HOMEWORK 2

POSTED 24/01/2020

Problem 1. Let S_1, S_2 be finite sets. Suppose $\mu : S_1 \mapsto \mathcal{P}(S_2)$ and $\nu : S_2 \mapsto \mathcal{P}(S_1)$ are stochastic kernels (i.e., μ_x, $x \in S_1$, are probability measures on S_2 and ν_y, $y \in S_2$, are probability measures on S_1). The question is whether there exists a probability distribution α on $S_1 \times S_2$ such that if $(X, Y) \sim \alpha$, then μ_x is the conditional distribution of Y given $X = x$ and ν_y is the conditional distribution of X given $Y = y$.

Show that the answer is yes if and only if the function $(x, y) \mapsto \frac{\mu_x\{y\}}{\nu_y\{x\}}$ factors as a function of x times a function of y.

Problem 2. Let X be a sub-martingale. Show that there is a unique pair of processes (M, A) such that M is a martingale, A is a predictable increasing process with $A_0 = 0$ and such that $X_n = M_n + A_n$. [*Remark:* Here the filtration is fixed throughout. Also, the uniqueness is up to sets of zero probability (make that precise).]

Problem 3. Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbf{P})$ be a filtered probability space and let X be a sub-martingale. Let $\tau_1 \le \tau_2$ be two stopping times such that $\tau_2 \le N$ $a.s.$ for some $N < \infty$.

(1) Show that $\mathbf{E}[X_{\tau_2} \bigm| \mathcal{F}_{\tau_1}] \ge X_{\tau_1}$.

(2) If $\{X_{\tau_2 \wedge n}\}$ is uniformly integrable, show the same conclusion without the hypothesis that τ_2 is a bounded random variable.

Problem 4. Let S be a random walk on $\mathbb{R}$ with i.i.d. $N(0, 1)$ steps.

(1) For $\theta \in \mathbb{R}$, show that $M_n^\theta := e^{\theta S_n - \frac{1}{2}\theta^2 n}$ is a martingale.

(2) Differentiate w.r.t θ repeatedly and set $\theta = 0$ to get martingales that are polynomials in S_n and n. Evaluate the first four of these explicitly (you may show that these are martingales directly or by justifying differentiation under expectation).

Problem 5. Let L_a denote the graph with vertices $\{0, 1, \dots, a\}$ with edges between i and $i + 1$ for $0 \le i \le a - 1$. Fix $a_1, \dots, a_k$ and let G be the graph got by merging the 0 vertex of $L_{a_1}, \dots, L_{a_k}$ (it is a tree with one root from which paths of lengths $a_1, \dots, a_k$ emanate). Let X be SRW on G started at the root 0. Let τ be the first time that the RW hits a leaf (a leaf is a degree 1 vertex). Find the probability distribution of X_τ. [*Hint:* First solve for the right harmonic function and use that to answer the question]

Problem 6. (Gambler's ruin problem on a regular tree). Let T_n be the regular binary tree up to n generations (this is the tree where generation k has $3 \times 2^{k-1}$ individuals, for $k = 1, 2, \ldots, n$, generation 0 has one individual, and every vertex has degree 3). Let B denote the vertices in the n-th generation. Solve for the harmonic measure on B from any vertex v (i.e., find $\mathbf{P}_v(X(\tau_B) = u$ for any $u \in B$).

Problem 7. $Y_0, Y_1, \ldots$ be random variables (assume real-valued, although that is not necessary) on $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\mathcal{F}_n = \sigma\{Y_0, \ldots, Y_n\}$. Let τ be a $\mathcal{F}_\bullet$-stopping time. Show that $\mathcal{F}_\tau$ is the same as the sigma-algebra generated by the stopped process $\{Y_{\tau \wedge n}\}_{n \geq 0}$.

Problem 8. Go back to the problem of finding $\mathbf{E}[\tau_{101}]$ in a sequence of fair coin tosses.

(1) Write the proof given in class in mathematical terms (without reference to gamblers and betting, etc.)

(2) Find $\mathbf{E}[e^{u\tau}]$ for small enough u (again, come up with an appropriate betting game).