MARTINGALES AND BROWNIAN MOTION: MIDTERM 1
October 6, 2015

Each question carries 10 marks.
Duration of the exam: 120 minutes.

1. Let ξ_n be i.i.d. $\text{Ber}_{\pm}(1/2)$ random variables. Let $S_n = \xi_1 + \ldots + \xi_n$ for $n \geq 1$ and $S_0 = 0$.

(1) Find a martingale of the form $X_n = S_n^4 + a(n)S_n^2 + b(n)$ where $a(n), b(n)$ are non-random. [*Hint:* If I had asked for a martingale of the form $S_n^2 + a(n)$, then what works is $a(n) = -n$].

(2) For a positive integer M, let $\tau_M = \min\{n \geq 1 : S_n = \pm M\}$. Find the first and second moments of τ_M.

2. Let W be a standard 1-dimensional Brownian motion.

(1) Let $X_t = W(1-t) - W(1)$ for $0 \leq t \leq 1$. Show that X is a standard Brownian motion (indexed only by $[0, 1]$, of course).

(2) You are given a standard Brownian bridge $W_b = (W_b(t))_{0\leq t\leq 1}$ and an independent standard Normal random variable ξ. Use W_b and ξ to construct a standard Brownian motion (indexed by $[0, 1]$).

3. (1) Let X be an $\text{Exp}(\lambda)$ random variable. If $\mathcal{G} = \sigma(X \wedge 1)$, find $\mathbf{E}[X \mid \mathcal{G}]$.

(2) Suppose $0 < t_1 < t_2 < \ldots < t_m$ and $t > 0$ are given (assume $m \geq 2$ and that $t_k < t < t_{k+1}$ for some k). Let W be a standard Brownian motion. Find the conditional distribution of W_t given $\mathcal{G} := \sigma\{W_{t_1}, W_{t_2}, \ldots, W_{t_m}\}$.

4. Consider a binary tree up to n generations (figure given below for $n = 3$).

(1) Find the harmonic function that takes the value 0 at the leaves and 1 at the root.

(2) Starting from a vertex v in the tree, what is the probability that the random walk reaches the root (vertex labelled 1) before hitting one of the leaves (in the picture they are vertices labelled $8 - 15$)?