Some Solutions

1. $\mathcal{P}(\mathbb{R})$ **is a complete metric space under the Levy metric.**

Sol. Let $\{\mu_n\}$ be a cauchy sequence, i.e., given $\epsilon > 0$, $\exists\, N_0$, s.t $\forall\, n, m \geq N_0$, $d(\mu_n, \mu_m) < \epsilon$. In particular $\forall n > N_0$

$$F_{N_0}(x - \epsilon) - \epsilon < F_N(x) < F_{N_0}(x + \epsilon) + \epsilon.$$

Consider $\mu_1, \mu_2, ...\mu_{N_0}$. Any finite sequence of measures is tight. So given $\epsilon > 0$, $\exists\, K > 0$, s.t for $n = 1, 2, ...N_0$

$$F_n(K) > 1 - \epsilon$$

$$F_n(-K) < \epsilon$$

Now $\forall\, n > N_0$

$$\begin{aligned} F_n(K+1) &> F_{N_0}(K + 1 - \epsilon) - \epsilon \\ &\geq F_{N_0}(K) - \epsilon \\ &> 1 - 2\epsilon. \end{aligned}$$

Similarly

$$\begin{aligned} F_n(-k-1) &< F_{N_0}(-k - 1 + \epsilon) + \epsilon \\ &\leq F_{N_0}(-k) + \epsilon \\ &< 2\epsilon \end{aligned}$$

Thus $\{\mu_n\}$ is tight and so has a convergent subsequence, i.e., $\exists$ a probability measure μ and a subsequence μ_{n_k} of μ_n s.t,

$$\mu_{n_k} \xrightarrow{d} \mu$$

Since $\{\mu_n\}$ is cauchy

$$\mu_n \xrightarrow{d} \mu$$

In terms of metric,

$$d(\mu_n, \mu) \to 0 \text{ as } n \to \infty$$

Hence the completeness.

2. **If** $\mu_n \xrightarrow{d} \mu$ **then** $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ **if** G **is open.**

Sol. Since every open set in $\mathbb{R}$ is a countable union of disjoint open intervals it is enough to check the above for open intervals.

$$\mu_n(a, b) = F_{\mu_n}(b-) - F_{\mu_n}(a)$$

Since $\mu_n \xrightarrow{d} \mu$,

$$\lim\ \sup_{n\to\infty} F_{\mu_n}(x) \geq F_\mu(x) \tag{0.1}$$

Also for all $u > 0$

$$\lim\ \inf_{n\to\infty} F_{\mu_n}(x-) \geq \lim\ \inf_{n\to\infty} F_{\mu_n}(x-u) > F_\mu(x-2u) - u$$

Now letting $u \to 0$ we get,

$$\lim\ \inf_{n\to\infty} F_{\mu_n}(x-) > F_\mu(x-) \tag{0.2}$$

Therefore,

$$\begin{aligned} \lim\ \inf_{n\to\infty} \mu_n(a,b) &= \lim\ \inf_{n\to\infty} (F_{\mu_n}(b-) - F_{\mu_n}(a)) \\ &\geq \operatorname{liminf}_{n\to\infty} F_{\mu_n}(b-) - \lim\ \sup\ _{n\to\infty} F_{\mu_n}(a) \\ &\geq F_\mu(b-) - F_\mu(a) \\ &= \mu(a,b) \end{aligned}$$

where the third inequality follows from (0.1) and (0.2).
Hence the proof.