Problem set 4

Problem 1. Show that the given functions are probability density functions and find the corresponding cumulative distribution functions.

1. $f(x) = 6x(1-x)$ if $0 < x < 1$ (and 0 otherwise).
2. $f(x) = \frac{1}{\pi\sqrt{1-x^2}}$ if $-1 < x < 1$.
3. $f(x) = xe^{-\frac{1}{2}x^2}$ for $x > 0$.

Problem 2. Show that the given functions are probability mass functions and find the corresponding CDFs.

1. $f(k) = \frac{1}{k(k+1)}$ for $k = 1, 2, \ldots$.
2. $f(k) = \log_{10}(1 + \frac{1}{k})$ for $k = 1, 2 \ldots, 9$.

Problem 3. Let X be a random variable with CDF F. What are the CDFs of (a) X^2, (b) X^3, (c) e^{-X}, (d) $\lfloor X \rfloor$?

Problem 4. Show that the following functions are CDFs and find the pdf or pmf, if it exists.

1. $F(x) = \frac{1}{\pi}\sin^{-1}(x)$ for $-1 < x < 1$, $F(x) = 0$ if $x \leq 0$ and $F(x) = 1$ if $x \geq 1$. Here $\sin^{-1}$ is taken to have values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
2. $F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$ for all x. Here $\tan^{-1}$ is taken to have values in $(-\pi/2, \pi/2)$.
3. $F(x) = 1 - 2^{-k}$ if $k \leq x < k+1$ for $k = 0, 1, 2 \ldots$ and $F(x) = 0$ for $x < 0$.
4. $$F(x) = \begin{cases} 0 & \text{if } x < -1, \\ \frac{1}{2}(x+1)^2 & \text{if } -1 \leq x \leq 0, \\ 1 - \frac{1}{2}(1-x)^2 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Problem 5. Let $X \sim \text{Exp}(1)$ and $Y = e^{-X}$. Show that $Y \sim \text{Unif}[0,1]$.

Problem 6. If F is a CDF, which of the following are necessarily CDFs?

1. $G(t) = F(t)^2$.
2. $G(t) = 1 - F(t)$.
3. $G(t) = F(t)$ if $F(t) < 0.7$ and $G(t) = 1$ if $F(t) \geq 0.7$.
4. $G(t) = F(t)$ if $F(t) \leq 0.7$ and $G(t) = 1$ if $F(t) > 0.7$.

Problem 7. (*) Give example of a CDF that is strictly increasing on the whole line but having a dense set of jumps.

Problem 8. (*) Let X be a random variable with CDF F. If F is continuous everywhere, then show that $\mathbf{P}\{X \in \mathbb{Q}\} = 0$.