MARTINGALES AND BROWNIAN MOTION: MIDTERM 2
14/NOV/2015

Duration of the exam: 180 minutes.
Maximum marks: 75

1 (5 marks). If X_n are independent r.v.s with zero means and $\sum_n \mathrm{Var}(X_n) < \infty$, show that $\sum_n X_n$ converges almost surely.

2 (5 marks). Let X_i be i.i.d. random variables. Then $\frac{1}{n(n-1)} \sum\limits_{1\le i\ne j\le n} \sin(X_i X_j^2)$ converges almost surely to a constant.

3 (5 marks). Let W be a standard 1-dimensional Brownian motion. Use the law of iterated logarithm as stated in lecture to show that for any $t \ge 0$, almost surely the set of limit points (as $h \downarrow 0$) of $\frac{W(t+h)-W(t)}{\sqrt{h \log\log \frac{1}{h}}}$ is equal $[-\sqrt{2}, \sqrt{2}]$ with probability 1.

4 (10 marks). Let $\theta \sim \mathrm{unif}[0,1]$ and conditional on θ, let $X_1, X_2, \ldots$ be i.i.d. Ber(θ). Show that $\mathbf{E}[\theta \mid X_1, \ldots, X_n] \stackrel{a.s.}{\to} \theta$.

5 (10 marks). Let W be a standard 1-dimensional Brownian motion and let τ be a stopping time for its natural filtration (or right continuous version of it). If $\mathbf{E}[\tau] < \infty$, show that $(W_{t\wedge\tau})_{t\ge 0}$ is uniformly integrable.

6 (15 marks). Let $L(t) = \mu t + a$ where $\mu > 0$ and $a > 0$. Let W be standard one dimensional Brownian motion statrted at 0. Let $\tau = \min\{t : W_t = L(t)\}$. Show that $\mathbf{P}\{\tau < \infty\} = e^{-2a\mu}$.

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7 (15 marks). Consider an urn that initially contains one red, one blue, one green ball. Following Pólya's scheme, a ball is drawn at random, and returned to the urn with another ball of the same colour. The process is repeated. Let R_n, B_n, G_n be the numbers of balls of each colour after after n steps.

(1) Show that the $\frac{1}{n}(R_n, B_n, G_n)$ converges almost surely to a random vector (r, b, g) taking values in the simplex $\Delta = \{(x_1, x_2, x_3) : x_i \geq 0$ and $x_1 + x_2 + x_3 = 1\}$.

(2) Find the joint density of (r, b). [*Hint:* Show exchangeability of the colours drawn].

8 (20 marks). Let $W_t = (X_t, Y_t)$ be standard 2-dimensional Brownian motion started at $(0, 0)$. Let $\tau = \min\{t : Y_t = -1\}$.

(1) Show that X_τ has a Cauchy distribution symmetric about zero, i.e., the density of X_τ is $\frac{\alpha}{\pi(\alpha^2+u^2)}$ for some $\alpha > 0$.

(2) Show that $\alpha = 1$.

[*Hint:* For the first part, either (a) use the density of τ OR (b) use the fact that Cauchy distribution is the unique distribution such that $X + Y \stackrel{d}{=} 2X$ for i.i.d. copies X, Y. From (b), the second part is trickier.]