HOMEWORK 1

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Problem 1. Let (X, Y) have density $f(x, y)$ on $\mathbb{R}^2$. Find the conditional distribution of X given Y. Find the conditional expectation of $\varphi(X)$ given Y for any bounded measurable $\varphi : \mathbb{R} \mapsto \mathbb{R}$.

Problem 2. Let $X \sim N_n(\mu, \Sigma)$. Write $X = (X_1, \ldots, X_n)^t$. For $m < n$, let $Y = (X_1, \ldots, X_m)^t$ and $Z = (X_{m+1}, \ldots, X_n)^t$. Find the conditional distribution of Z given Y as explicitly as possible. In particular, work out the case when $m = n - 1$.

Problem 3. Let $X_1, \ldots, X_n$ be i.i.d. Ber(p) random variables. Find the conditional distribution of $(X_1, \ldots, X_n)$ given $S := X_1 + \ldots + X_n$.

Problem 4. Let X, Y be integrable random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-sigma algebra and $A \in \mathcal{G}$. If $X = Y$ on A, show that $\mathbf{E}[X \mid \mathcal{G}] = \mathbf{E}[Y \mid \mathcal{G}]$ $a.e.[\mathbf{P}]$ on A. [*Remark:* This is called locality property of conditional expectation.]

Problem 5. Prove the following statements. The setting is $(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub-sigma algebra.

(1) Conditional Markov inequality: If $X \geq 0$ and integrable, then $\mathbf{P}(X \geq t \mid \mathcal{G}) \leq \frac{1}{t}\mathbf{E}[X \mid \mathcal{G})$ $a.s.[\mathbf{P}]$.

(2) Conditional Cauchy-Schwarz inequality: If X, Y are square integrable, then $\mathbf{E}[XY \mid \mathcal{G}]^2 \leq \mathbf{E}[X^2 \mid \mathcal{G}]\mathbf{E}[^2Y \mid \mathcal{G}]$ $a.s.[\mathbf{P}]$.

(3) Analysis of variance: If X has finite variance, then $\text{Var}(X) = \mathbf{E}[\text{Var}(X \mid \mathcal{G})] + \text{Var}(\mathbf{E}[X \mid \mathcal{G}])$.

Problem 6. (1) Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean. Let $S_n = X_1 + \ldots + X_n$. Show that $\mathbf{E}[X_1 \mid S_n] = \frac{1}{n}S_n$.

(2) If $X \sim \text{Exp}(\lambda)$, and $Y = \mathbf{1}_{X \geq t}$, find the conditional distribution of X given Y.

(3) If X, Y are i.i.d. Exp(λ), find the conditional distribution of $\frac{X}{X+Y}$ given $X + Y$.

Problem 7. Let $(\Omega_i, \mathcal{F}_i)$ be two measure spaces and let ν be a probability measure on Ω_1 and let $\kappa : \Omega_1 \times \mathcal{F}_2 \mapsto [0, 1]$ be a stochastic kernel (as defined in class). Define a probability measure μ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ by $\mu(A \times B) = \int_A \left[\int_B \kappa(x, dy)\right] \nu(dx)$ for cylinder sets $A \times B \in \mathcal{F}_1 \otimes \mathcal{F}_2$. Let X, Y be the projections from $\Omega_1 \times \Omega_2$ to Ω_1 and to Ω_2, respectively.

(1) Find the conditional distribution of Y given X.

(2) Find the conditional distribution of X given Y. (This is called *Bayes' rule*).

Problem 8. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\mathbf{Q}$ be another probability measure on $\mathcal{F}$ such that $d\mathbf{Q} = Y d\mathbf{P}$ (Y is the Radon Nikodym derivative). If X is a random variable integrable with respect to $\mathbf{Q}$, then XY is integrable with respect to $\mathbf{P}$ and for any $\mathcal{G} \subseteq \mathcal{F}$,

$$E_{\mathbf{Q}}[X \mid \mathcal{G}] = \frac{\mathbf{E}_{\mathbf{P}}[XY \mid \mathcal{G}]}{\mathbf{E}_{\mathbf{P}}[Y \mid \mathcal{G}]}.$$