HOMEWORK 5

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Problem 1. Show that the following kernels are positive semi-definite.

(1) $K(t,s) = t(1-s)$ for $0 \le t \le s \le 1$.

(2) $K(t,s) = e^{-|t-s|}$ for $t, s \in \mathbb{R}$.

(3) $K(t,s)^2$ and $e^{K(t,s)}$ where K is any positive definite kernel.

Problem 2. Let $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ denote the $N(0,t)$ density. Show that $\frac{\partial}{\partial t} p_t(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_t(x)$. How does this generalize to the standard Gaussian density on $\mathbb{R}^n$?

Problem 3. Let X be a Gaussian vector on $\mathbb{R}^n$ with $N(\mu, \Sigma)$ density where Σ is non-singular. Fix $k < 1$, write $Y = (X_1, \dots, X_k)^t$ and $Z = (X_{k+1}, \dots, X_n)^t$ and correspondingly write $\mu = (\mu_1, \mu_2)^t$ and $\Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{bmatrix}$.

Show that the conditional distribution of Y given Z is Gaussian on $\mathbb{R}^k$ with mean $\mu_1 + \Sigma_{1,2}\Sigma_{2,2}^{-1}(Z - \mu_2)$ and covariance matrix $\Sigma_{1,1} - \Sigma_{1,2}\Sigma_{2,2}^{-1}\Sigma_{2,1}$.

Problem 4. Let W be standard Brownian motion.

(1) Fix $0 < t_1 < \ldots < t_k < 1$. Find the conditional distribution of $(W(t_1), \dots, W(t_n))$ given $W(1) = x$. Specialize to the case $x = 0$.

(2) Fix $s < t < u$. What is the conditional distribution of $(W(s), W(u))$ given $W(t)$?

Problem 5. Let W be standard Brownian motion on $[0,1]$. Show that the regular conditional distribution of W given $W(1)$ is the same as the distribution of the random path $X(t) = tW(1) + W_0(t)$, $0 \le t \le 1$, where W_0 is the Brownian bridge (a Gaussian process on $[0,1]$ with continuous sample paths, zero means, and covariance $\mathbf{E}[W_0(t)W_0(s)] = t(1-s)$ for $0 \le t \le s \le 1$).

Problem 6. Let μ be the Wiener measure on $C[0,1]$.

(1) Show that it is tight, i.e., given any $\epsilon > 0$, there is a compact subset K_ϵ of $C[0,1]$ such that $\mu(K_\epsilon) > 1 - \epsilon$.

(2) Show that any open set in $C[0,1]$ has positive measure. (This is not trivial, but spend a few minutes thinking about it!).

Problem 7. Let W be standard Brownian motion on $[0,1]$. Let $f \in C[0,1]$. Show that $W + f$ is nowhere differentiable.

The two problems below are to indicate why it is not a good idea to work with the space of all functions. Stochastic processes are always thought of as random variables taking values in a space of functions with reasonable continuity properties:

Problem 8. Consider $\Omega = \mathbb{R}^{[0,1]}$, the space of all functions on $[0,1]$ and endow it with the cylinder sigma-algebra $\mathcal{F}$ generated by all the projections $\Pi_t : \Omega \mapsto \mathbb{R}$. If any event A is in the cylinder sigma-algebra, show that there is a *countable set* $I \subseteq \mathbb{R}_+$ such that $A \in \sigma\{\Pi_t : t \in I\}$. Hence show that $C[0,1]$ is not a measurable subset.

Problem 9. Consider $C[0,1]$ with the sup-norm distance. Show that the Borel sigma algebra is equal to the cylinder sigma-algebra (i.e., the one generated by all projections $\Pi_t(f) = f(t)$). Show that if D is any dense subset, then $\{\Pi_t : t \in D\}$ generates the Borel sigma algebra.