Problem set 10

[*Caution*: Unlike in previous homeworks, you may have to use your brain.]

Problem 1. Let $X = (X_t)_{t\geq 0}$ be a random walk on $\mathbb{Z}$ with transition probabilities $P_{i,i+1} = p = 1 - P_{i,i-1}$ where $0 < p < 1$ is fixed.

1. Find the multi-step transition probabilities $P^t_{i,j}$ explicitly.

2. Show that the chain is transient unless $p = \frac{1}{2}$ and find the expected number of returns to zero, $\mathbf{E}_0[N_0]$.

Problem 2. Is the Markov chain on $S = \{1,2,3,4,5\}$ with transition matrix P given below 1. Irreducible? 2. Aperiodic?

$$P = \begin{bmatrix} 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Problem 3. Consider a graph with four vertices $1,2,3,4$ with edges $1 \leftrightarrow 2$, $1 \leftrightarrow 3$, $3 \leftrightarrow 2$, $1 \leftrightarrow 4$.

Find the eigenvalues and eigenvectors (use a computer!) of the transition matrix and hence find the limit of $P^t_{i,j}$ for all i, j.

Problem 4. Let $X = (X_t)_{t\geq 0}$ be the simple symmetric random walk on the complete graph K_n (this is the graph with vertex set $\{1,2,\dots,n\}$ and edges between every pair of vertices (for this problem, include self-loops $i \leftrightarrow i$).

1. Write down the transition matrix of the chain.

2. Find $\mathbf{E}_i[T_1]$ for all i. Can you find the exact distribution of T_1 if the chain starts from i?

3. What can you say about $T = \max\{T_1,\dots,T_n\}$?

Problem 5. Let $X = (X_t)_{t \geq 0}$ be a MC on a state space S with transition P.

1. If T and S are stopping times, show that $T + S$ is a stopping time. Give explicit examples to show that $T - S$ need not be a stopping time, even if $T \geq S$ with probability 1.

2. Which of the following are stopping times?

   (a) $T = \min\{t \leq T_i : X_t = j\}$ where i, j are fixed vertices. Here we set $T = T_j$ if the above set is empty.

   (b) $T = \max\{t \leq T_i : X_t = j\}$ where i, j are fixed vertices. Here we set $T = T_j$ if the above set is empty.

   (c) T is the first time the chain hits i after hitting j at least 5 times and after hitting k at least 2 times.

Problem 6. Let X be the biased random walk on $\mathbb{Z}$ (see the first problem) with probability p of moving to the right. Assume that $p > \frac{1}{2}$.

1. If $\mathbf{P}_0\{T_{-1} < \infty\} = \alpha$, show that $\mathbf{P}\{T_{-k} < \infty\} = \alpha^k$.

2. Show that $\alpha = 1 - p + p\alpha^2$ and hence find the exact value of α.

Problem 7. Let X be the biased random walk on $\mathbb{Z}$ (see the first problem) with probability p of moving to the right. Assume that $p < \frac{1}{2}$.

1. If $\beta = \mathbf{E}_1[T_0]$, show that $E_k[T_0] = k\beta$ for any $k \geq 1$.

2. Show that $\beta = \frac{1-p}{1-2p}$.

Problem 8. (*) In the coupon collector problem, recall that we had defined $T_1 = 1$ and T_k to be the number of draws required to see a new coupon after having seen $k-1$ distinct coupons. We had appealed to intuition to say that T_ks are independent and that T_k has a Geometric distribution with mean $n/(n-k+1)$.

Use the strong Markov property to justify these statements.