Problem set 1

Caution: Write the probability space, event, random variable clearly before calculating probabilities or probability mass functions!

Problem 1. The numbers $1, 2, \ldots, n$ are arranged in random order. Find the probability that the digits

(a) 1 and 2 appear consecutively in the order named.

(b) $1, 2, 3$ appear consecutively in some order.

(c) All odd numbers appear in odd numbered locations.

Problem 2. A fair die with $1, 2, \ldots, 6$ on its faces is thrown and if the number k shows up, a fair coin is tossed k times. Let X be the number of heads seen. Find the probability mass function of X.

Problem 3. A group of $2N$ boys and $2N$ girls is divided into two equal groups. Let X be the number of girls in the first group. Find the probability mass function of X.

Problem 4. Suppose r balls are placed into n bins at random, with each ball going into bin number k with probability α_k. Here α_i are positive numbers with $\alpha_1 + \ldots + \alpha_m = 1$. Let A_k be the event that the kth bin is empty.

Find $\mathbf{P}(A_1)$, $\mathbf{P}(A_2)$ and $\mathbf{P}(A_1 \cap A_2)$.

Problem 5. A man has n keys in a bunch, of which only one key opens the door to his room. In a state of drunkenness, he randomly tries various keys. Let X be the number of wrong attempts before he opens the door. Find the probability mass function of X in the two cases:

1. He may try the same key many times (sampling with replacement).
2. He does not try keys that have been tried before (sampling without replacement).

Problem 6. If r balls are put into m bins, find the probability that some bin is empty.

Problem 7. (Feller, II.12.1) Prove the following identities for $n \geq 2$. [*Convention:* Let n be a positive integer. Then $\binom{n}{y} = 0$ if y is not an integer or if $y > n$ or if $y < 0$].

$$1 - \binom{n}{1} + \binom{n}{2} - \ldots = 0$$
$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots = n2^{n-1}$$
$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \ldots = 0$$
$$2.1\binom{n}{2} + 3.2\binom{n}{3} + 4.3\binom{n}{4} + \ldots = n(n-1)2^{n-2}$$
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$