Problem set 3

Problem 1. Decide intuitively whether the following pairs of random variables X and Y are independent or not, and then justify your answer mathematically.

1. Two fair dice are thrown 10 times each. Let X be the number of sixes shown by the first die and let Y be the number of ones shown by the second die.

2. A fair die is thrown. If it shows k, then two fair coins are each tossed k times. Let X and Y be the number of heads shown by the first coin and the second coin, respectively.

3. Two number U and V are drawn from the set $\{0, 1, 2, \ldots, 7\}$. Let $X = U$ and $Y = U + V \pmod 8$.

4. Three fair dice are thrown. The first die shows up number U, the second shows the number V and the third shows that number W. Set $X = U - W$ and $Y = V - W$.

Problem 2. Let n distinguishable balls be placed at random in b labelled bins. Assume that $b \geq 2$. If X_k denotes the number of balls that fall into the kth bin, then find the probability mass function of X_1 and the joint probability mass function of (X_1, X_2).

Problem 3. If X, Y are integer-valued random variables having joint probability mass function of the form $f(k, \ell) = g(k)h(\ell)$ for some functions g and h, then show that X and Y are independent.

Problem 4. Suppose X is an integer valued random variable such that for any $k \in \mathbb{Z}$, the events $\{X \leq k\}$ and $\{X \geq k\}$ are independent. What can you say about the probability mass function of X?

Problem 5. Let X, Y be independent random variables, both having $\mathrm{Bin}(n, 1/2)$ distribution. Let Z be a random variable with $\mathrm{Bin}(2n, 1/2)$ distribution. Show that $\mathbf{P}\{X = Y\} = \mathbf{P}\{Z = n\}$.

Problem 6. Let X, Y be independent random variables, both having $\text{Geo}(1/2)$ distribution. Find (a) $\mathbf{P}\{X = Y\}$, (b) $\mathbf{P}\{X = 3Y\}$, (c) $\mathbf{P}\{X > Y\}$.

Problem 7. Three fair dice are thrown. If the total of the three numbers that show up is 8, then what is the probability that one of the three numbers is 2?

Problem 8. A fair die with $1, 2, \ldots, 6$ on its faces is thrown and if the number Y shows up, a fair coin is tossed Y times. Let X be the number of heads seen.

1. Find $\mathbf{P}\{X = 3 \mid Y = 3\}$.

2. Find $\mathbf{P}\{Y = 3 \mid X = 3\}$.

Problem 9. From a shuffled deck of 52 cards, fours cards are dealt to player A and four cards are dealt to player B.

1. What is the chance that player A has all four suits?

2. Given that A has all four suits, what is the chance that player B has all four suits?