Problem set 5

Problem 1. Explain how to simulate random numbers from the following densities, assuming you have a random number generator that outputs uniform random numbers.

1. $f(x) = 6x(1-x)$ if $0 < x < 1$ (and 0 otherwise).
2. $f(x) = \frac{1}{\pi\sqrt{1-x^2}}$ if $-1 < x < 1$.
3. $f(x) = xe^{-\frac{1}{2}x^2}$ for $x > 0$.

Problem 2. Explain how to simulate random numbers from the following mass functions, assuming you have a random number generator that outputs uniform random numbers.

1. $f(k) = \frac{1}{k(k+1)}$ for $k = 1, 2, \ldots$.
2. $f(k) = \log_{10}(1 + \frac{1}{k})$ for $k = 1, 2 \ldots, 9$.

Problem 3. If F and G are distribution functions and $0 \le \alpha \le 1$, then show that $H = \alpha F + (1-\alpha)G$ is also a distribution function.

If you have a uniform random number generator and a method to generate random numbers from F and from G, how would you generate random numbers from H?

Problem 4. Given a random number generator that gives random numbers from Exp(1) distribution, how would you generate random numbers from Pois(1) distribution?

Problem 5. Let F be a distribution function. Define

$$F^{-}(t) = \inf\{x \, : \, F(x) \ge t\} \quad \text{for } 0 < t < 1.$$
$$F^{*}(t) = \inf\{x \, : \, F(x) > t\} \quad \text{for } 0 < t < 1.$$

1. Give examples of F where F^{-} and F^{*} are not the same.

2. Given a random number U from uniform$[0,1]$ distribution, what will be the distributions of $F^{-}(U)$ and $F^{*}(U)$? Discuss.

Problem 6. (*) Let f be a density on an interval $[a,b]$ with a uniform bound $f(t) \leq M$ for all $t \in [0,1]$. Show that the following procedure generates random numbers from the density f.

1. Pick U,V be independent uniform$[0,1]$ random variables.
2. If $V \leq \frac{1}{M}f(U)$, then set $X = U$. If $U > \frac{1}{M}f(V)$, return to step-1 and pick two independent uniforms again.

[*Remark:* This is called the method of rejection sampling.]

Problem 7. (*) Let f and g be two densitites and assume that for some M, we have $g(t) \leq Mf(t)$ for all t. Given a random number generator that gives independent outputs according to density f, explain how you would generate random numbers according to density g.