Problem set 2

Problem 1. A fair coin is tossed 6 times.

1. What is the probability that there are three consecutive heads?
2. What is the probability that there are three consecutive heads or three consecutive tails?

[**Note:** When we say A or B we always mean $A \cup B$].

Problem 2. A deck of 52 cards is shuffled completely and placed face down on a table. A person claiming to be a psychic writes down his/her guess for the order of all the cards in the deck. The guesses are tested against the actual deck and the number of correct guesses is counted.

1. Write down the probability space and the random variable under discussion.
2. Find the probability that all guesses were wrong.

Problem 3. Suppose kb labelled balls are placed at random in b labelled bins. Show that the probability that some bin is empty is at most be^{-k}.

Problem 4. A box contains coupons labelled $1, 2, \dots, n$. Draw $(1+\delta)n \log n$ coupons from the box one after another, with replacement. Let $p_{n,\delta}$ be the probability that all coupons have shown up at least once. Show that $p_{n,\delta} \to 1$ as $n \to \infty$ for any fixed $\delta > 0$. [**Hint:** How is this question related to the previous one?]

Problem 5. Suppose X is a $\text{Bin}(n, p)$ random variable on a probability space.

1. Find k such that $\mathbf{P}\{X = k\}$ is maximized.
2. Find the distribution of the random variable $Y = n - X$.

Problem 6. Let X be a $\text{Geo}(p)$ random variable on a probability space. Find $\mathbf{P}\{X \geq m\}$ for any $m \geq 1$. Also find $\mathbf{P}\left\{X \geq m + \ell \;\middle|\; X \geq \ell\right\}$.

Problem 7. Let X_n be a $\text{Bin}(n, \frac{\lambda}{n})$ random variable where $\lambda > 0$ is fixed (this makes sense only for n sufficiently large). Show that

$$\mathbf{P}\{X_n = k\} \to e^{-\lambda}\frac{\lambda^k}{k!}.$$

Problem 8. Let $f(k) = \frac{1}{k(k+1)}$ for $k = 1, 2, \ldots$. Show that f is a probability mass function. If X is a random variable with pmf f, what is $\mathbf{P}\{X \geq m\}$?