1 < p_2$, then $L^{p_1}(X, \mathcal{F}, \mu) \supseteq L^{p_2}(X, \mathcal{F}, \mu)$.

(3) If $p_1 < p_2 < p_3$ then $L^{p_1}(\mu) \cap L^{p_3}(\mu) \subseteq L^{p_2}(\mu)$.

(4) If μ is a finite measure, then for any measurable f, the set $\{p \ge 1 : f \in L^p(\mu)\}$ is an interval of the form $[1, r]$ or $[1, r)$ for some $r \le \infty$.

(5) If f is a measurable function on $(X, \mathcal{F}, \mu)$, then the set $\{p \ge 1 : f \in L^p(\mu)\}$ is an interval (Extra: Give exmples to show that this interval can be open or closed on the left or right).

(6) $L^1(\mu) \cap L^2(\mu)$ is complete in the norm $\| \cdot \|_1 + \| \cdot \|_2$.

Problem 2. Show that $L^\infty(X, \mathcal{F}, \mu)$ is complete (This was skipped in class).

Problem 3. Let $f \in L^p(X, \mathcal{F}, \mu)$ with $p < \infty$.

(1) Show that $\|f\|_p^p = \int_0^\infty pt^{p-1}\mu\{|f| > t\}dt$. [*Note:* This was shown for $p = 1$ and simple non-negative f in an earlier exercise.]

(2) Show that that if $f \in L^p(\mu)$, then $\mu\{|f| > t\} \le \|f\|_p^p t^{-p}$.

Problem 4. Let μ be a finite measure. Then show that the following are equivalent.

(1) $f \in L^\infty(\mu)$.

(2) $f \in L^p(\mu)$ for all $p < \infty$ and $\sup_p \|f\|_p < \infty$.

If these equivalent conditions hold, show that $\|f\|_p \to \|f\|_\infty$.

Problem 5. (1) If $f, g \in L^2$, show that $\|f - g\|^2 + \|f + g\|^2 = 2\|f\|^2 + 2\|g\|^2$.

(2) (*) If $f_1, \ldots, f_n \in L^2(\mu)$, show that $(\langle f_i, f_j \rangle_{L^2(\mu)})_{1 \le i,j \le n}$ is a positive semi-definite matrix.

(3) Give examples to show that $L^p(X, \mathcal{F}, \mu)$ is not a Hilbert space if $p \neq 2$.

Problem 6. Let μ be a probability measure on X. If f, g are non-negative measurable and $fg \ge 1$ a.e.[μ], then $\left(\int_X f d\mu\right)\left(\int_X g d\mu\right) \ge 1$.