HOMEWORK 4

POSTED 29/03/2020

Problem 1. If $X_1, X_2, \dots$ is an infinite sequence of exchangeable random variables having finite variance, show that $\text{Cov}(X_i, X_j) \geq 0$ for all i, j. What if the sequence is finite, i.e., $X_1, \dots, X_n$ are exchangeable?

Problem 2. Let $X_1, X_2, \dots$ be an infinite sequence of exchangeable random variables with jointly Gaussian distribution. Then show that the sequence has the same distribution as $a + bY + cY_i$, where $a, b, c \in \mathbb{R}$ and Y, Y_i are i.i.d. $N(0, 1)$ random variables.

Problem 3. Let $d\mu_n(x) = f_n(x)dx$ and $d\nu_n(x) = g_n(x)dx$ be probability measures on $\mathbb{R}$. Show that $\mu_1 \otimes \mu_2 \otimes \dots$ is absolutely continuous to $\nu_1 \otimes \nu_2 \otimes$ if and only if $\sum_n \int_{\mathbb{R}} (\sqrt{f_n(x)}\sqrt{g_n(x)})^2 dx$ is finite.

Problem 4. Let μ_n and ν_n be probability measures on $\mathbb{R}$ that are mutually absolutely continuous. Let μ and ν be the corresponding product measures on $\mathbb{R}^{\mathbb{N}}$. Is it necessarily true that μ and ν are either singular to each other or mutually absolutely continuous? (In other words, can it happen that $\mu \ll \nu$ but $\nu \not\ll \mu$?)

Problem 5. Let μ be the Cantor measure (for the standard $1/3$-Cantor set) and let ν be the uniform measure on $(0, 1]$. Let $\mathcal{F}_n = \sigma\{(\frac{k}{3^n}, \frac{k+1}{3^n}] : 0 \leq k \leq 3^n - 1\}$. Find the Radon-Nikodym derivative of ν w.r.t. μ, when both are restricted to $\mathcal{F}_n$. How do they behave as $n \to \infty$, under μ and under ν?

Problem 6. Let $\mu = N(\alpha_1, 1) \otimes N(\alpha_2, 1) \otimes \dots$ and $\nu = N(0, 1) \otimes N(0, 1) \otimes \dots$. Find precise conditions on $(\alpha_n)_n$ that guarantee that $\mu \ll \nu$. In such cases is it also true that $\nu \ll \mu$?